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Method of Successive Projections for Finding a Common Point of Sets in Metric Spaces

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Abstract. Many problems in applied mathematics can be abstracted into finding a common point of a finite collection of sets. If all the sets are closed and convex in a Hilbert space, the method of successive projections (MOSP) has been shown to converge to a solution point, i.e., a point in the intersection of the sets. These assumptions are however not suitable for a broad class of problems. In this paper, we generalize the MOSP to collections of approximately compact sets in metric spaces. We first define a sequence of successive projections (SOSP) in such a context and then proceed to establish conditions for the convergence of a SOSP to a solution point. Finally, we demonstrate an application of the method to digital signal restoration.

Key Words. Successive projections, convergence, nonlinear optimization, set-valued projections, metric spaces.

1. Introduction

In pure and applied mathematics, ideas can often most simply and concisely be expressed in terms of set concepts and set notations. In this paper, we address the broad class of applied problems whose basic formulation is as follows: given a finite collection of sets in an abstract space, find a point which belongs to their intersection. Among the numerous problems which have been formalized within this general framework, we can specifically mention the solution of systems of linear equations (Ref. 1), multi-constrained optimization (Ref. 2), band-limited extrapolation

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(Ref. 3), control (Ref. 4), signal restoration (Refs. 5, 6, 7), tomographic image reconstruction (Ref. 8), and electron microscopy (Ref. 9). Additional references can be found in Refs. 10 and 11.

For purpose of illustration, let us give a nonlinear programming example with m constraints in $\mathbb{R}^n$. Suppose that it is desired to find a feasible point y which gives some objective function J a value at most J_0, i.e., find y in $\mathbb{R}^n$ such that

$$J(y) \leq J_0 \quad \text{and} \quad f_i(y) \leq 0, i = 1, \ldots, m, \tag{1}$$

where the functionals $\{f_1, \ldots, f_m\}$ represent the constraints. The set theoretic formulation of this problem reads: Find

$$y \in S = \bigcap_{i=0}^{m} S_i, \tag{2}$$

where

$$S_0 = \{x \in \mathbb{R}^n \,|\, J(x) \leq J_0\}, \tag{3a}$$

$$S_i = \{x \in \mathbb{R}^n \,|\, f_i(x) \leq 0\}, \qquad i = 1, \ldots, m. \tag{3b}$$

Let $\{S_1, \ldots, S_m\}$ be a collection of sets in an abstract space X, with a nonempty intersection S. It is sought to establish a mathematical procedure which, under proper assumptions, will produce a solution to our basic problem, i.e., a point in S. If all the sets are closed and convex in a Hilbert space X, then the method of successive projections (MOSP) has been shown to yield a solution, as will be seen in Section 3. There is however a vast body of problems whose set-theoretic formulation does not comply with the requirements stated above, which narrows considerably the scope of the MOSP. For instance, one of the sets may not be convex or the underlying space may not be hilbertian.

The objective of this paper is to extend the MOSP to collections of approximately compact sets in metric spaces and to set forth conditions for the convergence of the method to a solution. The paper is organized as follows. Section 2 is devoted to preliminary definitions and results. In Section 3, a brief review of the MOSP in Hilbert spaces is given. In Section 4, the MOSP is generalized to metric spaces and convergence results are established. An application of the method to the problem of digital signal restoration is demonstrated in Section 5. Finally, our conclusions appear in Section 6.

2. Projections in Metric Spaces

We devote this section to providing some elements of the theory of projections in metric spaces. In the following, X is a metric space with distance d. The closed ball of center x and radius r in X is denoted by $B(x, r)$. The closure of a set S is denoted by $\bar{S}$.

2.1. Brief Review of Set-Valued Maps. Let X_1, X_2, and X_3 be topological spaces. The class of nonempty closed subsets of X_i is denoted by 2^{X_i}. By a set-valued map from X_1 into 2^{X_2}, we mean a function T which assigns to each point x in X_1 a set $T(x)$ in 2^{X_2}. Following Kuratowski (Ref. 12), a set-valued map T from X_1 into 2^{X_2} is said to be upper semicontinuous (u.s.c.) at a point x_0 in X_1 if, for every open neighborhood V of $T(x_0)$, there exists an open neighborhood U of x_0 such that $T(x) \subset V$, $\forall x \in U$. T is said to be u.s.c. if it is u.s.c. at every point in X_1. If T is u.s.c. and if we further assume that X_2 is a metric space, then T is closed in the sense that the set $\{(x, y) \in X_1 \times X_2 \mid y \in T(x)\}$ is closed in the topological product $X_1 \times X_2$. Berge (Ref. 13) defines u.s.c. maps in a slightly different manner by imposing that, in addition to the above, the set $T(x)$ be compact in X_2 for all x in X_1. Thereafter, we shall call such a map upper Berge semicontinuous (u.B.s.c.).

If $T_1: X_1 \to 2^{X_2}$ and $T_2: X_2 \to 2^{X_3}$ are two set-valued maps, then the image under the composition $T = T_2 \circ T_1$ of a point x of X_1 is defined as

$$T(x) = \bigcup_{y \in T_1(x)} T_2(y).$$

Theorem 2.1. See Ref. 13. Let X_1, X_2, and X_3 be metric spaces. Let T_1 be a u.s.c. [respectively u.B.s.c.] set-valued map from X_1 into 2^{X_2}, and let T_2 be a u.s.c. [respectively u.B.s.c.] set-valued map from X_2 into 2^{X_3}. Then, the composition product $T = T_2 \circ T_1$ is a u.s.c. [respectively u.B.s.c.] set-valued map from X_1 into 2^{X_3}.

