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Solution Concepts in Two-Person Multicriteria Games

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Abstract. In this paper, we propose new solution concepts for multicriteria games and compare them with existing ones. The general setting is that of two-person finite games in normal form (matrix games) with pure and mixed strategy sets for the players. The notions of efficiency (Pareto optimality), security levels, and response strategies have all been used in defining solutions ranging from equilibrium points to Pareto saddle points. Methods for obtaining strategies that yield Pareto security levels to the players or Pareto saddle points to the game, when they exist, are presented. Finally, we study games with more than two qualitative outcomes such as combat games. Using the notion of guaranteed outcomes, we obtain saddle-point solutions in mixed strategies for a number of cases. Examples illustrating the concepts, methods, and solutions are included.

Key Words. Game theory, vector-valued optimization, multicriteria games, combat games, Pareto optimality, equilibrium points, security levels, saddle points.

1. Introduction

Games with multiple noncomparable criteria are called multicriteria games or games with vector payoffs. Such games have attracted limited attention in the game theory literature, perhaps due to a paucity of useful applications of such a theory. Another important reason is that many of the intuitively appealing results in scalar criterion games seem to have no counterparts in the existing theory of multicriteria games. The limited results obtained to date for these games also do not seem to possess any straightforward and logical game theoretic interpretation. Recently certain generalizations of the well-known theory of pursuit-evasion games (Ref. 1), called combat games (Ref. 2), have been formulated as zero-sum bicriterion

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differential games (Ref. 3 and 4). Though these games are dynamic in nature, we hope to get some insight into these problems by first analyzing multicriteria two-person zero-sum matrix games.

Blackwell's (Ref. 5) was the first paper which dealt with multicriteria games as a generalization of the scalar criterion games. An asymptotic analog of the minmax theorem in scalar criterion games was established for repeated games with vector payoffs. The analysis is based upon an approachability-excludability theory to answer the question as to whether a player will be able to force his average payoff to approach or exclude a given subset in the payoff space, if the game is repeated a large number of times. Shapley (Ref. 6) defined the concept of equilibrium points in games with vector payoffs and proved that the solution of a certain scalarized nonzero-sum game (a bimatrix game) also gives the equilibrium solution in mixed strategies of the original game. More recently, Nieuwenhuis (Ref. 7) presented a possible generalization of the notions of minmax, maxmin, and saddle points for vector-valued functions using the well-known concepts of efficiency or Pareto optimality. Corley (Ref. 8) used identical generalizations in the case of matrix games with vector payoffs and indicated a way of obtaining the minmax, maxmin, and equilibrium (saddle) points. The approaches adopted in Refs. 6-8 are more or less the same.

In the present paper, we use the notion of Pareto optimality (efficiency) and that of guaranteed security levels simultaneously to define solution concepts in multicriteria two-person zero-sum games; we show their relationships with the existing solution concepts in scalar and multicriteria games. This generalization is different from the generalizations presented in Refs. 6-8. We have attempted to compare these different kinds of generalizations and point out their conceptual differences. Our main effort has been to show that, in multicriteria games, a single solution concept in terms of only equilibrium points is not sufficient. So, we propose and discuss various solution concepts associated with multicriteria games.

Lastly, we abandon the usual representation of the payoff matrices in terms of numerical quantities and replace them with outcomes in a qualitative sense. Such matrices are then analyzed for the existence of saddle points under pure and mixed strategies based upon the preference order of the players. This approach is meaningful in the qualitative analysis of combat games.

2. Some Definitions and Remarks

Let $\mathscr{D} \subseteq R^n$ be a compact subset of the n-dimensional real space. Each element $v \in \mathscr{D}$ is of the form $(v_1, \ldots, v_n)'$. Let us define a preference cone

$Z \subseteq R^n$ with its apex coinciding with the origin. The preference cone Z is assumed closed and convex with nonempty interior and has the property that $Z \cap (-Z) = \{0\}$. The interior of Z is denoted by int Z and its boundary by ∂Z. Let $v^1, v^2 \in \mathscr{D}$, $v^i = (v_1^i, \ldots, v_n^i)$, $i = 1, 2$; and let $Z^0 = Z \backslash \{0\}$. Then:

$v^1 = v^2$, if and only if $v^1 - v^2 = 0$,
$v^1 \geqq v^2$, if and only if $v^1 - v^2 \in Z$,
$v^1 > v^2$, if and only if $v^1 - v^2 \in Z^0$,
$v^1 \not> v^2$, if and only if $v^1 - v^2 \notin Z^0$.

Definition 2.1. A vector $v \in \mathscr{D}$ is said to be a Pareto minimum in $\mathscr{D}$ with respect to Z iff, for all $u \in \mathscr{D}$, $(v - u) \in Z$ implies that $(v - u) = 0$; i.e., there exists no $u \in \mathscr{D}$, such that $(v - u) \in Z^0$. Similarly, a vector $v \in \mathscr{D}$ is said to be a Pareto maximum in $\mathscr{D}$ with respect to Z iff, for all $u \in \mathscr{D}$, $(u - v) \in Z$ implies $(u - v) = 0$; i.e., there exists no $u \in \mathscr{D}$ such that $(u - v) \in Z^0$.

There are other ways of defining Pareto-optimal vectors. For details, see Ref. 9. In the remainder of the paper, we assume that $Z = Z_+$, the positive orthant in R^n. Then, we have our usual definition of Pareto maximum and Pareto minimum, which are denoted by Pmax and Pmin respectively in this paper.

Remark 2.1. For $n = 1$, Pmax and Pmin become identical to the usual definition of max and min in scalar criterion games. Note that the relation $\geqslant$ is a partial order on $\mathscr{D} \subseteq R^n$ for $n \geq 2$, but not a total order. For $n = 1$, $\geq$ is not only a partial order but also a total order. The relations $\not>$ and $\not<$ are not partial orders for $n \geq 2$. But for $n = 1$, they are equivalent to $\leq$ and $\geq$, respectively (and hence total orders).

Consider a matrix $A = \{a_{ij}\}$, with p number of rows and q number of columns. Each element a_{ij} of the matrix is an n-tuple $(a_{ij}(1), \ldots, a_{ij}(n))$. We define individual matrices of dimension $p \times q$ as

$$A(k) = \{a_{ij}(k)\}, \qquad k = 1, \ldots, n.$$

There are two players, P1 (the minimizer, who chooses rows) and P2 (the maximizer, who chooses columns).

The mixed strategy spaces of the players P1 and P2 are

$$\Gamma^1 = \left\{ \gamma^1 : \sum_{i=1}^{p} \gamma_i^1 = 1;\ \gamma_i^1 \geq 0,\ i = 1, \ldots, p \right\}, \tag{1}$$

$$\Gamma^2 = \left\{ \gamma^2 : \sum_{j=1}^{q} \gamma_j^2 = 1;\ \gamma_j^2 \geq 0,\ j = 1, \ldots, q \right\}, \tag{2}$$

where an element $\gamma^i \in \Gamma^i$, $i = 1, 2$, is of the form

$$\gamma^1 = (\gamma_1^1, \ldots, \gamma_p^1)^t, \tag{3}$$

$$\gamma^2 = (\gamma_1^2, \ldots, \gamma_q^2)^t. \tag{4}$$

The pure strategy sets are denoted by Γ_p^1 and Γ_p^2 and are the extreme points or vertices of Γ^1 and Γ^2, respectively.

This implies that, when a player P_i follows a strategy γ^i, he chooses a row (or column) j with probability γ_j^i. The expected payoff of the game is denoted by

$$J(\gamma^1, \gamma^2) = (J_1(\gamma^1, \gamma^2), \dots, J_n(\gamma^1, \gamma^2)), \tag{5}$$

where

$$J(\gamma^1, \gamma^2) = \gamma^{1'} A \gamma^2, \tag{6}$$

$$J_k(\gamma^1, \gamma^2) = \gamma^{1'} A(k) \gamma^2, \qquad k = 1, \dots, n. \tag{7}$$

Thus, we have a two-person zero-sum n-criterion matrix game.

