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Discrete Approximation of Relaxed Optimal Control Problems

I. CHRYSSOVERGHI[1] AND A. BACOPOULOS[2]

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Abstract. We consider a general nonlinear optimal control problem for systems governed by ordinary differential equations with terminal state constraints. No convexity assumptions are made. The problem, in its so-called relaxed form, is discretized and necessary conditions for discrete relaxed optimality are derived. We then prove that discrete optimality [resp., extremality] in the limit carries over to continuous optimality [resp., extremality]. Finally, we prove that limits of sequences of Gamkrelidze discrete relaxed controls can be approximated by classical controls.

Key Words. Optimal control, nonlinear systems, discretization, nonconvexity, relaxed controls, approximation.

1. Introduction

It is well known that optimal control problems, without any convexity assumptions, generally do not have classical solutions. Generalized or relaxed controls have been used by several authors (Refs. 1-4) to prove existence theorems and derive necessary conditions for optimality for non-convex problems. Moreover, iterative methods have been developed for these problems, which use relaxed controls (Refs. 3 and 5). One must, of course, discretize the optimal control problems to implement the numerical methods on a computer (Ref. 6). Accordingly, we study in this paper properties of discrete relaxed optimality and extremality and their behavior in the limit, as well as the approximation of relaxed controls by classical ones.

[1] Assistant Professor, Department of Mathematics, National Technical University, Athens, Greece.

[2] Professor, Department of Mathematics, National Technical University, Athens, Greece.

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2. Continuous Relaxed Optimal Control Problem

We state in this section some background information regarding the continuous problem which is seen in Section 4 to be the limit of the discrete analogue. For the relevant theory, see Ref. 2.

Consider the following optimal control problem. The state equations are

$$x'(t) = f(t, x(t), u(t)), \qquad 0 \leq t \leq T < \infty, \tag{1a}$$

$$x(0) = x_0, \tag{1b}$$

where $x = x^u$, $x(t) \in \mathbb{R}^p$, and $u(t) \in U \subset \mathbb{R}^q$.

We may have also state constraints of the form

$$G_1(u) := g_1(x(T)) = 0, \qquad g_1 : \mathbb{R}^p \to \mathbb{R}^{m_1}, \tag{2a}$$

$$G_2(u) := g_2(x(T)) \leq 0, \qquad g_2 : \mathbb{R}^p \to \mathbb{R}^{m_2}. \tag{2b}$$

The cost functional is given by

$$G_0(u) := g_0(x(T)). \tag{3}$$

The classical optimal control problem is to minimize $G_0(u)$ subject to the above constraints.

We suppose that U is a compact (not necessarily convex) subset of $\mathbb{R}^q$. We set $I := [0, T]$, and define the set of classical controls by

$$C := \{u: t \mapsto u(t) \mid u\text{-measurable from } I \text{ to } U\}.$$

We then define the set of relaxed controls by

$$R := \{t \mapsto r(t) \mid r \text{ measurable from } I \text{ to the set of probability measures } M_1(U) \text{ on } U\}.$$

$M_1(U)$ is a closed subset of $M(U)$, the space of finite regular Borel measures on U, which is the dual $C^0(U)^*$ of the space of continuous functions, with the weak-star topology. R is a closed subset of $B(I \times U)^*$, where $B(I \times U)$ is the space of Caratheodory functions on $I \times U$. A sequence $\{r_k\}$ in R converges to r if

$$\lim_{k \to \infty} \int_{I \times U} F(t, u) r_k(t)(du)\, dt = \int_{I \times U} F(t, u) r(t)(du)\, dt,$$

for every $F \in B(I \times U)$ or $F \in C^0(I \times U)$. For simplicity of notation, we write

$$F(t, r(t)) := \int_U F(t, u) r(t)(du).$$

The sets $M_1(U)$ and R are metrizable, convex, and compact. We also identify every classical control $u(t)$ with its associated Dirac relaxed control $\delta_{u(t)}$. Thus, we have $C \subset R$.

The continuous relaxed optimal control problem (CRP) may now be formulated as follows. We replace Eq. (1) by

$$x'(t) = f(t, x(t), r(t)), \qquad \text{on } I, \tag{4a}$$

$$x(0) = x_0, \tag{4b}$$

where $r \in R$, $x = x^r$, the constraints (2) by

$$G_1(r) := g_1(x(T)) = 0, \tag{5a}$$

$$G_2(r) := g_2(x(T)) \leqslant 0, \tag{5b}$$

and the cost (3) by

$$G_0(r) := g_0(x(T)). \tag{6}$$

The continuous relaxed problem (CRP) is to minimize $G_0(r)$ subject to the above constraints.