2.2. Projections and Set Properties in Metric Spaces. Let S be a nonempty subset of (X, d). The distance from a point x in X to S is given by

$$\phi_S(x) = \inf\{d(x, y) \mid y \in S\}.$$

It is well known that ϕ_S is continuous and that

$$\phi_S(x) = 0 \Leftrightarrow x \in \bar{S}, \quad \forall x \in X$$

(Ref. 14). If x is a point in X, we shall call y a projection of x onto S if y belongs to S and $\phi_S(x) = d(x, y)$. S is said to be proximinal if every point in X has at least one projection onto S, and S is called a Chebyshev set if every point of X has exactly one projection onto S. S is said to be boundedly compact if its intersection with an arbitrary closed ball is compact, and S is called approximately compact if, for every x in X, every sequence $\{y_n\}_{n \geq 0}$ of points in S such that $\{d(x, y_n)\}_{n \geq 0}$ converges to $\phi_S(x)$ possesses a subsequence converging to a point in S. The notion of approximative compactness was introduced by Efimov and Stechkin for real Banach spaces (Ref. 15) and was naturally extended to arbitrary metric spaces by Singer (Ref. 16).

Theorem 2.2. Let S be a nonempty subset of a metric space X. Then, each property in the following list implies the next:

(i) S is compact;
(ii) S is boundedly compact;
(iii) S is approximately compact;
(iv) S is proximinal;
(v) S is closed.

In addition, if X is a finite-dimensional normed vector space, properties (ii) through (v) are equivalent.

Proof. (i)⇒(ii) is trivial; (ii)⇒(iii) and (iii)⇒(iv) are due to Efimov and Stechkin (Ref. 15); (iv)⇒(v) because no point in $\bar{S}-S$ admits a projection onto S. The last assertion follows from the fact that, in such a metric space X, all the closed balls are compact and therefore every closed set is boundedly compact. □

2.3. Projection Maps. By the projection operator onto a Chebyshev subset S of X, we mean the function π_S from X onto S which maps every point x into its unique projection onto S. By the projection map onto a proximinal subset S of X, we mean the set-valued map Π_S defined by

$$\Pi_S : X \to 2^S$$
$$x \mapsto \{y \in S \mid \phi_S(x) = d(x, y)\}. \tag{4}$$

Theorem 2.3. The projection map onto a nonempty approximately [respectively boundedly] compact subset S of a metric space X is u.s.c. [respectively u.B.s.c.] from X into 2^S.

Proof. The first part of the theorem is due to Singer (Ref. 16) and the part in brackets is simply proven by noting that

$$\Pi_S(x) = S \cap B(x, \phi_S(x)), \qquad \forall x \in X. \tag{5}$$

Hence, if S is boundedly compact, the set $\Pi_S(x)$ is compact for all x in X. □

3. Method of Successive Projections in Hilbert Spaces

The theory of successive projections in Hilbert spaces rests on the following result.

Theorem 3.1. See Ref. 17. Let S be a nonempty convex and complete subset of a pre-Hilbert space X with scalar product $(\cdot|\cdot)$. Then,

(i) S is a Chebyshev set;
(ii) $\mathrm{Re}(x-\pi_S(x)|y-\pi_S(x))\leq 0$, $\forall x\in X$, $\forall y\in S$.

Definition 3.1. Let $\{S_1,\ldots,S_m\}$ be an ordered collection of nonempty closed and convex subsets of a Hilbert space X. For every i in $\{1,\ldots,m\}$ we denote by π_i the projection operator onto the (Chebyshev) set S_i, and by π the composition $\pi_1\circ\cdots\circ\pi_m$. Given a point x_0 in X, we shall call a sequence of successive projections (SOSP) the sequence $\{x_n\}_{n\geq 0}$ constructed according to the recursion

$$x_{n+1}=\pi(x_n)=\pi^{n+1}(x_0),\qquad \forall n\in\mathbb{N}. \tag{6}$$

Proposition 3.1. Suppose that, in addition to the hypotheses of Definition 3.1, the intersection S of the sets is nonempty. Then, if the SOSP converges, it is to a point in S.

Proof. By a corollary of Theorem 3.1, each π_i is continuous (Ref. 17). Hence, π is continuous and, if the SOSP $\{x_n\}_{n\geq 0}$ converges to a point x, $\{\pi(x_n)\}_{n\geq 0}$ converges to $\pi(x)$. Consequently, by (6), $\{x_{n+1}\}_{n\geq 0}$ converges to $\pi(x)$. Hence, $x=\pi(x)$. But since every fixed point of π is in S (Ref. 5), it follows that x is in S. □

The main convergence results of the MOSP in Hilbert spaces can now be stated.

Theorem 3.2. Let $\{S_1,\ldots,S_m\}$ be an ordered collection of closed and convex subsets of a Hilbert space X whose intersection S is not empty. Then, for every x_0 in X:

(i) the SOSP converges weakly to a point in S;
(ii) the SOSP converges strongly to a point in S if one of the sets is boundedly compact;
(iii) the SOSP converges strongly to a point in S if all the sets are linear varieties.

The first assertion is due to Brègman (Ref. 18) and the second to Stiles (Ref. 19). Assertion (iii) was proved by Halperin (Ref. 20) for vector subspaces, but his proof can routinely be extended to linear varieties. For $m = 2$, Halperin's result is known as the alternating projection theorem and is due to Von Neumann (Ref. 21). If the dimension of X is finite, (i) and (ii) are equivalent and (iii) is a particular case of (i), because then a closed set is boundedly compact and the notions of weak and strong convergence coincide.