3. Generalization with Pareto Optimality Concepts

In this section, we shall briefly present the generalizations proposed by Shapley (Ref. 6), Nieuwenhuis (Ref. 7), and Corley (Ref. 8). An equilibrium point (or a saddle point) is said to exist, if there exists a strategy pair $(\gamma^{1*}, \gamma^{2*})$, $\gamma^{1*} \in \Gamma^1$, $\gamma^{2*} \in \Gamma^2$, such that

$$J(\gamma^{1*}, \gamma^2) \ngtr J(\gamma^{1*}, \gamma^{2*}), \qquad \forall \gamma^2 \in \Gamma^2, \tag{8a}$$

$$J(\gamma^1, \gamma^{2*}) \nless J(\gamma^{1*}, \gamma^{2*}), \qquad \forall \gamma^1 \in \Gamma^1. \tag{8b}$$

Note that this reduces to the usual saddle-point definition for scalar games when $n = 1$. Generalizations of minmax and maxmin points can also be obtained. For given $\gamma^1 \in \Gamma^1$ and $\gamma^2 \in \Gamma^2$, we define sets of vectors,

$$J(\gamma^1, \Gamma^2) \triangleq \bigcup_{\gamma^2 \in \Gamma^2} \{J(\gamma^1, \gamma^2)\}, \tag{9}$$

$$J(\Gamma^1, \gamma^2) \triangleq \bigcup_{\gamma^1 \in \Gamma^1} \{J(\gamma^1, \gamma^2)\}, \tag{10}$$

$$g(\gamma^1) \triangleq \{x \in J(\gamma^1, \Gamma^2) \colon (x + Z_+^0) \cap J(\gamma^1, \Gamma^2) = \varnothing\}, \qquad \gamma^1 \in \Gamma^1, \tag{11}$$

$$g(\Gamma^1) \triangleq \bigcup_{\gamma^1 \in \Gamma^1} g(\gamma^1). \tag{12}$$

Similarly,

$$h(\gamma^2) \triangleq \{x \in J(\Gamma^1, \gamma^2): (x - Z_+^0) \cap J(\Gamma^1, \gamma^2) = \varnothing\}, \qquad \gamma^2 \in \Gamma^2, \quad (13)$$

$$h(\Gamma^2) \triangleq \bigcup_{\gamma^2 \in \Gamma^2} h(\gamma^2), \quad (14)$$

$$G \triangleq \operatorname*{Pmin}_{\gamma^1 \in \Gamma^1} \operatorname*{Pmax}_{\gamma^2 \in \Gamma^2} J(\gamma^1, \gamma^2)$$

$$= \{x \in g(\Gamma^1): (x - Z_+^0) \cap g(\Gamma^1) = \varnothing\}, \quad (15)$$

$$H \triangleq \operatorname*{Pmax}_{\gamma^2 \in \Gamma^2} \operatorname*{Pmin}_{\gamma^1 \in \Gamma^1} J(\gamma^1, \gamma^2)$$

$$= \{x \in h(\Gamma^2): (x + Z_+^0) \cap h(\Gamma^2) = \varnothing\}. \quad (16)$$

One can use the same definitions with respect to the pure strategy sets by replacing Γ^1 and Γ^2 by Γ_p^1 and Γ_p^2. The sets $G \subseteq R^n$ and $H \subseteq R^n$ are sets of minmax and maxmin points as defined in Refs. 7 and 8.

There is no apparent relationship between the minmax, maxmin, and saddle (equilibrium) points, except that a point which is both minmax and maxmin is obviously a saddle (equilibrium) point (Ref. 8). One main drawback of such a generalization is that the ordered interchangeability property among pairs of strategies does not hold. Thus, the achievement of equilibrium depends on some amount of cooperation with the opponent. These characteristics are somewhat similar to those of the Nash equilibrium points in nonzero-sum scalar games. These equilibrium points, in general, do not offer the best security levels in terms of the individual criteria. Illustrative examples of these will be presented later.

4. Generalizations Using Security Levels

In scalar criterion games, the concepts of security levels and security strategies are inherent in the definition of maxmin and minmax points. We shall exploit some of these concepts in the following discussion related to multicriteria games.

Associated with every strategy $\gamma^i \in \Gamma^i$ for Player P_i, there exist security levels in each of its criteria $J_j, j = 1, \ldots, n$. They are denoted by $\bar{J}_j^1(\gamma^1)$ and $\underline{J}_j^2(\gamma^2)$ for players P1 and P2, respectively, and are defined as the payoff with respect to the jth criterion when P_i plays $\gamma^i \in \Gamma^i$ and his opponent does his best to minimize (or maximize, as the case may be) the jth criterion. Hence,

$$\bar{J}_j^1(\gamma^1) = \max_{\gamma^2 \in \Gamma^2} J_j(\gamma^1, \gamma^2), \qquad j = 1, \ldots, n, \quad (17)$$

$$\underline{J}_j^2(\gamma^2) = \min_{\gamma^1 \in \Gamma^1} J_j(\gamma^1, \gamma^2), \qquad j = 1, \ldots, n. \quad (18)$$

The security levels are n-tuples of the form

$$\bar{J}^1(\gamma^1) = (\bar{J}_1^1(\gamma^1), \ldots, \bar{J}_n^1(\gamma^1)),$$

$$\underline{J}^2(\gamma^2) = (\underline{J}_1^2(\gamma^2), \ldots, \underline{J}_n^2(\gamma^2)),$$

which represent the guaranteed payoffs in each of the criteria to the players P1 and P2, respectively.

Definition 4.1. A strategy $\tilde{\gamma}^1 \in \Gamma^1$ is said to be a Pareto-optimal (or efficient) security strategy (POSS) for P1 iff, for all $\gamma^1 \in \Gamma^1$,

$$\bar{J}^1(\gamma^1) \leq \bar{J}^1(\tilde{\gamma}^1) \text{ implies } \bar{J}^1(\gamma^1) = \bar{J}^1(\tilde{\gamma}^1).$$

Similarly, a strategy $\tilde{\gamma}^2 \in \Gamma^2$ is said to be a POSS for P2 iff, for all $\gamma^2 \in \Gamma^2$,

$$\underline{J}^2(\gamma^2) \geq \underline{J}^2(\tilde{\gamma}^2) \text{ implies } \underline{J}^2(\gamma^2) = \underline{J}^2(\tilde{\gamma}^2).$$

This way of defining Pareto optimality in multicriteria games is also used in Ref. 10.

The set of Pareto-optimal security strategies for Player P_i is defined as

$$\Gamma_{\mathrm{sp}}^i = \{\gamma^i \in \Gamma^i \colon \gamma^i \text{ is a POSS}\}, \qquad i = 1, 2. \tag{19}$$

The members of these sets are analogous to the security strategies in scalar criterion games. In fact, for $n = 1$, $\bar{J}^1(\gamma^1)$ and $\underline{J}^2(\gamma^2)$ for $\gamma^1 \in \Gamma_{\mathrm{sp}}^1$ and $\gamma^2 \in \Gamma_{\mathrm{sp}}^2$ are precisely the upper and lower values of the game. But, unlike the scalar criterion games, where these values are unique, there could be multiple upper and lower values in multicriteria games. We can prove that the upper values of the game are never less than the lower values of the game. Hence, we have the following lemma.

Lemma 4.1. For every $\gamma^1 \in \Gamma_{\mathrm{sp}}^1$ and $\gamma^2 \in \Gamma_{\mathrm{sp}}^2$,

$$\underline{J}^2(\gamma^2) \leq \bar{J}^1(\gamma^1). \tag{20}$$

Proof. By definition,

$$\underline{J}^2(\gamma^2) \leq J(\gamma^1, \gamma^2) \leq \bar{J}^1(\gamma^1), \tag{21}$$

which proves the lemma. □

This is analogous to a result in scalar criterion games which states that the loss ceiling is never less than the gain floor. In fact, Γ_{sp}^i are analogous to the minmax and maxmin strategies in scalar games, but different from those defined in Section 3.