The following are relatively weak assumptions concerning general nonlinear control problems, including the discretization which follows.

Assumption A1. The function f is continuous on the set

$$D := \{(t, x, u) \mid 0 \leqslant t \leqslant a, \|x - x_0\| \leqslant b, u \in U\},$$

where $a, b > 0$ and

$$0 < T \leqslant \min(a, b/M),$$

where

$$M := \max_D \|f(t, x, u)\|.$$

Proposition 2.1. Under Assumption A1, for every $r \in R$, there exists an absolutely continuous solution x of Eqs. (4) which satisfies

$$\|x - x_0\|_\infty \leqslant c := MT.$$

If f is also Lipschitzian w.r.t. x on D, then Eqs. (4) have a unique solution.

Assumption A2. Equations (4) have a unique solution $x = x^r$, for every $r \in R$.

Assumption A3. There exists an admissible control $r \in R$, i.e., which satisfies the constraints (5).

Assumption A4. The functions g_0, g_1, g_2 are continuous for $\|x\| \leqslant c$.

Theorem 2.1. Under Assumptions A1 and A2, the mapping $r \mapsto x^r$, from R to $C^0(I)$, is continuous. Under Assumptions A1–A4, there exists an optimal control for the CRP.

Now, define the following set, for given c':

$$D' := \{(t, x, u) \mid t \in I, \|x - x_0\| \leqslant c', u \in U\}.$$

Assumption A5. There exists $c' > c$ such that f and f_x are continuous on D' and g_0, g_1, g_2 are differentiable for $\|x\| < c$.

Theorem 2.2. Continuous Relaxed Pontryagin Minimum Principle. Under Assumptions A1-A5, if r is optimal for the CRP, then r is extremal, i.e., there exist multipliers $\lambda_0 \in \mathbb{R}$, $\lambda_1 \in \mathbb{R}^{m_1}$, $\lambda_2 \in \mathbb{R}^{m_2}$; $\lambda_0 \geq 0$, $\lambda_2 \geq 0$, with $\lambda_0 + \|\lambda_1\| + \|\lambda_2\| = 1$, such that

$$z(t) \cdot f(t, x(t), r(t)) = \min_{u \in U} z(t) \cdot f(t, x(t), u), \qquad \text{a.e. in } I, \tag{7}$$

where $x = x^r$ and the adjoint state z is given by the equations

$$z'(t) = -z(t) \cdot f_x(t, x(t), r(t)), \tag{8a}$$

$$z(T) = \sum_{l=0}^{2} \lambda_l \cdot g_{lx}(x(T)), \tag{8b}$$

and is such that the following transversality condition holds:

$$\lambda_2 \cdot g_2(x(T)) = 0. \tag{9}$$

3. Discrete Relaxed Optimal Control Problem

We now discretize the continuous relaxed problem CRP. For each $n \in \mathbb{N}$, choose an integer $k = k(n)$, $k+1$ points in $I := [0, T]$ with

$$0 = t_{n0} < t_{n1} < \cdots < t_{nk} = T,$$

and set

$$h_{ni} := t_{n,i+1} - t_{ni}, \qquad i = 0, \ldots, k-1,$$

$$h_n := \max_i h_{ni},$$

$$I_{ni} := [t_{ni}, t_{n,i+1}), \qquad i = 0, \ldots, k-2,$$

$$I_{n,k-1} := [t_{n,k-1}, t_{nk}].$$

Let R_n be the set of piecewise constant relaxed controls relative to the partition $\{I_{ni}\}_{i=0}^{k-1}$,

$$R_n := \{r_n \in R \mid r_n(t) = r_{ni} \in M_1(U), \text{ on } I_{ni},\ i = 0, \ldots, k-1\};$$

let C_n be the set of piecewise constant classical controls

$$C_n := \{u_n \in C \mid u_n(t) = u_{ni}, \text{ on } I_{ni},\ u_{ni} \in U\};$$

and let R_n^G be the set of piecewise constant Gamkrelidze controls

$$R_n^G := \left\{ \bar{r}_n \in R \mid \bar{r}_n(t) = r_{in} = \sum_{j=0}^{p} \alpha_{nij} \delta_{u_{nij}}, \text{ on } I_{ni}, \right.$$

$$\left. \text{with } u_{nij} \in U, \ \sum_{j=0}^{p} \alpha_{nij} = 1, \ \alpha_{nij} \geqslant 0 \right\}.$$

Clearly, $C_n \subset R_n^G \subset R_n \subset R$, for every n. R_n is convex and compact for the weak-star topology of $[M_1(U)]^k$. However, note that C_n and R_n^G are not compact.