For completeness, let us mention that, in order to improve the speed of convergence of the MOSP, Gubin *et al.* (Ref. 10) have introduced relaxed projection operators which extend the projections beyond the boundary of the sets. It is also noted that, in the MOSP presented above, the sets are activated in cyclic order at each iteration. Other schemes have been considered in the literature, which aim at an optimal speed of convergence. In that respect, Ottavy (Ref. 11) has recently presented a unified framework for the study of a very broad class of projection algorithms, along with strong convergence results.

4. Method of Successive Projections in Metric Spaces

As was seen in the introduction, the framework of the MOSP as described in the previous section is unsuitable for a wide class of problems. In this section, we shall broaden the scope of the MOSP by placing ourselves in the general setting of a metric space where $\{S_1, \ldots, S_m\}$ is a collection of proximinal sets.

4.1. Cyclic Projection Map

Definition 4.1. Let $\Gamma = \{S_1, \ldots, S_m\}$ be an ordered collection of proximinal sets in a metric space X. For every i in $\{1, \ldots, m\}$, we denote by Π_i the projection map onto S_i, regarded as a set-valued map from X into 2^X. Then, the composition map $\Pi = \Pi_1 \circ \cdots \circ \Pi_m$ will be called the cyclic projection map of Γ.

Theorem 4.1. The cyclic projection map of an arbitrary ordered finite collection of nonempty approximately [respectively boundedly] compact sets in a metric space X is a u.s.c. [respectively u.B.s.c.] map from X into 2^X.

Proof. It is done by Theorem 2.3 and by invoking Theorem 2.1 inductively. □

4.2. Sequence of Successive Projections

Definition 4.2. Let Π be the cyclic projection map of an ordered collection of proximinal sets $\Gamma = \{S_1, \ldots, S_m\}$ in a metric space X. Then, given a point x_0 in X, we shall call a SOSP (relative to Γ and x_0) any sequence $\{x_n\}_{n\geq 0}$ constructed according to the recursion

$$x_{n+1} \in \Pi(x_n), \qquad \forall n \in \mathbb{N}. \tag{7}$$

In words, one selects a projection y_m of x_0 onto S_m, then a projection y_{m-1} of y_m onto S_{m-1}, and so on. The projection of y_2 onto S_1 which has been selected is x_1. A SOSP $\{x_n\}_{n\geq 0}$ is constructed by continuing this cyclic process *ad infinitum*. It is noted that such a sequence exists, since $\Pi(x) \neq \varnothing$, $\forall x \in X$. Moreover, since the values of Π are subsets of S_1, the sequence $\{x_n\}_{n\geq 1}$ lies in S_1.

For a given collection Γ and a point x_0, the uniqueness of a SOSP depends on the properties of the sets in the region where the iterations are performed. Clearly, for every starting point x_0, the cyclic projection map of a collection of Chebyshev sets generates a unique SOSP. In connection with the question of uniqueness, let us mention two important properties of a space which is frequently encountered in applications, namely the finite-dimensional Euclidean space E. First, the class of Chebyshev sets and the class of nonempty closed and convex sets coincide in E (Ref. 22). Second, the points which admit more than one projection onto a nonempty closed subset of E form a set of Lebesgue measure zero (Ref. 23).

Proposition 4.1. Let Π be the cyclic projection map of an ordered finite collection of nonempty approximately compact sets in a metric space. Then, if a SOSP converges, it is to a fixed point of Π.

Proof. Let $\{x_n\}_{n\geq 0}$ be a SOSP that converges to a point x. Then, $\{x_{n+1}\}_{n\geq 0}$ converges to x and $x_{n+1} \in \Pi(x_n)$, $\forall n \in \mathbb{N}$. Π is u.s.c. by Theorem 4.1 and *a fortiori* closed. Hence, $x \in \Pi(x)$. □

Unlike Proposition 3.1, the above proposition does not guarantee that the limit of a convergent SOSP is a solution point. Indeed, Π may admit fixed points outside the solution set. In this respect, Fig. 1 shows a system of two closed sets in the Euclidean plane, where x is a fixed point of the cyclic projection map Π which lies outside the intersection S. In that example, the set of fixed points of Π is $F = \{x\} \cup S$.

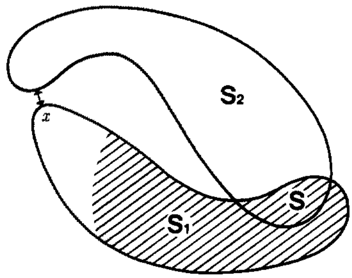

Fig. 1. Fixed points and region of attraction.

Definition 4.3. Let $\Gamma=\{S_1,\ldots,S_m\}$ be an ordered collection of proximinal sets in a metric space X whose intersection S is not empty. Let Π be the cyclic projection map of Γ, and let Y be the set of points in S_1-S from which an iteration step may fail to reduce ϕ_S,

$$Y=\{x\in S_1-S \mid \exists x'\in\Pi(x) \text{ such that } \phi_S(x')\geq\phi_S(x)\}. \tag{8}$$

We shall define the radius of attraction of Γ as

$$\rho=\begin{cases}\inf\{\phi_S(x)\mid x\in Y\}, & \text{if } Y\neq\emptyset,\\ +\infty, & \text{otherwise,}\end{cases} \tag{9}$$

and the region of attraction of Γ as

$$R=S\cup\{x\in S_1\mid \phi_S(x)<\rho\}. \tag{10}$$

Two direct consequences of Definition 4.3 are

$$\phi_S(x')<\phi_S(x), \qquad \forall x\in R-S,\ \forall x'\in\Pi(x), \tag{11}$$

and

$$\Pi(x)\subset R, \qquad \forall x\in R. \tag{12}$$

It is important to note that, under our assumptions on the collection of sets Γ, the radius of attraction ρ can assume all the values in $[0,+\infty]$. For instance, the case $\rho=+\infty$ occurs when Γ consists of two intersecting straight lines in the Euclidean plane. Then, the region of attraction is S_1 itself. Figure 1 illustrates a case where $0<\rho<+\infty$ for a collection $\Gamma=\{S_1,S_2\}$ of closed subsets of the Euclidean plane. The shaded area represents the region of attraction R of Γ. In the case where $\rho=0$, the region of attraction reduces to the solution set S. The following example illustrates this case.