Definition 4.2. A strategy $\hat{\gamma}^1 \in \Gamma^1$ is said to be a Pareto-optimal (or efficient) response strategy (PORS) for P1 against a strategy $\gamma^2 \in \Gamma^2$ of P2 iff, for all $\gamma^1 \in \Gamma^1$,

$$J(\gamma^1, \gamma^2) \leq J(\hat{\gamma}^1, \gamma^2) \text{ implies } J(\gamma^1, \gamma^2) = J(\hat{\gamma}^1, \gamma^2).$$

Similarly, a strategy $\hat{\gamma}^2 \in \Gamma^2$ is a PORS for P2 against a strategy $\gamma^1 \in \Gamma^1$ of P1 iff, for all $\gamma^2 \in \Gamma^2$,

$$J(\gamma^1, \gamma^2) \geq J(\gamma^1, \hat{\gamma}^2) \text{ implies } J(\gamma^1, \gamma^2) = J(\gamma^1, \hat{\gamma}^2).$$

The set of Pareto-optimal response strategies for player $P1$ against an opponent's given strategy is defined as

$$\Gamma^1_{\mathrm{rp}}(\gamma^2) = \{\gamma^1 \in \Gamma^1 \colon \gamma^1 \text{ is PORS against a } \gamma^2 \in \Gamma^2\}, \tag{22}$$

$$\Gamma^2_{\mathrm{rp}}(\gamma^1) = \{\gamma^2 \in \Gamma^2 \colon \gamma^2 \text{ is PORS against a } \gamma^1 \in \Gamma^1\}. \tag{23}$$

Based upon these definitions, we can define different solution concepts for multicriteria games.

4.1. Efficiency in the Mutual Response Mode (Equilibrium Points). A pair of strategies $(\gamma^{1*}, \gamma^{2*})$, $\gamma^{1*} \in \Gamma^1$, $\gamma^{2*} \in \Gamma^2$, is said to be an efficient pair of strategies in the mutual response mode if

$$\gamma^{1*} \in \Gamma^1_{\mathrm{rp}}(\gamma^{2*}) \quad \text{and} \quad \gamma^{2*} \in \Gamma^2_{\mathrm{rp}}(\gamma^{1*}). \tag{24}$$

This definition is equivalent to the definition of equilibrium points in Section 3, since

$$\begin{aligned} &\gamma^{1*} \in \Gamma^1_{\mathrm{rp}}(\gamma^{2*}) \\ &\Leftrightarrow \exists \text{ no } \gamma^1 \in \Gamma^1 \ni J(\gamma^1, \gamma^{2*}) < J(\gamma^{1*}, \gamma^{2*}) \\ &\Leftrightarrow J(\gamma^1, \gamma^{2*}) \not< J(\gamma^{1*}, \gamma^{2*}), \qquad \forall \gamma^1 \in \Gamma^1. \end{aligned}$$

Similarly,

$$\begin{aligned} &\gamma^{2*} \in \Gamma^2_{\mathrm{rp}}(\gamma^{1*}) \\ &\Leftrightarrow J(\gamma^{1*}, \gamma^2) \not> J(\gamma^{1*}, \gamma^{2*}), \qquad \forall \gamma^2 \in \Gamma^2, \end{aligned}$$

which proves the equivalence.

4.2. Equilibrium Points with Pareto-Optimal (Efficient) Security Levels. A pair of strategies $(\gamma^{1*}, \gamma^{2*})$, $\gamma^{1*} \in \Gamma^1_{\mathrm{sp}}$, $\gamma^{2*} \in \Gamma^2_{\mathrm{sp}}$, is said to be an equilibrium point with Pareto-optimal (efficient) security levels if they also satisfy the equilibrium condition (24). Clearly, they also offer Pareto-optimal security levels. As we shall see later, such strategies may not always exist.

4.3. Pareto-Optimal (Efficient) Saddle Points. A strategy pair $(\gamma^{1*}, \gamma^{2*})$, $\gamma^{1*} \in \Gamma^1$, $\gamma^{2*} \in \Gamma^2$, is said to be in Pareto-optimal saddle-point equilibrium if

$$\bar{J}^1(\gamma^{1*}) = \underline{J}^2(\gamma^{2*}). \tag{25}$$

Such a saddle point satisfies the conditions of the previous two solution concepts as well. It is the Pareto saddle-point concept for multicriteria games that is truly equivalent to the saddle-point condition in scalar games (i.e., minmax = maxmin). However, the existence of such a saddle point is rare even under mixed strategies. But if it exists, it has all the good properties of saddle points in scalar games (ordered interchangeability, equilibrium, efficiency, security, etc.).

Remark 4.1. All the three solution concepts described here reduce to the saddle-point condition in zero-sum scalar criterion games when $n = 1$. The existence of equilibrium points (as defined in Section 4.1) is guaranteed by Theorem 3.1 in Ref. 7. But the other two kinds of equilibrium points may not exist. In fact, the existence of equilibrium strategies which are also POSS can be easily determined by taking the intersection of the set of equilibrium strategies and the set of POSS for the same player. The existence of the Pareto-optimal saddle points can be checked by determining the security levels associated with the strategies in the POSS set and checking if Eq. (25) holds. In view of these observations, it is clear that the determination of POSS is important to solve a game using the concepts proposed here. Sometimes POSS are found to be more useful than just equilibrium strategies without Pareto-optimal security associated with them. A good example is the combat game mentioned earlier (Ref. 4).

Example 4.1. See Ref. 8. Consider the game matrix

$$A = \begin{bmatrix} (0,0) & (2,-1) \\ (1,-2) & (0,0) \end{bmatrix},$$

$$\gamma^i = (\gamma_1^i, \gamma_2^i), \quad \gamma_1^i + \gamma_2^i = 1, \quad \gamma_1^i, \gamma_2^i \geq 0, \qquad i = 1, 2.$$

The equilibrium strategies (Section 4.1) are given by

$$\begin{aligned} &\{(\gamma^1, \gamma^2)\colon 0 \leq \gamma_1^1 < 1/3, 2/3 < \gamma_1^1 \leq 1, \gamma_2^1 = 1 - \gamma_1^1; \\ &\qquad\qquad\qquad\qquad 0 \leq \gamma_1^2 < 1/3, 2/3 < \gamma_1^2 \leq 1, \gamma_2^2 = 1 - \gamma_1^2\} \\ &\cup \{(\gamma^1, \gamma^2)\colon 1/3 \leq \gamma_1^1 \leq 2/3, \gamma_2^1 = 1 - \gamma_1^1; \gamma_1^2 = 0, \gamma_2^2 = 1\} \\ &\cup \{(\gamma^1, \gamma^2)\colon \gamma_1^1 = 1, \gamma_2^1 = 0; 1/3 \leq \gamma_1^2 \leq 2/3, \gamma_2^2 = 1 - \gamma_1^2\}, \end{aligned}$$

whereas the POSS sets are

$$\Gamma_{sp}^1 = \{\gamma^1 \in \Gamma^1\colon 1/3 \leq \gamma_1^1 \leq 2/3, \gamma_2^1 = 1 - \gamma_1^1\},$$
$$\Gamma_{sp}^2 = \{\gamma^2 \in \Gamma^2\colon 1/3 \leq \gamma_1^2 \leq 2/3, \gamma_2^2 = 1 - \gamma_1^2\}.$$

Thus the set of equilibrium points with efficient security levels (Section 4.2) is empty. Of course, each strategy in Γ^1_{sp} and Γ^2_{sp} offers security to each of the players individually, but do not form equilibrium pairs. Clearly, no Pareto saddle points (Section 4.3) exist for the game.