The discrete relaxed optimal control problem (DRP_n) is now formulated as follows. The state equations are given by the Euler scheme (for simplicity)

$$x_{n,i+1} = x_{ni} + h_{ni} f(t_{ni}, x_{ni}, r_{ni}), \qquad i = 0, \ldots, k-1, \tag{10a}$$

$$x_{n0} = x_0, \tag{10b}$$

for $r_n \in R_n$, the state constraints by

$$G_{1n}(r_n) := g_1(x_{nk}) = \epsilon_{1n}, \tag{11a}$$

$$G_{2n}(r_n) := g_2(x_{nk}) \leqslant \epsilon_{2n}, \tag{11b}$$

where $\epsilon_{1n} \in \mathbb{R}^{m_1}$, $\epsilon_{2n} \in \mathbb{R}^{m_2}$, $\epsilon_{2n} \geqslant 0$, are chosen vectors, and the cost by

$$G_{0n}(r_n) := g_0(x_{nk}) = \min. \tag{12}$$

Theorem 3.1. Under Assumption A1, the mapping $r_n \mapsto x_n$ is continuous from $R_n = [M_1(U)]^k$ to $R^{p(k+1)}$. Under Assumptions A1 and A4, and if there exists an admissible control for the DRP_n, then there exists an optimal control.

Proof. Let $r_n^m \to r_n$ in R_n. By induction on i, if $x_{ni}^m \to x_{ni}$, then

$$\lim_{m\to\infty} x_{n,i+1}^m = \lim_{m\to\infty} [x_{ni}^m + h_{ni} f(t_{ni}, x_{ni}^m, r_{ni}^m)]$$

$$= x_{ni} + h_{ni} f(t_{ni}, x_{ni}, r_{ni}) = x_{n,i+1}.$$

Since $x_{n0}^m = x_{n0} = x_0$, it follows that $x_n^m \to x_n$. Therefore, $r_n \mapsto x_n$ is sequentially continuous, hence continuous. Clearly, $\|x_{nk}\| \leqslant MT = c$, for all $r_n \in R_n$. Since R_n is compact and g_0, g_1, g_2 are continuous for $\|x\| \leqslant c$, $G_{0n}(r_n) := g_0(x_{nk})$ attains its minimum on the nonempty compact set

$$\{r_n \in R_n \mid \|x_{nk}\| \leqslant c, \ g_1(x_{nk}) = \epsilon_{1n}, \ g_2(x_{nk}) \leqslant \epsilon_{2n}\}. \qquad \square$$

Theorem 3.2. Discrete Relaxed Minimum Principle. Under Assumptions A1, A4, A5, if r_n is optimal for the DRP_n, then r_n is extremal, i.e., there exists multipliers $\lambda_{0n} \in \mathbb{R}$, $\lambda_{1n} \in \mathbb{R}^m$, $\lambda_{2n} \in \mathbb{R}^m$, $\lambda_{0n} \geq 0$, $\lambda_{2n} \geq 0$, with $\lambda_{0n} + \|\lambda_{1n}\| + \|\lambda_{2n}\| = 1$, such that

$$z_{n,i+1} \cdot f(t_{ni}, x_{ni}, r_{ni}) = \min_{u \in U} z_{n,i+1} \cdot f(t_{ni}, x_{ni}, u), \qquad i = 0, \ldots, k-1, \tag{13}$$

where x_n is given by (10) and the adjoint discrete state is described by

$$z_{ni} = z_{n,i+1} + h_{ni} z_{n,i+1} \cdot f_x(t_{ni}, x_{ni}, r_{ni}), \qquad i = 0, \ldots, k-1, \tag{14a}$$

$$z_{nk} = \sum_{l=0}^{2} \lambda_{ln} \cdot g_{lx}(x_{nk}), \tag{14b}$$

and is such that

$$\lambda_{2n} \cdot [g_2(x_{nk}) - \varepsilon_{2n}] = 0. \tag{15}$$

Proof. By the general multiplier theorem (Ref. 2, p. 303), if r_n is optimal, there exist multipliers λ_{ln} such that