Example 4.1. Let X be the Euclidean real line. For every n in $\mathbb{N}$, we define

$$a_n=1/n \quad\text{and}\quad b_n=(3n+2)/(3n(n+1)), \qquad \text{if } n>0, \tag{13a}$$

$$a_0=b_0=0. \tag{13b}$$

Consider the collection $\Gamma = \{S_1, S_2\}$ in X where

$$S_1 = \bigcup_{n \geq 0} \{a_n\} \quad \text{and} \quad S_2 = \bigcup_{n \geq 0} \{b_n\}. \tag{14}$$

We claim that the radius of attraction of Γ is zero.

Proof. First of all, we check that Γ satisfies the assumptions of Definition 4.3. It is readily seen that, $\forall n \in \mathbb{N}$,

$$n > 0 \Rightarrow a_{n+1} < b_n < a_n. \tag{15}$$

Thus, $S = \{0\} \neq \varnothing$ and, since the sets S_1 and S_2 are closed, they are proximinal in X by Theorem 2.2. Now, let ϵ be a fixed positive real number. Then, there exists an n in $\mathbb{N}$ such that $1/(n+1) < \epsilon$. Whence, there exists an $x = a_{n+1}$ in $S_1 - S$ such that $\phi_S(x) < \epsilon$. Let $\Pi = \Pi_1 \circ \Pi_2$ and ρ be respectively the cyclic projection map and the radius of attraction of Γ. From (13a) and (15),

$$\Pi_2(x) = \{b_{n+1}\} \quad \text{and} \quad \Pi_1(b_{n+1}) = \{a_{n+1}\} = \{x\}.$$

We conclude that

$$\Pi(x) = \{x\},$$

and hence x lies in the set Y of (8). Thus, necessarily, $\rho < \epsilon$. Since ϵ can be arbitrarily small, it follows that $\rho = 0$. □

Definition 4.4. Let $\Gamma = \{S_1, \ldots, S_m\}$ be an ordered collection of proximinal sets in a metric space X whose intersection S is nonempty. Let R and Π be respectively the region of attraction and the cyclic projection map of Γ. We shall say that a point x_0 in X is a point of attraction of Γ if, for every SOSP $\{x_n\}_{n \geq 0}$, there exists a nonnegative integer ν such that x_ν belongs to R. We shall call the smallest such ν the index of attraction of a given SOSP.

It follows readily from Definition 4.4, (11), and (12) that, if x_0 is a point of attraction and $\{x_n\}_{n \geq 0}$ a SOSP with index of attraction ν, then $\{\phi_S(x_n)\}_{n \geq \nu}$ is a nonincreasing sequence and

$$x_{\nu+n} \in \{x \in S_1 \mid \phi_S(x) \leq \phi_S(x_\nu)\} \subset R, \qquad \forall n \in \mathbb{N}. \tag{16}$$

In words, the tail of every SOSP starting at a point of attraction lies in the region of attraction. Moreover, (11) implies that all the fixed points of Π in R belong to the solution set. Thus, if all the sets are approximately compact in Definition 4.4, the limit of every convergent SOSP starting at a point of attraction is a solution point on account of Proposition 4.1.

4.3. Convergence Result. We remind the reader that a topological space is said to be connected if it is not the union of two disjoint nonempty closed sets and that a compact connected Hausdorff topological space is said to be a continuum (Ref. 12). Moreover, we shall say that a continuum is nontrivial if it does not reduce to $\varnothing$ or a singleton.

Theorem 4.2. Let $\{x_n\}_{n\geq 0}$ be a sequence of points in a compact subset of a metric space (X, d) such that $\{d(x_n, x_{n+1})\}_{n\geq 0}$ converges to zero. Then, either $\{x_n\}_{n\geq 0}$ converges or its set of cluster points is a nontrivial continuum.

Proof. This is a straightforward generalization of a theorem given in Ref. 24 for Euclidean spaces. □

The following lemma will be needed subsequently.

Lemma 4.1. Let S and S_1 be respectively a nonempty bounded and a nonempty boundedly compact set in a metric space, and let α be a nonnegative real number. Then, the set $\{x \in S_1 \mid \phi_S(x) \leq \alpha\}$ is compact.

Proof. Let

$$A = \{x \in S_1 \mid \phi_S(x) \leq \alpha\}.$$

Clearly, by continuity of ϕ_S, A is closed. Since S is bounded, it is contained in a closed ball centered at some point z in S, say $B(z, r)$. Now let x be an arbitrary point in A. Then, there exists a point y in S such that

$$d(x, y) \leq \alpha + \tfrac{1}{2}.$$

Hence,

$$d(z, x) \leq d(z, y) + d(x, y) \leq r + \alpha + \tfrac{1}{2} = r'.$$

Therefore, x belongs to the closed ball B of center z and radius r', and it follows that $A \subset B \cap S_1$. Hence, A is a closed subset of $B \cap S_1$ which is compact, since S_1 is boundedly compact. Thus, A is compact. □

We are now ready to present the main result.