Example 4.2. Consider the game matrix,

$$A = \begin{bmatrix} (-1, 1) & (-2, 3) \\ (1, -1) & (3, -2) \end{bmatrix}.$$

The equilibrium strategies are given by

$$\begin{aligned} \{(\gamma^1, \gamma^2)\colon 0 \le \gamma^1_1 < 1/3, 2/3 < \gamma^1_1 \le 1,\ \gamma^1_2 = 1 - \gamma^1_1;& \\ 0 \le \gamma^2_1 < 1,\ \gamma^2_2 = 1 - \gamma^2_1\}& \\ \cup \{(\gamma^1, \gamma^2)\colon 1/3 \le \gamma^1_1 \le 2/3,\ \gamma^1_2 = 1 - \gamma^1_1;\ \gamma^2_1 = 0,\ \gamma^2_2 = 1\}.& \end{aligned}$$

The POSS sets are

$$\Gamma^1_{sp} = \{\gamma^1 \in \Gamma^1\colon 0 \le \gamma^1_1 \le 1,\ \gamma^1_2 = 1 - \gamma^1_1\},$$
$$\Gamma^2_{sp} = \{\gamma^2 \in \Gamma^2\colon \gamma^2_1 = 1,\ \gamma^2_2 = 0\}.$$

Then, the strategy pairs that yield efficient security levels to players and are in equilibrium form the set

$$\{(\gamma^1, \gamma^2)\colon 0 \le \gamma^1_1 < 1/3, 2/3 < \gamma^1_1 \le 1,\ \gamma^1_2 = 1 - \gamma^1_1;\ \gamma^2_1 = 1,\ \gamma^2_2 = 0\}.$$

This game also does not have a Pareto saddle point.

Example 4.3. Consider the game matrix

$$A = \begin{bmatrix} (2, 3) & (3, 2) \\ (4, 1) & (2, 3) \end{bmatrix}.$$

The POSS sets are

$$\Gamma^1_{sp} = \{\gamma^1 \in \Gamma^1\colon \gamma^1_1 = 2/3,\ \gamma^1_2 = 1/3\},$$
$$\Gamma^2_{sp} = \{\gamma^2 \in \Gamma^2\colon \gamma^2_1 = 1/3,\ \gamma^2_2 = 2/3\}.$$

These are also the equilibrium and Pareto-optimal saddle point strategies for the game with

$$\bar{J}^1(\gamma^{1*}) = \underline{J}^2(\gamma^{2*}) = (8/3, 7/3).$$

4.4. Existence of Pareto Saddle Points. From the previous discussion, it is clear that the existence of a Pareto saddle point, as defined in Section 4.3, is desirable due to its excellent properties of efficiency, security, and stability. But as we have seen, not all multicriteria zero-sum matrix games have this kind of equilibrium point. It would be interesting to know the class of matrices which has this property. In order to do so, we state and prove the following theorem.

Theorem 4.1. If there exists a pair of Pareto saddle-point strategies $(\gamma^{1*}, \gamma^{2*})$, $\gamma^{1*} \in \Gamma^1$, $\gamma^{2*} \in \Gamma^2$, then the following statements are equivalent:

(i) $J(\gamma^{1*}, \gamma^2) \leq J(\gamma^{1*}, \gamma^{2*}) \leq J(\gamma^1, \gamma^{2*})$, $\gamma^1 \in \Gamma^1$, $\gamma^2 \in \Gamma^2$; (26)

(ii) $(\gamma^{1*}, \gamma^{2*})$ are saddle-point strategies of scalar criterion game matrices $A(k)$, $k = 1, \ldots, n$;

(iii) $\gamma^{1*} \in \Gamma^1_{sp}$, $\gamma^{2*} \in \Gamma^2_{sp}$, and $\bar{J}^1(\gamma^{1*}) = \underline{J}^2(\gamma^{2*})$. (27)

Proof. We shall first prove that (i)$\Rightarrow$(ii). From the inequality (26),

$$J_k(\gamma^{1*}, \gamma^2) \leq J_k(\gamma^{1*}, \gamma^{2*}) \leq J_k(\gamma^1, \gamma^{2*}),$$
$$k = 1, \ldots, n, \qquad \gamma^1 \in \Gamma^1, \qquad \gamma^2 \in \Gamma^2, \tag{28}$$

which implies that $(\gamma^{1*}, \gamma^{2*})$ is also an equilibrium pair of strategies for the matrix $A(k)$, $k = 1, \ldots, n$.

Next, we prove that (ii)$\Rightarrow$(iii). From (28),

$$\max_{\gamma^2 \in \Gamma^2} J_k(\gamma^{1*}, \gamma^2) = J_k(\gamma^{1*}, \gamma^{2*}) = \min_{\gamma^1 \in \Gamma^1} J_k(\gamma^1, \gamma^{2*}).$$

Thus,

$$\bar{J}^1(\gamma^{1*}) = \underline{J}^2(\gamma^{2*}) = J(\gamma^{1*}, \gamma^{2*}). \tag{29}$$

In order to prove that $\gamma^{1*} \in \Gamma^1_{sp}$, assume that there exists $\tilde{\gamma}^1 \in \Gamma^1$ such that

$$\bar{J}^1(\tilde{\gamma}^1) < \bar{J}^1(\gamma^{1*}),$$

which implies that

$$J(\tilde{\gamma}^1, \gamma^{2*}) < J(\gamma^{1*}, \gamma^{2*}).$$

This is a contradiction of (28). Thus no such $\tilde{\gamma}^1$ can exist and $\gamma^{1*} \in \Gamma^1_{sp}$. Similarly, $\gamma^{2*} \in \Gamma^2_{sp}$.

Next, we prove that (iii)$\Rightarrow$(i). For $k = 1, \ldots, n$,

$$\min_{\gamma^1 \in \Gamma^1} J_k(\gamma^1, \gamma^{2*}) \leq J_k(\gamma^{1*}, \gamma^{2*}) \leq \max_{\gamma^2 \in \Gamma^2} J_k(\gamma^{1*}, \gamma^2). \tag{30}$$

Using (27), the inequality signs in (30) can be replaced by equality signs. This implies that

$$J_k(\gamma^{1*}, \gamma^2) \leq J_k(\gamma^{1*}, \gamma^{2*}) \leq J_k(\gamma^1, \gamma^{2*}),$$

and therefore,

$$J(\gamma^{1*}, \gamma^2) \leq J(\gamma^{1*}, \gamma^{2*}) \leq J(\gamma^1, \gamma^{2*}).$$

This completes the proof. □

Corollary 4.1. If a Pareto-optimal saddle point exists, then for all $\gamma^1 \in \Gamma^1_{sp}$ and for all $\gamma^2 \in \Gamma^2_{sp}$,

$$\bar{J}^1(\gamma^1) = \underline{J}^2(\gamma^2). \tag{31}$$

Proof. From Theorem 4.1, we know that there exists $\gamma^{1*} \in \Gamma^1_{sp}$ and $\gamma^{2*} \in \Gamma^2_{sp}$, such that

$$\bar{J}^1(\gamma^{1*}) = \underline{J}^2(\gamma^{2*}) = a \text{ (say).}$$

Let there be $\hat{\gamma}^1 \in \Gamma^1_{sp}$ such that

$$\bar{J}^1(\hat{\gamma}^1) \neq a.$$

But, since $\hat{\gamma}^1 \in \Gamma^1_{sp}$, by Lemma 4.1,

$$\bar{J}^1(\hat{\gamma}^1) > \underline{J}^2(\gamma^{2*}) \Rightarrow \bar{J}^1(\hat{\gamma}^1) > \bar{J}^1(\gamma^{1*}),$$

which implies that $\hat{\gamma}^1 \notin \Gamma^1_{sp}$. This is a contradiction. Thus, no such $\hat{\gamma}^1$ can exist in Γ^1_{sp}. A similar proof can be constructed for $\hat{\gamma}^2 \in \Gamma^2_{sp}$. □

The corollary shows that, when a Pareto saddle point exists, all POSS are also Pareto saddle-point strategies and vice versa. The properties of ordered interchangeability and best security levels are also satisfied. The game has a unique value.