$$\sum_{l=0}^{2} \lambda_{ln} \cdot DG_{ln}(r_n, r'_n - r_n) \geq 0, \qquad \forall r'_n \in R_n,$$

where DG_{ln} is the directional derivative of G_{ln}, or after some calculations

$$\sum_{i=0}^{n-1} h_{ni} z_{n,i+1} \cdot f(t_{ni}, x_{ni}, r'_{ni} - r_{ni}) \geq 0, \qquad \forall r'_n \in R_n,$$

which is equivalent to the discrete pointwise minimum principle (13). Equality (15) is the transversality condition. □

4. Convergence

From now on, we suppose that the partitions $\{I_{ni}\}$ are chosen such that $h_n := \max_i h_{ni} \to 0$, as $n \to \infty$.

The following lemma shows that the sets C_n, hence R_n^G, R_n, approximate R, as $n \to \infty$.

Lemma 4.1. Given $r \in R$, there exists a sequence $\{u_n \in C_n\}$ such that $u_n \to r$ in R.

Proof. It is proved in Ref. 2, pp. 275–276, that given $r \in R$, there exists a sequence $\{\bar{u}_m\}$ of piecewise constant classical controls such that $\bar{u}_m \to r$

in R. Let $\varphi \in C^0(I)$, $\psi \in C^0(U)$, and let $\epsilon > 0$ be given. Let $\{\epsilon_m\}$ be a sequence of positive numbers such that $\epsilon_m \to 0$. For each m, define the sequence of controls $\{u_m^n \in C_n\}$ by

$$u_m^n(t) = \bar{u}_m(t_{ni}) \text{ on } I_{ni}, \qquad i = 0, \ldots, k-1.$$

It follows easily that, for every fixed m, there exists $N(m)$ such that

$$\|\varphi\|_\infty \int_I |\psi(u_m^n(t)) - \psi(\bar{u}_m(t))|\, dt \leq \epsilon_m, \qquad \text{for } n \geq N(m),$$

and we can suppose that $N(m) < N(m+1)$, for all m. Now, set

$$u_n(t) := u_m^n(t), \qquad \text{for } N(m) \leq n < N(m+1).$$

Then,

$$a_m^n := \|\varphi\|_\infty \int_I |\psi(u_n) - \psi(\bar{u}_m)|\, dt \leq \epsilon_m,$$
$$\text{for } N(m) \leq n < N(m+1).$$

Now, since $\bar{u}_m \to r$ and $\epsilon_m \to 0$, there exists $M(\epsilon)$ such that

$$b_m := \left| \int_I \varphi[\psi(\bar{u}_m) - \psi(r)]\, dt \right| \leq \epsilon/2,$$

and $\epsilon_m \leq \epsilon/2$, for $m \geq M(\epsilon)$. Hence,

$$\left| \int_I \varphi(t)[\psi(u_n) - \psi(r)]\, dt \right| \leq a_m^n + b_m \leq \epsilon, \qquad \text{for } n \geq N(M(\epsilon)).$$

Since the linear combinations of functions $\varphi \cdot \psi$ are dense in $C^0(I \times U)$, it follows that $u_n \to r$ in R. □

For $r_n \in R_n$, define the functions

$$\bar{x}_n(t) = x_{ni}, \qquad t \in I_{ni},\ i = 0, \ldots, k-1, \tag{16a}$$

$$x_n(t) = x_{ni} + (t - t_{ni}) f(t_{ni}, x_{ni}, r_{ni}), \qquad t \in I_{ni},\ i = 0, \ldots, k-1, \tag{16b}$$

where $\{x_{ni}\}_{i=0}^k$ corresponds to r_n by (10).

Lemma 4.2. Under Assumptions A1 and A2, if $r_n \to r$ in R, then $x_n \to x^r$ and $\bar{x}_n \to x^r$ uniformly on I.