Theorem 4.3. Let $\Gamma = \{S_1, \ldots, S_m\}$ be an ordered collection of approximately compact sets in a metric space X, whose intersection S is nonempty and bounded, and such that S_1 is boundedly compact. Let x_0 be a point of attraction of Γ, let $\{x_n\}_{n\geq 0}$ be an arbitrary SOSP, and let C be the set of its cluster points. Then, either $\{x_n\}_{n\geq 0}$ converges to a point in S or C is a nontrivial continuum in S. A sufficient condition for the former is that

$$\sum_{n\geq 0} \phi_S(x_n) < +\infty.$$

Proof. Let $\Pi = \Pi_1 \circ \cdots \circ \Pi_m$ be the cyclic projection map of Γ, and let R be its region of attraction. Since x_0 is a point of attraction of Γ, the SOSP $\{x_n\}_{n\geq 0}$ possesses an index of attraction ν. From (16), the sequence $\{x_n\}_{n\geq\nu}$ lies in the set

$$A = \{x \in S_1 \mid \phi_S(x) \leq \phi_S(x_\nu)\},$$

which is compact by Lemma 4.1. Hence, $\{x_n\}_{n\geq 0}$ admits a cluster point y. By continuity of ϕ_S, $\phi_S(y)$ is a cluster point of $\{\phi_S(x_n)\}_{n\geq 0}$. But, as $\{\phi_S(x_n)\}_{n\geq\nu}$ is a nonincreasing sequence bounded from below, it must converge to $\phi_S(y)$. Moreover,

$$\phi_S(x_{\nu+n}) \geq \phi_S(y), \qquad \forall n \in \mathbb{N}.$$

Now, suppose that $y \notin S$. Then, since $y \in A \subset R$, (11) yields

$$\phi_S(y') < \phi_S(y), \qquad \forall y' \in \Pi(y).$$

Consider the open neighborhood

$$V = \{z \in X \mid \phi_S(z) < \phi_S(y)\}$$

of $\Pi(y)$. Π is u.s.c. at y by Theorem 4.1. It follows that there exists an open neighborhood U of y such that $\Pi(x) \subset V$, $\forall x \in U$. But y is a cluster point of $\{x_n\}_{n\geq\nu}$ and, hence, there exists a positive integer n such that $x_{\nu+n-1} \in U$. The successor $x_{\nu+n}$ of $x_{\nu+n-1}$ in the SOSP belongs to $\Pi(x_{\nu+n-1})$ and therefore to V. Consequently, $\phi_S(x_{\nu+n}) < \phi_S(y)$, which contradicts a previous inequality statement. Hence, $y \in S$. Thus, $\{\phi_S(x_n)\}_{n\geq 0}$ converges to zero and $\varnothing \neq C \subset S$.

Now, let n be fixed in $\mathbb{N}$, and let $y_0 = x_n$. For every j in $\{0, \ldots, m-1\}$, we denote by y_{j+1} the projection of y_j onto S_{m-j}, which has been selected in the process of obtaining x_{n+1}. We have $x_{n+1} = y_m$. It is noted that S is boundedly compact as a closed subset of S_1. Hence, by Theorem 2.2, there exists a point z in S which is a projection of y_0 onto S. Since z belongs to each S_{m-j}, we have

$$\begin{aligned} d(y_j, y_{j+1}) &= \inf\{d(y_j, y) \mid y \in S_{m-j}\} \\ &\leq d(y_j, z), \qquad \forall j \in \{0, \ldots, m-1\}. \end{aligned} \tag{17}$$

Let $d = d(y_0, z)$. Let us prove that

$$d(y_j, z) \leq 2^j d, \qquad \forall j \in \{0, \ldots, m\}. \tag{18}$$

The statement is clearly true for $j = 0$. For any integer j in $\{0, \ldots, m-1\}$ for which (18) holds, (17) yields

$$d(y_{j+1}, z) \leq d(y_j, y_{j+1}) + d(y_j, z) \leq 2d(y_j, z) \leq 2^{j+1} d, \tag{19}$$

which completes the proof by induction. By using (17) and (18), we get

$$d(x_n, x_{n+1}) \leq \sum_{j=0}^{m-1} d(y_j, y_{j+1}) \leq \sum_{j=0}^{m-1} d(y_j, z)$$

$$\leq \sum_{j=0}^{m-1} 2^j d = (2^m - 1)d. \tag{20}$$

It is easy to see that $d = \phi_S(x_n)$. Therefore,

$$d(x_n, x_{n+1}) \leq (2^m - 1)\phi_S(x_n), \qquad \forall n \in \mathbb{N}. \tag{21}$$

Since $\{\phi_S(x_n)\}_{n\geq 0}$ converges to zero, so does $\{d(x_n, x_{n+1})\}_{n\geq 0}$. Thus, since $C \subset S$, it follows from Theorem 4.2 that either $\{x_n\}_{n\geq 0}$ converges to a point in S or C is a nontrivial continuum in S.

To prove the last assertion, it is enough to show that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. Let p and q be any two nonnegative integers such that $p < q$. We have

$$d(x_p, x_q) \leq \sum_{n=p}^{q-1} d(x_n, x_{n+1}) \leq \sum_{n=p}^{+\infty} d(x_n, x_{n+1})$$

$$\leq (2^m - 1) \sum_{n=p}^{+\infty} \phi_S(x_n). \tag{22}$$

Since the rightmost expression in (22) is nothing but the tail of a convergent series, it must go to zero as p goes to infinity, which concludes the proof. □

Comment 4.1. In the proof of Theorem 4.3, it is shown that, without any summability assumption on $\{\phi_S(x_n)\}_{n\geq 0}$, the sequence $\{d(x_n, x_{n+1})\}_{n\geq 0}$ converges to zero. It is noted that, in an ultrametric space, i.e., a metric space (X, d) such that

$$d(x, z) \leq \sup\{d(x, y), d(y, z)\}, \qquad \forall (x, y, z) \in X^3, \tag{23}$$

this condition guarantees that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence (Ref. 14), which would establish at once its convergence. Ultrametric spaces are however seldom encountered in applications; see Bourbaki (Refs. 14, 17) and Dieudonné (Ref. 25) for examples and properties.