5. Determination of POSS

The importance of POSS as a solution concept has been discussed earlier. The sets of POSS for the players must be obtained in order to check for the existence of solutions based upon the idea of security levels (Sections 4.2 and 4.3). For this, we define two scalar criterion games, one for each player, and prove that the minmax (or maxmin, as the case may be) solutions of these games are also POSS for the corresponding player. Similar sufficient conditions were proved for dynamic games in Ref. 11. Here, we also prove a necessary condition for matrix games. The following discussion relates to the strategies of Player P1. Identical arguments hold for P2, for which similar results can be obtained by suitable modifications.

5.1. P1 Game. In this game, P1 has a strategy $\gamma^1 \in \Gamma^1$, but P2 has n strategies $\gamma^{21}, \dots, \gamma^{2n} \in \Gamma^2$. The payoff function for the game is defined as

$$\hat{J}^1(\gamma^1, \underline{\gamma}^2) = \alpha_1 J_1(\gamma^1, \gamma^{21}) + \cdots + \alpha_n J_n(\gamma^1, \gamma^{2n}), \tag{32}$$

where

$$\underline{\gamma}^2 = (\gamma^{21}, \dots, \gamma^{2n}) \in \underline{\Gamma}^2 = \prod_{k=1}^{n} \Gamma^2, \tag{33}$$

$$J_k(\gamma^1, \gamma^{2k}) = \gamma^{1'} A(k) \gamma^{2k}, \qquad k = 1, \dots, n, \tag{34}$$

and $\alpha_1, \dots, \alpha_n$ are scalar real numbers.

This is a scalar criterion game in which P1 tries to minimize $_J1$ and P2 tries to maximize $_J1$. Since P2 has n strategies, it can obviously devote each of its strategies for the individual maximization of J_k, $k=1,\ldots,n$. One may construct a P2 game in a similar fashion.

Definition 5.1. A strategy $\gamma^{1*}\in\Gamma^1$ is said to be a minmax strategy for P1 in the P1 game if, for all $\gamma^1\in\Gamma^1$,

$$\max_{\underline{\gamma}^2\in\underline{\Gamma}^2} \hat{J}^1(\gamma^{1*},\underline{\gamma}^2)\le\max_{\underline{\gamma}^2\in\underline{\Gamma}^2}\hat{J}^1(\gamma^1,\underline{\gamma}^2). \tag{35}$$

Theorem 5.1. A strategy $\gamma^{1*}\in\Gamma^1$ is a Pareto-optimal security strategy for P1 in the original game (i.e., $\gamma^{1*}\in\Gamma^1_{\mathrm{sp}}$), if γ^{1*} is a minmax strategy for P1 in a P1 game with $\alpha_k>0$, $k=1,\ldots,n$, $\alpha_1+\cdots+\alpha_n=1$.

Proof. Since γ^{2k} affects only $J_k(\gamma^1,\gamma^{2k})$, we have

$$\max_{\underline{\gamma}^2\in\underline{\Gamma}^2}\hat{J}^1(\gamma^1,\underline{\gamma}^2)=\sum_{k=1}^{n}\alpha_k\max_{\gamma^2\in\Gamma^2}J_k(\gamma^1,\gamma^2). \tag{36}$$

Then from (35), we have, for all $\gamma^1\in\Gamma^1$,

$$\sum_{k=1}^{n}\alpha_k\max_{\gamma^2\in\Gamma^2}J_k(\gamma^{1*},\gamma^2)\le\sum_{k=1}^{n}\alpha_k\max_{\gamma^2\in\Gamma^2}J_k(\gamma^1,\gamma^2). \tag{37}$$

This is enough to prove that $\gamma^{1*}\in\Gamma^1_{\mathrm{sp}}$ (Ref. 11). □

Theorem 5.1 proves only a sufficiency condition for POSS. Below, we shall also obtain a necessary condition. In order to do this, we use the following theorem, which is a weaker version of Theorem 4.2 given by Lin (Ref. 12), and therefore can be proved similarly.

Theorem 5.2. If a set $L\subseteq R^n$ is closed and convex, and if $z^0\in L$ is a noninferior point in L (Pareto minimum, in the sense of Definition 2.1), then there exists scalars $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_k\ge0$, $k=1,\ldots,n$, $\alpha_1+\cdots+\alpha_n=1$, such that $\alpha' z$, $z\in L$, reaches its minimum at z^0.

In Section 4, we defined security levels $\bar{J}^1(\gamma^1)$ for each $\gamma^1\in\Gamma^1$. Let $S\subseteq R^n$ be defined as the set of all security levels associated with P1's strategies

$$S=\{\bar{J}^1(\gamma^1)\colon \gamma^1\in\Gamma^1\}. \tag{38}$$

We define an extension of the set S, denoted by S^{E}, as follows:

$$S^{\mathrm{E}}=\{x\in R^n\colon \exists y\in S\ni y\le x\}. \tag{39}$$

Obviously, $S\subseteq S^{\mathrm{E}}$, and S^{E} is a closed set, since R^n is closed and S itself is closed by its very definition.

Theorem 5.3. The set S^{E} is convex.

Proof. Let $a \in S^{\mathrm{E}}$ and $b \in S^{\mathrm{E}}$. Then, for an arbitrary $\lambda \in [0, 1]$, define

$$c = \lambda a + (1-\lambda) b. \tag{40}$$

Since $a \in S^{\mathrm{E}}$, there exists $\gamma_a^1 \in \Gamma^1$ such that

$$\bar{J}^1(\gamma_a^1) \leq a. \tag{41}$$

Similarly, there exists $\gamma_b^1 \in \Gamma^1$ such that

$$\bar{J}^1(\gamma_b^1) \leq b. \tag{42}$$

Therefore,

$$\lambda \bar{J}^1(\gamma_a^1) + (1-\lambda)\bar{J}^1(\gamma_b^1) \leq \lambda a + (1-\lambda) b. \tag{43}$$

Let

$$\gamma_c^1 = \lambda \gamma_a^1 + (1-\lambda)\gamma_b^1. \tag{44}$$

Obviously, $\gamma_c^1 \in \Gamma^1$. So,

$$\bar{J}^1(\gamma_c^1) \in S. \tag{45}$$

Consider

$$\begin{aligned}
&\bar{J}_k^1(\lambda \gamma_a^1 + (1-\lambda)\gamma_b^1) \\
&= \max_{\gamma^2 \in \Gamma^2} (\lambda \gamma_a^1 + (1-\lambda)\gamma_b^1)' A(k) \gamma^2 && (46) \\
&\leq \max_{\gamma^2 \in \Gamma^2} (\lambda \gamma_a^1)' A(k) \gamma^2 + \max_{\gamma^2 \in \Gamma^2} ((1-\lambda)\gamma_b^1)' A(k) \gamma^2 \\
&= \lambda \bar{J}_k^1(\gamma_a^1) + (1-\lambda)\bar{J}_k^1(\gamma_b^1). && (47)
\end{aligned}$$

This is true for all $k = 1, \ldots, n$. From (44) and (46), we have

$$\bar{J}^1(\gamma_c^1) \leq c; \tag{48}$$

and, since (45) is true, it proves that $c \in S^{\mathrm{E}}$. Thus, S^{E} is a convex set. □

Theorem 5.4. If $\gamma^{1*} \in \Gamma^1$ is a POSS for P1 (i.e., $\gamma^{1*} \in \Gamma_{\mathrm{sp}}^1$), then there exists $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_k \geq 0$, $k = 1, \ldots, n$, $\alpha_1 + \cdots + \alpha_n = 1$, such that γ^{1*} is a minmax solution of the P1 game with this α.