Proof. Let $\epsilon > 0$. Since f is uniformly continuous on the compact set D, there exists δ such that

$$\|f(t', x', u) - f(t'', x'', u)\| \leq \epsilon,$$

for $|t'-t''|\leq\delta$, $\|x'-x''\|\leq\delta$, and $u\in U$. Clearly, this implies that

$$\|f(t',x',\rho)-f(t'',x'',\rho)\|\leq\epsilon,$$

for every $\rho\in M_1(U)$, $|t'-t''|\leq\delta$, and $\|x'-x''\|\leq\delta$. Now, choose n such that

$$h_n=\max_i h_{ni}\leq\min(\delta,\delta/M).$$

Then, by construction of $x_n(t)$, we see that

$$\|x_n(t')-x_n(t'')\|\leq M|t'-t''|,\qquad t',t''\in I,$$

and

$$\|x_n(t)-x_0\|\leq MT\leq b,$$

which show that the functions $x_n(t)$ are equicontinuous and bounded on I. For $t\in I_{ni}$, we also have

$$\|x_n(t)-x_{ni}\|\leq\delta,$$

hence,

$$\|x_n'(t)-f(t,x_n(t),r_n(t))\|$$
$$=\|f(t_{ni},x_{ni},r_{ni})-f(t,x_n(t),r_n(t))\|\leq\epsilon,\qquad \text{for } t\in I_{ni},\ i=0,\ldots,k-1.$$

Therefore,

$$x_n'(t)=f(t,x_n(t),r_n(t))+\alpha_n(t),$$

where $\alpha_n\to 0$ uniformly on I. Now, we have

$$x_n(t)=x_0+\int_0^t[f(s,x_n(s),r_n(s))+\alpha_n(s)]\,ds.$$

By Ascoli's theorem (Ref. 2, p. 109), there exists a subsequence $\{x_n\}$ (same notation) such that $x_n\to x$ uniformly. We have

$$x_n(t)=x_0+\int_0^t[f(s,x_n,r_n)-f(s,x,r_n)]\,ds$$
$$+\int_0^t[f(s,x,r_n)-f(s,x,r)]\,ds$$
$$+\int_0^t f(s,x,r)\,ds+\int_0^t\alpha_n(s)\,ds.$$

Since f is uniformly continuous and $r_n\to r$ in R, we find that, in the limit,

$$x(t)=x_0+\int_0^t f(s,x(s),r(s))\,ds,$$

which shows that $x = x^r$. The convergence of the whole sequence $\{x_n\}$ follows from the uniqueness of the limit x^r. Finally, it follows easily that also $\bar{x}_n \to x^r$ uniformly. □

Lemma 4.3. Under Assumptions A1-A4, we can choose the sequences $\{\epsilon_{1n}\}$, $\{\epsilon_{2n}\}$ of vectors in (11) such that the DRP$_n$ have an admissible control.

Proof. Let r be admissible for the CRP. By Lemma 4.1, there exists a sequence $\{r_n \in R_n\}$ converging to r. By Lemma 4.2 and the continuity of g_1, g_2 for $\|x\| \leq c$, we have

$$\lim_{n\to\infty} G_{1n}(r_n) = \lim_{n\to\infty} g_1(x_{nk}) = g_1(x^r(T)) = 0,$$

$$\lim_{n\to\infty} G_{2n}(r_n) = \lim_{n\to\infty} g_2(x_{nk}) = g_2(x^r(T)) \leq 0.$$

Now, for each n, choose any solution r_n^* of the minimization problem

$$\min_{r'_n \in R_n} \{\|G_{1n}(r'_n)\|^2 + \|\max[0, G_{2n}(r'_n)]\|^2\},$$

where the max between vectors is taken componentwise, and set

$$\epsilon_{1n} = G_{1n}(r_n^*), \qquad \epsilon_{2n} = \max[0, G_{2n}(r_n^*)].$$

Then, r_n^* is admissible for the DRP$_n$, and clearly $\epsilon_{1n}, \epsilon_{2n} \to 0$. □

From now on, we suppose that the sequences $\{\epsilon_{1n}\}$, $\{\epsilon_{2n}\}$ are chosen as in Lemma 4.3.

Theorem 4.1. Under Assumptions A1-A4, let r_n be optimal for the DRP$_n$, for $n = 1, 2, \ldots$. Then, the sequence $\{r_n\}$ has cluster points and every cluster point is optimal for the CRP.