4.4. Questions Relative to the Implementation of the MOSP. In the previous sections, the MOSP has been developed as a conceptual mathematical procedure for obtaining a point in the intersection S of a collection $\Gamma = \{S_1, \ldots, S_m\}$ of sets in a metric space X. We shall now address the main issues pertaining to its implementation. In the following discussion, it is assumed that $X = \mathbb{R}^n$, as is the case in most applied problems. Moreover, because of its computational advantages, the Euclidean norm $\|\cdot\|$ is chosen to metrize $\mathbb{R}^n$. By Theorem 2.2, in such a metric space, Theorem 4.3 applies to finite collections of closed sets whose intersection is nonempty and bounded.

Closedness and Boundedness Restrictions on the Sets. We have

$$\bar{S}_i = \{x \in X \mid d(x, S_i) = 0\}.$$

Hence, the requirement that the sets be closed is not restrictive, since we can always replace a set by its closure. Moreover, the condition that the solution set be bounded should not cause concern for, in practice, there are always boundedness constraints on the components of a feasible solution. Thus, the conditions of closedness and boundedness on the sets are quite mild and can always be satisfied in applied problems.

Computation of the Projections. For a fixed x in $\mathbb{R}^n$, let ϕ_x denote the functional $\phi_x : y \mapsto \|x - y\|^2$. Each elementary step of the MOSP involves the computation of a projection y_p of a point x onto a proximinal subset S_i of $\mathbb{R}^n$, i.e., a global minimum of ϕ_x over S_i. As there exists no universal method to solve efficiently this quadratic minimization problem, a complete discussion is neither possible nor intended. In practice, the projection onto a given S_i should be considered on a case-by-case basis.

If S_i is convex, any local minimum of ϕ_x is a global one. Specific algorithms have been established for special cases such as polyhedrons (Ref. 26) or polytopes (Ref. 27). Generally speaking, quadratic programming algorithms are of interest when S_i is specified by functional inequalities. In applied problems, an equation is often available for the boundary of S_i, e.g.,

$$\partial S_i = \{x \in \mathbb{R}^n \mid f_i(x) = 0\}.$$

Since the projection of a point in S_i^c onto S_i belongs to ∂S_i, the problem can be approached via the method of Lagrange multipliers. If S_i is characterized only by a contact function, the quadratic programming algorithm of Ref. 28 can be used. If S_i is not convex, ϕ_x may admit several local minima and, thus, the problem of finding a projection of x is one of global optimization. In some problems, there may not be criteria for deciding whether a local solution is global and global optimization methods must be employed. Both deterministic and stochastic procedures have been proposed in the literature to solve the global optimization problem, and we refer the reader to Ref. 29 for a detailed survey. Recent developments of the stochastic approach can be found in Ref. 30 and in the references therein. Stochastic methods should be used when there is no certainty that deterministic algorithms will produce a global minimum to some acceptable degree of accuracy. They are regarded as very reliable tools which, under relatively mild conditions, offer an asymptotic guarantee of convergence (in some probabilistic sense) to a global solution.

Finding a Point of Attraction. In Theorem 4.3, it is stated that the iterations should be started at a point of attraction. Since our definition of a point of attraction is not constructive, such a point may be difficult to find. Indeed, because of the geometrical complexity of the system of sets, it is usually impractical to characterize *a priori* the region of attraction and to establish whether or not a point is a point of attraction. This potential limitation of the MOSP should however be mitigated by noting that in a practical application an approximate solution is often available. Since points of attraction are more likely to be found in the vicinity of the solution set, this approximate solution is a good candidate for a starting point of a SOSP. Then, the point y produced by the algorithm is accepted as a solution if it belongs to all the sets. If it does not, then none of the points in the corresponding SOSP belong to the region of attraction. Hence, loosely speaking, a new starting point should be chosen outside the path followed by that unsuccessful SOSP. A heuristic way to do this is to start the iterations at the symmetric x_0' of x_0 with respect to y, i.e., $x_0' = 2y - x_0$. In general, this method does not ensure that a point of attraction will be found. Nonetheless, it has been used successfully in most of our applications.

Stopping Rule. In practical problems, the MOSP presented above consists in generating a sequence of points $\{x_n\}_{n\geq 0}$, where x_0 is a point of attraction of Γ, according to the following algorithm:

$$\forall n \in \mathbb{N}, \text{ stop if } x_n \in S_i, \forall i \in \{1, \ldots, m\};$$

$$\text{else, let } x_{n+1} \text{ be any point in } \Pi(x_n). \tag{24}$$

Under the hypotheses of Theorem 4.3, if a solution is not obtained in a finite number of iterations, it is any cluster point or the limit of $\{x_n\}_{n\geq 0}$.

Since we are interested in obtaining a solution in finite time, it remains to discuss a practical convergence criterion for the MOSP.

In the proof of Theorem 4.3, it was seen that, if x_0 is a point of attraction of Γ and $\{x_n\}_{n\geq 0}$ a SOSP, then the sequence $\{\|x_{n+1}-x_n\|\}_{n\geq 0}$ converges to zero. Therefore, the following criterion can be used as a stopping rule:

$$\|x_{n+1}-x_n\| \leq \epsilon. \tag{25}$$

Such a criterion has notorious drawbacks. It may for instance stop the algorithm prematurely in case of slow convergence. Another problem is that the choice of the parameter ϵ is somewhat arbitrary. We shall however adopt it, because it has proven useful in practice. It is noted that the MOSP does not necessarily yield a solution point in a literal sense since, because of the truncation of the SOSP, it may produce a point which does not exactly lie in all the S_i's. From a practical standpoint, this should not raise concern, because slight deviations are usually allowable in specifying the boundaries of the S_i's.