Proof. The set S^{E} is closed and convex. The vector $\bar{J}^1(\gamma^{1*}) \in S^{\mathrm{E}}$ and is a noninferior point. Therefore, from Theorem 5.2, there exists $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_k \geq 0$, $k = 1, \ldots, n$, $\alpha_1 + \cdots + \alpha_n = 1$, such that $\alpha' z$, $z \in S^{\mathrm{E}}$, attains its minimum at $J^1(\gamma^{1*})$, which implies that

$$\alpha' \bar{J}^1(\gamma^{1*}) \leq \alpha' z, \qquad \text{for all } z \in S^{\mathrm{E}}. \tag{49}$$

Since for all $\gamma^1 \in \Gamma^1$, $\bar{J}^1(\gamma^1) \in S^{\mathrm{E}}$, we have

$$\alpha' \bar{J}^1(\gamma^{1*}) \leq \alpha' \bar{J}^1(\gamma^1), \qquad \text{for all } \gamma^1 \in \Gamma^1. \tag{50}$$

from which we get, for all $\gamma^1 \in \Gamma^1$,

$$\begin{aligned} &\alpha_1 \max_{\gamma^2 \in \Gamma^2} J_1^1(\gamma^{1*}, \gamma^2) + \cdots + \alpha_n \max_{\gamma^2 \in \Gamma^2} J_n^1(\gamma^{1*}, \gamma^2) \\ &\leq \alpha_1 \max_{\gamma^2 \in \Gamma^2} J_1^1(\gamma^1, \gamma^2) + \cdots + \alpha_n \max_{\gamma^2 \in \Gamma^2} J_n^1(\gamma^1, \gamma^2). \end{aligned} \tag{51}$$

Thus,

$$\max_{\underline{\gamma}^2 \in \underline{\Gamma}^2} \hat{J}^1(\gamma^{1*}, \underline{\gamma}^2) \leq \max_{\underline{\gamma}^2 \in \underline{\Gamma}^2} \hat{J}^1(\gamma^1, \underline{\gamma}^2), \qquad \text{for all } \gamma^1 \in \Gamma^1. \tag{52}$$

Therefore, by Definition 5.1, γ^{1*} is a minmax strategy for the P1 game. This proves the theorem. □

Remark 5.1. We proved a sufficiency condition and a necessary condition above for POSS. According to a theorem by Arrow, Barankin, and Blackwell (Ref. 13), the set of all Pareto-optimal points in S^{E} is the closure of the set of points obtained by scalarization with strictly positive weights, and thus we obtain almost all the Pareto-optimal strategies. Further, if the set S^{E} were polyhedral, then strictly positive scalarization would have been both a necessary and a sufficient condition. Then, only a finite number of such scalarizations would have been sufficient to obtain all the Pareto-optimal security strategies. But the polyhedrality of S^{E} is not proved here.

The above theorems and results can also be proved for player P2 with suitable modifications.

5.2. Solution Method. We have shown that the POSS for player P1 can be obtained by solving the P1 game. Here, we outline a method by which the P1 game can be first transformed into a zero-sum matrix game and then solved by the usual linear programming technique.

Let B be a matrix with p rows and q^n columns. Each column j is identified by an n-tuple $(s_1, \ldots, s_n)$, $s_i \in \{1, \ldots, q\}$, and each element b_{ij} is defined as

$$b_{ij} = \sum_{k=1}^{n} \alpha_k a_{i,s_k}(k), \tag{53}$$

for a given set of $(\alpha_1, \ldots, \alpha_n)$. Here, $a_{i,s_k}(k)$ is the element in the ith row and s_kth column of the matrix $A(k)$. Actually, b_{ij} is precisely the payoff in the P1 game when P1 chooses the ith row and P2 chooses the s_kth column in the matrix $A(k)$.

Thus, solving the zero-sum game with matrix B will give the solution to the P1 game. Since the former has a saddle-point solution, the P1 game will also have a saddle-point solution with minmax equal to maxmin value. The zero-sum game with matrix B can be solved by the usual linear programming technique for player P1 only. The solution to the P2 game can be obtained by following a similar line of reasoning.

6. Solution Concept Based on Outcome Maps

The operation of minimization and maximization in a multicriteria game imposes a preference structure on the payoff space which is not particularly well defined, the main reason being the absence of a total order relationship between the various outcomes. A special situation, in which this preference ordering can be expressed as a total order, can be constructed by partitioning the payoff space itself into a finite number of disjoint sets and associating a well-defined outcome, in a qualitative sense, to each of these sets. Here, we use the concept of ordinal utility function (Ref. 14) and define outcomes in terms of specific set of values of the performance index. It is a general concept which can be applied to both scalar and multicriteria games.

Let the payoff space $\mathscr{D}$ be partitioned into m subsets, denoted by $\mathscr{D}_i$, $i = 1, \ldots, m$, such that

$$\mathscr{D} = \bigcup_{i=1}^{m} \mathscr{D}_i, \qquad \mathscr{D}_i \cap \mathscr{D}_j = \varnothing,\ i \neq j. \tag{54}$$

Associated with each $\mathscr{D}_i$ is an outcome O_i. Thus, we have a set of outcomes $\{O_1, \ldots, O_m\}$. Each player will rank these outcomes differently depending upon their preference. Let us denote the preference relation for player P_i as $\succeq_i$. If $O_k \succeq_i O_l$, then P_i prefers O_k at least as much as O_l. The relation $\succ_i$ indicates a strict preference of one outcome over another by player P_i. We assume that $\succeq_i$ is a total order as well on the outcome set. To reflect the preferences of the players, we define two sets of preference orderings PO1 and PO2 for Players P1 and P2, such that each set is an ordered set with elements as the outcomes O_i, $i = 1, \ldots, m$, ordered with respect to the relations $\succeq_1$ and $\succeq_2$, respectively. In these sets, the first element is the most preferred outcome and the last element is the least preferred. Note that, if the set PO1 had been the reverse of the set PO2, then the game would have been a perfectly antagonistic one with no element of cooperation possible between players. Otherwise, for some outcomes, the preferences of both the players may match and some element of cooperation is possible. The former is similar to a zero-sum scalar criterion game, and the latter to a nonzero-sum one.

6.1. Pure Strategy Solutions. Consider the situation in which the players use pure strategies only. The pair of strategies (γ_p^1, γ_p^2), $\gamma_p^1 \in \Gamma_p^1$, $\gamma_p^2 \in \Gamma_p^2$, is such that γ_p^1 corresponds to the ith row and γ_p^2 to the jth column of the matrix A. Then,

$$J(\gamma_p^1, \gamma_p^2) = a_{ij} = (a_{ij}(1), \ldots, a_{ij}(n)). \tag{55}$$

Obviously, $a_{ij} \in \mathscr{D}$, and so there exists $\mathscr{D}_k \subset \mathscr{D}$, such that $a_{ij} \in \mathscr{D}_k$. Let O_k be the outcome associated with the subset $\mathscr{D}_k$. Then, the outcome resulting from the choice of the pair of strategies (γ_p^1, γ_p^2) is denoted by $O(\gamma_p^1, \gamma_p^2)$ and is

$$O(\gamma_p^1, \gamma_p^2) = O(i, j) = O_k, \tag{56}$$

where O is a function from the pure strategy space $\Gamma_p^1 \times \Gamma_p^2$ into the set of outcomes.

Definition 6.1. An outcome map $O(A)_{p\times q}$ for a payoff matrix $A_{p\times q}$ is defined as a matrix with elements from the set of outcomes $\{O_1, \ldots, O_m\}$ such that, if $a_{ij} \in \mathscr{D}_k$, then the (i, j)th element of $O(A)$ is O_k.