Proof. Since R is compact, let $\{r_n\}$ (same notation) be a subsequence such that $r_n \to r$. By Assumptions A1, A4 and Lemma 4.2,

$$\lim G_{ln}(r_n) = \lim g_l(x_{nk}) = g_l(x^r) = G_l(r), \qquad \text{for } l = 0, 1, 2.$$

Since r_n is optimal for the DRP$_n$, we have

$$G_{0n}(r_n) \leq G_{0n}(r'_n), \qquad \forall r' \in R_n.$$

Let $r' \in R$ and, by Lemma 4.1, a sequence $\{r'_n \in R_n\}$ converging to r'. Then,

$$G_0(r) = \lim G_{0n}(r_n) \leq \lim G_{0n}(r'_n) = G_0(r'),$$

$$G_1(r) = \lim G_{1n}(r_n) = \lim \epsilon_{1n} = 0,$$

$$G_2(r) = \lim G_{2n}(r_n) \leq \lim \epsilon_{2n} = 0,$$

i.e., r is optimal for the CRP. □

Now, for z_n given by (14), define

$$\bar{z}_n(t) = z_{n,i+1} \qquad \text{on } I_{ni},\ i=0,\dots,k-1, \tag{17a}$$

$$z_n(t) = z_{n,i+1} + (t_{n,i+1} - t) z_{n,i+1} \cdot f(t_{ni}, x_{ni}, t_{ni}), \qquad \text{on } I_{ni},\ i=0,\dots,k-1. \tag{17b}$$

Lemma 4.4. Under Assumptions A1, A4, A5, if $r_n \to r$ and $\lambda_{ln} \to \lambda_l$, $l=0,1,2$, then $z_n \to z$ and $\bar{z}_n \to z$ uniformly on I, where z [resp., z_n] is given by (8) [resp., (14)].

Proof. Setting

$$M' := \max_{D'} \|f_x\|,$$

from (14) and Lemma 4.2, we get

$$\begin{aligned} \|z_{ni}\| &\leq (1+h_{ni}M')\|z_{n,i+1}\| \leq \prod_{j=i}^{k-1} (1+h_{nj}M')\|z_{nk}\| \\ &\leq \exp\left[\sum_{j=0}^{k-1} h_{nj}M'\right]\|z_{nk}\| \leq \exp(TM')\|z_{nk}\| \\ &\leq \exp(TM')\left\|\sum_{l=0}^{2} \lambda_{ln}\cdot g_{lx}(x_{nk})\right\| \leq c_1, \qquad \text{for } i=0,\dots,k-1. \end{aligned}$$

Hence, by (17) we have

$$\|z_n(t') - z_n(t'')\| \leq c_1 M'(t'-t'')$$

and

$$\|z_n(t) - z_{nk}\| \leq c_1 M'T,$$

which show that the $z_n(t)$ are equicontinuous and bounded. As in Lemma 4.2, it follows that

$$z_n(t) = \sum_{l=0}^{2} \lambda_{ln}\cdot g_{lx}(x_n(T)) + \int_t^T [z_n(s)\cdot f_x(s, x_n(s), r_n(s)) + \beta_n(s)]\, ds,$$

where $\beta_n \to 0$ uniformly, and we can pass to the limit in this equation, since, by Lemma 4.2, $x_n \to x$ uniformly. □

Theorem 4.2. Under Assumptions A1, A4, A5, let r_n be admissible and extremal for the DRP_n, for $n=1,2,\dots$. Then, the sequence $\{r_n\}$ has cluster points and every cluster point is admissible and extremal for the CRP.

Proof. Setting $\bar{t}_n(t) = t_{ni}$, $t \in I_{ni}$, $i = 0, \dots, k-1$, the discrete necessary conditions for optimality can be written as

$$\int_I \bar{z}_n(t) \cdot f(\bar{t}(t), \bar{x}_n(t), r'_n(t) - r_n(t))\, dt \geq 0, \qquad \forall r'_n \in R_n.$$

Let $\{r_n\}$, $\{\lambda_{ln}\}$ be subsequences converging to r, λ_{ln}, respectively (note that the λ_{ln} are bounded). Let any $r' \in R$ and, by Lemma 4.1, a sequence $\{r'_n \in R_n\}$ converging to r'. By Lemmas 4.2 and 4.3, we can pass to the limit in the above inequality,

$$\int_I z(t) \cdot f(t, x(t), r'(t) - r(t))\, dt \geq 0, \qquad \forall r' \in R,$$

which is in fact equivalent to the pointwise minimum principle (7), and in the transversality condition (15),

$$\lambda_2 \cdot g_2(x(T)) = 0.$$

Therefore, r is extremal for the CRP. It is easily seen that r is also admissible. □