5. Application to Digital Signal Restoration

5.1. Introduction. The purpose of this section is to illustrate an application of the method developed above to digital signal restoration. The digital restoration problem is to estimate the original form of a blurred and noise-corrupted discrete signal. It is a common problem of various fields of engineering, from image processing to seismology.

Conventional restoration methods seek to produce an estimate of the original signal which is optimum in terms of a predefined criterion [e.g., least-squares (Wiener filtering), constrained least-squares, maximum *a posteriori*, and maximum entropy]; see Ref. 31. In recent years, set-theoretic signal restoration has been reported to outperform these conventional methods (Refs. 5, 6, 7). In that approach, consistency with all the available *a priori* knowledge serves as an estimation criterion. Each piece of *a priori* knowledge is represented by a property set in $\mathbb{R}^n$. A restored signal is any point in the intersection S of these sets and, therefore, the set theoretic restoration problem takes the form of (2). It is important to note that the problems presented in Refs. 5, 6, 7 were solved by using the algorithm of Theorem 3.2 in the Euclidean space (in the signal recovery literature, this algorithm is often referred to as POCS, for projection onto convex sets). Therefore, only convex property sets could be considered in these studies, which precluded the incorporation of some useful pieces of *a priori* information.

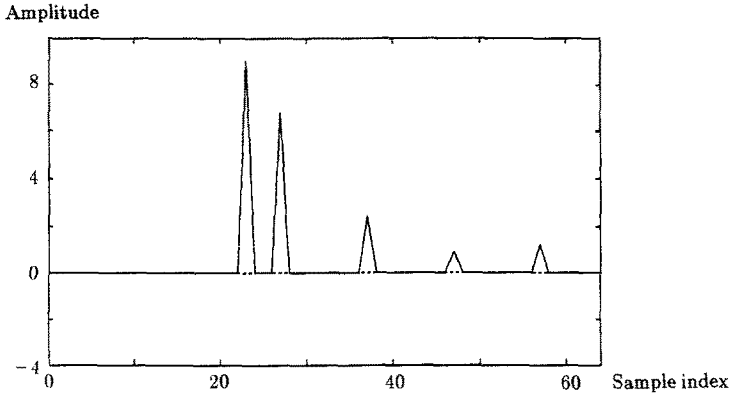

Fig. 2. Original signal.

5.2. Signal Degradation. In these simulations, we employ the standard shift-invariant linear degradation model (Ref. 31)

$$d = Ha + w, \tag{26}$$

where d is the $n \times 1$ degraded signal vector, H the $n \times n$ known blur matrix, a the $n \times 1$ original signal vector, and w the $n \times 1$ noise vector. In the results shown here, $n = 64$ and the original signal a is the simulated X-ray fluorescence spectrum displayed in Fig. 2. Such signals feature high-resolution patterns together with large zero regions and are often used to test restoration methods. This signal was blurred by convolution with a Gaussian-shaped impulse response with a standard deviation of two points. This impulse

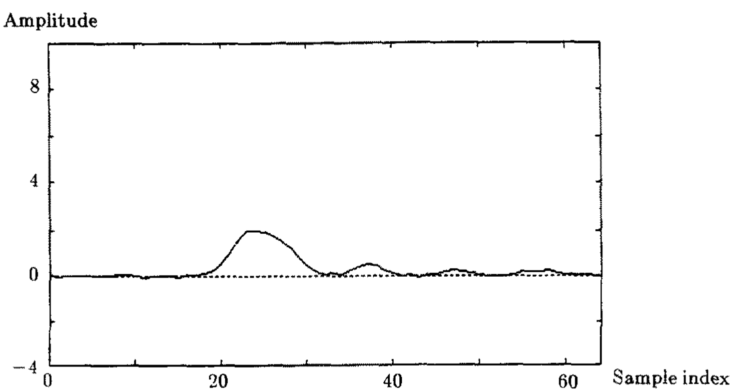

Fig. 3. Degraded signal.

response constitutes a good model for the finite resolution of the measurement instruments. Zero-mean Gaussian white noise with variance $\sigma^2 = 0.002$ was then added to obtain the degraded signal d seen in Fig. 3.

5.3. Set Theoretic Restoration. $\mathbb{R}^n$ is equipped with the Euclidean norm $\|\cdot\|$. In order to perform a set-theoretic restoration, we first need to construct sets in $\mathbb{R}^n$ from the available *a priori* knowledge. As was shown in Ref. 6, property sets can be constructed by constraining the sample statistics of the estimation residual

$$r = r(\hat{a}) = d - H\hat{a}$$

to agree, within some confidence coefficient, with those known properties of the noise w. Let r_i be the ith component of r. From the knowledge that the noise is zero mean white and Gaussian with power σ^2, we can construct the set S_1 of estimates which give residual points within some confidence interval

$$S_1 = \bigcap_{i=1}^{n} C_i, \qquad \text{where } C_i = \{\hat{a} \in \mathbb{R}^n \,|\, |r_i| \le \delta_0\}, \tag{27}$$

the set S_2 of estimates which give a residual sample variance consistent with the noise power

$$S_2 = \{\hat{a} \in \mathbb{R}^n \,|\, \|d - H\hat{a}\|^2 \le \delta_v\}, \tag{28}$$

and the set S_3 of estimates which give a residual periodogram consistent with the whiteness and the normality of the noise

$$S_3 = \{\hat{a} \in \mathbb{R}^n \,|\, |R_k|^2 \le \delta_p, \forall k \in \{1, \dots, n/2 - 1\}\}, \tag{29}$$

where R_k is the kth component of the discrete Fourier transform of r. The expressions of δ_0, δ_v, and δ_p along with the derivations of the projection operators onto the closed and convex sets C_i, S_2, and S_3 can be found in Ref. 6. In the simulations, the confidence coefficient was fixed to 95%.