Now, the equilibrium strategies may be defined similarly to the Nash equilibrium concept. A pair of pure strategies $(\gamma_p^{1*}, \gamma_p^{2*})$ is said to be in equilibrium if

$$O(\gamma_p^{1*}, \gamma_p^{2*}) \succeq_1 O(\gamma_p^1, \gamma_p^{2*}), \qquad \forall \gamma_p^1 \in \Gamma_p^1, \tag{57}$$

$$O(\gamma_p^{1*}, \gamma_p^{2*}) \succeq_2 O(\gamma_p^{1*}, \gamma_p^2), \qquad \forall \gamma_p^2 \in \Gamma_p^2. \tag{58}$$

If the ordered set PO1 had been the reverse of PO2, then it would have implied that $\succeq_1$ and $\succeq_2$ are the opposite of each other, and a pair of strategies $(\gamma_p^{1*}, \gamma_p^{2*})$ would be in equilibrium if

$$O(\gamma_p^{1*}, \gamma_p^2) \succeq_1 O(\gamma_p^{1*}, \gamma_p^{2*}) \succeq_1 O(\gamma_p^1, \gamma_p^{2*}), \qquad \forall \gamma_p^1 \in \Gamma_p^1, \gamma_p^2 \in \Gamma_p^2. \tag{59}$$

This is analogous to the definition of saddle points in pure strategies for zero-sum scalar criterion games. Clearly, there could be outcome maps for which no Nash (or saddle point) equilibrium strategies may exist.

In zero-sum games, the concept of security was brought in through the minmax and maxmin solutions of the game. Similar notions also exist in nonzero-sum games. One can extend these results to the present context quite easily. Let

$$W^i = \bigcup_{\gamma_p^i \in \Gamma_p^i} W^i(\gamma_p^i), \tag{60}$$

where $W^i(\gamma_p^i)$ is the worst outcome that P_i can expect if he chooses γ_p^i. For an outcome $O_k \in W^i$, define

$$\Gamma_p^i(O_k) = \{\gamma_p^i \in \Gamma_p^i \colon W^i(\gamma_p^i) = O_k\}. \tag{61}$$

If W^i is ordered according to the relation $\succeq_i$, then the first element $\mathscr{F}(W^i)$ is the best outcome (security level) that P_i can expect for his choice of strategies restricted to Γ_p^i. The corresponding set of security strategies will be $\Gamma_p^i(\mathscr{F}(W^i))$. The security level of each player is unique, and there exists at least one security strategy for each player.

Lemma 6.1. $\mathscr{F}(W^2) \succeq_1 \mathscr{F}(W^1)$ and $\mathscr{F}(W^1) \succeq_2 \mathscr{F}(W^2)$. (62)

Proof. Let $\mathscr{F}(W^1) = \mathrm{PO1}(k)$. Then, P2 can never guarantee an outcome lying in the set $\{\mathrm{PO1}(k+1), \ldots, \mathrm{PO1}(m)\}$. Thus,

$$\mathscr{F}(W^2) \in \{\mathrm{PO1}(1), \ldots, \mathrm{PO1}(k)\} \Rightarrow \mathscr{F}(W^2) \succeq_1 \mathscr{F}(W^1).$$

Using exactly similar arguments, it can be proved that

$$\mathscr{F}(W^1) \succeq_2 \mathscr{F}(W^2).$$ □

This lemma is similar to a result in scalar-criterion zero-sum games, which says that the minmax value is never lower than the maxmin value. Only, in this case, the notions of minmax and maxmin have been replaced by the notions of assured or guaranteed outcomes based upon the preference structures imposed upon the payoff space by the individual preference orders PO1 and PO2.

Let

$$\mathscr{F}(W^1) = \mathrm{PO1}(r), \qquad \mathscr{F}(W^2) = \mathrm{PO2}(s).$$

Then,

$$\mathrm{PO1}(r) \in \{\mathrm{PO2}(1), \ldots, \mathrm{PO2}(s)\},$$

$$\mathrm{PO2}(s) \in \{\mathrm{PO1}(1), \ldots, \mathrm{PO1}(r)\}.$$

If the players decide to use only their security strategies, then the outcomes that are possible must belong to the set

$$\{\mathrm{PO2}(1), \ldots, \mathrm{PO2}(s)\} \cap \{\mathrm{PO1}(1), \ldots, \mathrm{PO1}(r)\}.$$

This set is nonempty, since both $\mathrm{PO1}(r)$ and $\mathrm{PO2}(s)$ belong to it.

6.2. Extension to Mixed Strategies. Extensions of the concepts given above to mixed strategies are not as straightforward as in other cases. The reason is that we are classifying the payoffs in terms of outcomes which are not really quantified, but are only a qualitative description of payoffs in the game. As a result, the averaging effect brought about by expanding the strategy set to include mixed strategies becomes meaningless. Note that the mixed strategies that we are employing are with respect to the outcome matrix $O(A)$ and not the original payoff matrix A. Thus, when mixed strategies are used, one can only determine the probability of occurrence of a particular outcome. We define m $p\times q$ matrices $M_1, \dots, M_m$ such that

$$M_k(i,j)=1, \qquad \text{if } O(A)(i,j)=O_k, \tag{63a}$$

$$M_k(i,j)=0, \qquad \text{otherwise.} \tag{63b}$$

If the players use mixed strategies $\gamma^1 \in \Gamma^1$ and $\gamma^2 \in \Gamma^2$, then the probability of occurrence of an outcome O_k is

$$P(O_k, (\gamma^1, \gamma^2)) = \gamma^{1'} M_k \gamma^2, \tag{64}$$

where P is a function from the set of outcomes and the mixed strategy spaces $\Gamma^1 \times \Gamma^2$ of the players to the segment of the real line $[0, 1]$. Obviously,

$$\sum_{k=1}^{m} P(O_k, (\gamma^1, \gamma^2)) = \sum_{k=1}^{m} \gamma^{1'} M_k \gamma^2 = 1. \tag{65}$$

Even with this interpretation, in general, it is not possible to define an equilibrium point except for some special cases.

6.3. Combat Game in Matrix Form. Combat games are generalizations of pursuit-evasion problems in differential games (Ref. 2). They can be modelled as bicriterion dynamic games (Refs. 3 and 4). Here, we define a bicriterion matrix game with outcomes similar to combat games. The four outcomes in the $J_1 J_2$-space are defined as follows:

$$\mathscr{D}_1\colon J_1 \le 0, J_2 < 0\ (O_1, \text{win for } P1), \tag{66a}$$

$$\mathscr{D}_2\colon J_1 > 0, J_2 \ge 0\ (O_2, \text{win for P2}), \tag{66b}$$

$$\mathscr{D}_3\colon J_1 > 0, J_2 < 0\ (O_3, \text{draw}), \tag{66c}$$

$$\mathscr{D}_4\colon J_1 \le 0, J_2 \ge 0\ (O_4, \text{mutual kill}). \tag{66d}$$

Obviously, the preference ordering for the two players is

$$\text{PO1} = \{O_1, O_3, O_4, O_2\}, \tag{67}$$

$$\text{PO2} = \{O_2, O_3, O_4, O_1\}. \tag{68}$$

Note that the game is not perfectly antagonistic in this formulation. Using the ideas of security levels in individual criteria, as presented in Section 4, and the concept of preference orderings on the outcomes, one can obtain some useful results in certain special cases.

From here onwards, we shall denote the subset of the payoff space, and the outcome associated with it, with the same symbol O_i.

Case 1. See Fig. 1. Let the player P1 be assured of a guaranteed payoff $\bar{J}^1(\gamma^1) \in O_3$ (draw), for some $\gamma^1 \in \Gamma^1_{sp}$. In addition, there exists no $\tilde{\gamma}^1 \in \Gamma^1$ such that $\bar{J}^1(\tilde{\gamma}^1) \in O_1$. On the other hand, P2 is only assured of a guaranteed payoff $\underline{J}^2(\gamma^2) \in O_1$ (win for P1) for some $\gamma^2 \in \Gamma^2_{sp}$, and there exists no other $\tilde{\gamma}^2 \in \Gamma^2$ such that $\underline{J}^2(\tilde{\gamma}^2) \in O_2 \cup O_3 \cup O_4$. This implies that the best outcome that P1 can guarantee is O_3 (draw), but that the best that P2 can guarantee is O_1 (win for P1) in the outcome map analysis. In fact, the converse is also true here.