5. Approximation by Classical Controls

In relaxed numerical methods for solving nonconvex optimal control problems, it seems computationally more efficient to use Gamkrelidze controls (cf. Ref. 3). Since one must discretize anyway these problems to implement these methods on a computer, it is natural to use discrete Gamkrelidze controls R_n^G in the DRP_n. Note that, by Caratheodory's theorem (Ref. 2, p. 139), for every $r_n \in R_n$, there exists a control $\bar{r}_n = \{\bar{r}_{ni}\}_{i=0}^{k-1} \in R_n^G$, where

$$\bar{r}_{ni} = \sum_{j=0}^{p} \alpha_{nij} \delta_{u_{nij}}, \tag{18}$$

which has the same effect on the discrete state equation (10) (and hence gives the same cost),

$$x_{n,i+1} = x_{ni} + h_{ni} f(t_{ni}, x_{ni}, r_{ni}) = x_{ni} + h_{ni} \sum_{j=0}^{p} \alpha_{nij} f(t_{ni}, x_{ni}, u_{nij}),$$

since $f(t_{ni}, x_{ni}, r_{ni}) \in \mathrm{Cof}(t_{ni}, x_{ni}, U)$.

Now, given $\bar{r}_n \in R_n^G$, as defined by (18), we construct an associated approximate discrete classical control as follows. Subdivide each I_{ni}, $i = 0, \dots, k-1$, into $p+1$ subintervals I_{nij} of length $\alpha_{nij} h_{ni}$, $j = 0, \dots, p$, and define $\bar{u}_n$ by

$$\bar{u}_n(t) = u_{nij}, \qquad \text{on } I_{nij},\ j = 0, \dots, p,\ i = 0, \dots, k-1.$$

Theorem 5.1. Let $r \in R$, and let $r_n \in R_n$ be a sequence converging to r in R. For each n, let $\bar{r}_n^m \in R_n^G$ be a sequence converging to r_n in R_n. Let $\bar{u}_n^m$ be the discrete classical control associated to $\bar{r}_n^m$. Then, there exists an integer function $M(n)$ such that

$$\lim_{\substack{n,m\to\infty \\ m \geq M(n)}} \bar{u}_n^m = r, \qquad \text{in } R.$$

Proof. Let $\varphi \in C^0(I)$, $\psi \in C^0(U)$, and $\epsilon > 0$ be given. Define $\bar{\varphi}_n$ by

$$\bar{\varphi}_n(t) = \varphi(t_{ni}), \qquad \text{on } I_{ni},\ i = 0, \ldots, k-1.$$

Clearly, $\bar{\varphi}_n \to \varphi$ uniformly on I. Now, write

$$e_n^m = \int_I \varphi[\psi(\bar{u}_n^m) - \psi(r)]\, dt = a_n^m + b_n^m + c_n^m + d_n,$$

where

$$|a_n^m| = \left| \int_I (\varphi - \bar{\varphi}_n)[\psi(\bar{u}_n^m) - \psi(r_n)]\, dt \right| \leq 2T\|\psi\|_\infty \cdot \|\varphi - \bar{\varphi}_n\|_\infty,$$

$$b_n^m = \int_I \bar{\varphi}_n[\psi(\bar{u}_n^m) - \psi(\bar{r}_n^m)]\, dt = 0,$$

by construction of $\bar{u}_n^m$,

$$|c_n^m| = \left| \int_I \bar{\varphi}_n[\psi(\bar{r}_n^m) - \psi(r_n)]\, dt \right| \leq h_n \|\varphi_n\|_\infty \sum_{i=0}^{k-1} |\psi(\bar{r}_n^m) - \psi(r_n)|,$$

and

$$d_n = \int_I \varphi[\psi(r_n) - \psi(r)]\, dt.$$

It follows that there exists N and $M(n)$, for each n, such that

$$|e_n^m| \leq \epsilon,$$

for $n \geq N$ and $m \geq M(n)$. □

In practice, r may be an optimal [resp., admissible and extremal] control for the CRP, r_n an optimal [resp., admissible and extremal] control for the DRP$_n$, and the sequences $\{r_n^m\}_{m=0}^\infty$ are computed by applying some relaxed optimization method (descent method, penalty method, etc.) on the DRP$_n$ using discrete Gamkrelidze controls. The discrete classical controls $\bar{u}_n^m$ thus approximate the relaxed control r for n, m sufficiently large.

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