As an X-ray fluorescence spectrum, the original signal is nonnegative and does not possess more than a few nonzero values, say z (given that $n = 64$, a standard value is $z = 9$). The set of all vectors with nonnegative components whose number of nonzero values does not exceed z will be denoted by S_4. If $\{e_i \,|\, 1 \le i \le n\}$ denotes the standard orthonormal basis of $\mathbb{R}^n$ and S_5 the set of vectors with nonnegative components, S_4 can be

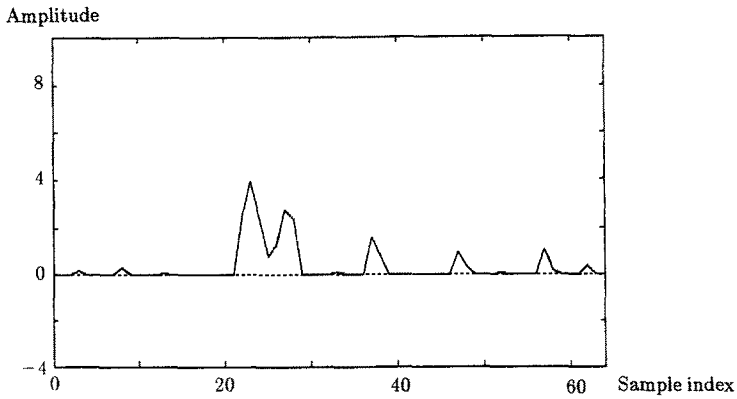

Fig. 4. Restoration by the POCS.

described as the intersection of S_5 with the union of the $n!/z!(n-z)!$ vector subspaces of $\mathbb{R}^n$ generated by z distinct e_i's. Clearly, S_4 is proximinal but not convex. A projection of a vector a in $\mathbb{R}^n$ onto S_4 is simply obtained by retaining z of the largest nonnegative components of a and by setting the remaining components to zero.

In implementing the POCS, S_4 cannot be used because it is not convex. We can however use its closed and convex superset S_5. The degraded signal served as a starting point for the iterations and sequential projections onto the C_i's, S_2, S_3, and S_5 were carried out. The coefficient in the stopping

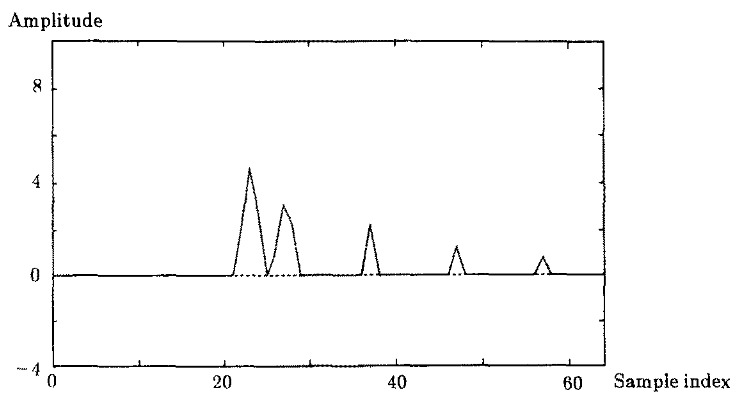

Fig. 5. Restoration by the MOSP.

rule of (25) was set to $\epsilon = 10^{-2}$. Convergence to the estimate displayed in Fig. 4 was obtained in 45 iterations.

The MOSP of Theorem 4.3 was then used. It allows us to incorporate the nonconvex set S_4. As noted in Section 4, points of attraction are more likely to be found in the vicinity of the solution set S. Because the degraded signal x still retains some of the features of the original signal, it constitutes a sensible choice for a starting point. The MOSP was implemented by projecting sequentially onto the C_i's, S_2, S_3, and S_4. As above, ϵ was set to 10^{-2}. Convergence to the feasible signal displayed in Fig. 5 was achieved in 60 iterations. Since more *a priori* knowledge has been used, it is not surprising that the MOSP gave a better restoration than the POCS. Indeed, the three peaks on the left are more sharply recovered and the separation between the two main peaks has been improved. Moreover, all the artifacts which appeared in the flat regions of the signal have been removed.

5.4. Other Applications. There are other problems in which the authors have successfully employed the MOSP with the inclusion of nonconvex sets. For instance, remaining in the framework of signal restoration, the nonconvex set of signals whose Euclidean norm is bounded from below (minimum energy constraint) and that of signals with a prescribed Fourier transform magnitude have been used. Another application is the problem of estimating the parameters of an autoregressive time series of order p from a finite sample path. In that problem, an important constraint is the stationarity of the autoregressive process, which gives rise to a nonconvex set in the coefficient space for $p > 2$. This set is also encountered in the identification of purely recursive stable discrete systems.

6. Conclusions

A method for obtaining a common point of a finite collection of sets in a metric space has been presented. This method was developed by generalizing the method of successive projections for closed and convex sets in Hilbert spaces to approximately compact sets in metric spaces. Sequences of successive projections were constructed via the composition of the set-valued projection maps onto the individual sets. It was shown that, under certain hypotheses, a solution could be obtained as any cluster point or the limit of any such sequence. Potential applications of this result are found in any problem whose set-theoretic format cannot be reduced to that of closed and convex sets in a Hilbert space (e.g., nonconvex programming in arbitrary normed vector spaces).

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