In the outcome map $O(A)$, there will be at least one row containing only O_1 or O_3, or only O_3. Each column will contain at least one O_1. These outcomes are the security levels corresponding to the pure security strategies

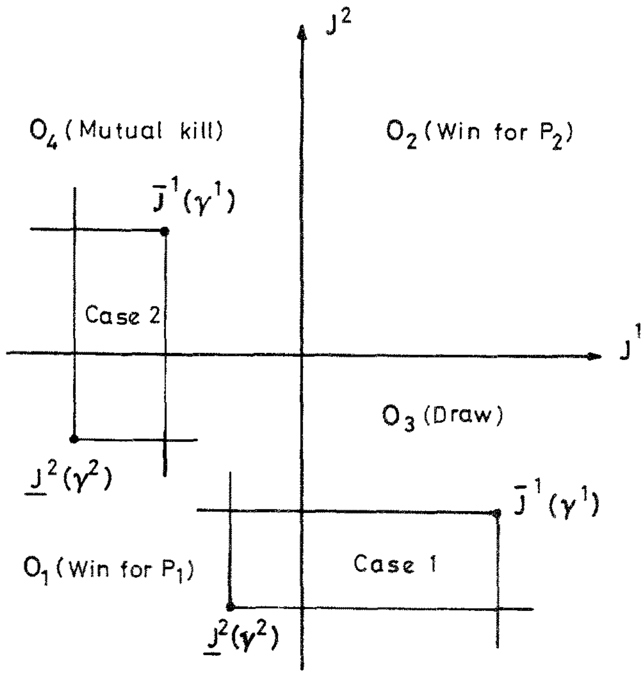

Fig. 1. Security levels in Cases 1 and 2.

of P1 and P2, respectively. It is reasonable to expect that P1 will mix only among his security strategies in order to improve his probability of win (O_1) without any risk of being destroyed. He might improve his probability of win further by expanding his strategy set beyond his security strategies, but only at the risk of occurrence of outcomes O_2 and O_4. Player P2 has no choice and will mix among all his available strategies to increase the chances of draw (O_3). Using only security strategies, we have a reduced outcome map $\hat{O}(A)$, made up of elements from the set $\{O_1, O_3\}$ only.

In this reduced matrix, a pair of mixed strategies $(\gamma^{1*}, \gamma^{2*})$ is said to be in saddle-point equilibrium iff

$$P(O_1, (\gamma^{1*}, \gamma^2)) \geq P(O_1, (\gamma^{1*}, \gamma^{2*})) \geq P(O_1, (\gamma^1, \gamma^{2*}))$$
$$\Leftrightarrow P(O_3, (\gamma^{1*}, \gamma^2)) \leq P(O_3, (\gamma^{1*}, \gamma^{2*})) \leq P(O_3, (\gamma^1, \gamma^{2*})).$$

The equilibrium pair of strategies can be found by considering a matrix $\hat{M}_1$ derived from $\hat{O}(A)$ using (63), and finding its saddle-point solution with P1 as the maximizing player and P2 as the minimizing player, or by finding the saddle point of the matrix $\hat{M}_3$, with P1 as the minimizer and P2 as the maximizer.

Remark 6.1. Exactly similar arguments can be used to obtain saddle-point solutions when the best outcome that P2 can assure is O_3 and the best outcome that P1 can assure is O_2.

Case 2. See Fig. 1. Let the player P1 be assured of a guaranteed payoff $\bar{J}^1(\gamma^1) \in O_4$ (mutual kill), for some $\gamma^1 \in \Gamma^1_{sp}$. In addition, there exists no $\tilde{\gamma}^1 \in \Gamma^1$ such that $\bar{J}^1(\tilde{\gamma}^1) \in O_1 \cup O_3$. On the other hand, player P2 is assured only of a guaranteed payoff $\underline{J}^2(\gamma^2) \in O_1$ (win for P1) for some $\gamma^2 \in \Gamma^2_{sp}$, and there exists no other $\tilde{\gamma}^2 \in \Gamma^2$ such that $\underline{J}^2(\tilde{\gamma}^2) \in O_2 \cup O_3 \cup O_4$. This implies that the best outcome that P1 can guarantee is O_4 (mutual kill), but that the best that P2 can guarantee is O_1 (win for P1) in the outcome map analysis. Unlike Case 1, the converse is not true here, since the reduced outcome matrix may also contain O_3 when we use the assured outcomes; but, when we use assured security levels, O_3 cannot occur, and only O_1 and O_4 will occur.

The outcome map $O(A)$ will contain at least one row with elements O_1 and O_4 only, or only O_4, and each column will contain at least one O_1. Exactly as in Case 1, the reduced outcome map $\hat{O}(A)$, containing O_1 and O_4 only, is obtained by using security strategies, and the saddle point is determined through $\hat{M}_1$ and $\hat{M}_4$.

Remark 6.2. Again, similar arguments can be used to obtain saddle-point solutions when $\bar{J}^1(\gamma^1) \in O_2$ and $\underline{J}^2(\gamma^2) \in O_4$, for some $\gamma^1 \in \Gamma^1_{sp}$ and

$\gamma^2 \in \Gamma^2_{sp}$, are the best security levels that P1 and P2 can guarantee, respectively.

Remark 6.3. Whenever the outcome matrix can be reduced, using security strategies, to a matrix with only two outcomes as its elements such that one of the outcomes is favored by P1 and the other by P2, we can find a saddle-point solution. In other cases, one can only speak of the highest probability of occurrence of one or a collection of outcomes.

Example 6.1. Consider a game matrix A as follows:

$$A = \begin{bmatrix} (-2,-3) & (-1,-2) & (1,2) \\ (2,-1) & (-1,-1) & (-2,-1) \\ (-3,-1) & (1,-2) & (-2,-2) \end{bmatrix}.$$

The corresponding outcome map $O(A)$ is given as

$$O(A) = \begin{bmatrix} O_1 & O_1 & O_2 \\ O_3 & O_1 & O_1 \\ O_1 & O_3 & O_1 \end{bmatrix}.$$

Security strategies for P1 are the rows 2 and 3, whereas security strategies for P2 are columns 1, 2, and 3. The reduced outcome map is

$$\hat{O}(A) = \begin{bmatrix} O_3 & O_1 & O_1 \\ O_1 & O_3 & O_1 \end{bmatrix},$$

corresponding to which we have

$$\hat{M}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \hat{M}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The saddle-point strategies for these two games will be $(1/2, 1/2)$ for P1 and $(1/2, 1/2, 0)$ for P2. This will yield the saddle-point probabilities of win and draw for player P1 as $1/2$ and $1/2$, respectively.

Suppose that P1 uses all his strategies in a bid to maximize his probability of win and P2 uses his strategies to maximize his own chances of win (O_2) or draw (O_3). Then, we have

$$M_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

which P1 maximizes and P2 minimizes. The saddle-point strategies for both players are $(1/3, 1/3, 1/3)$. The probability of win for P1 (O_1) for this set of strategies is $2/3$. But P1 runs the risk of P2 winning (O_2) with a probability

1/9. The probability of draw is 2/9 here. This was the situation described in Case 1.

7. Conclusions

In earlier works, the concept of equilibrium points in a two-person multicriteria game has mainly been defined using Pareto-optimal (efficient) equilibrium strategies which do not, in general, satisfy the conditions of Pareto-optimal security levels. A combination of these two notions gives rise to equilibrium points with efficient security levels. Also, a particularly desirable solution is the Pareto saddle point, which ensures the equality of the security levels of both the players. But these solutions do not exist for all cases. In this paper, conditions for their existence and methods for their determination are presented.

This notion of security levels is also extended to games with qualitative outcomes, and some solution concepts are proposed. An attempt is being made at present to obtain more general results for these games.

We hope that this contribution will stimulate further research into the conceptual framework, theory, and computational methods for the solution of two-person multicriteria games.

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