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Sensitivity Analysis in Multiobjective Optimization[1,2]

T. TANINO[3]

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Abstract. Sensitivity analysis in multiobjective optimization is dealt with in this paper. Given a family of parametrized multiobjective optimization problems, the perturbation map is defined as the set-valued map which associates to each parameter value the set of minimal points of the perturbed feasible set in the objective space with respect to a fixed ordering convex cone. The behavior of the perturbation map is analyzed quantitatively by using the concept of contingent derivatives for set-valued maps. Particularly, it is shown that the sensitivity is closely related to the Lagrange multipliers in multiobjective programming.

Key Words. Sensitivity analysis, multiobjective optimization, perturbation maps, contingent derivatives.

1. Introduction

Stability and sensitivity analysis is not only theoretically interesting but also practically important in optimization theory. A number of useful results have been obtained in usual scalar optimization. See, for example, Fiacco (Ref. 1) and Rockafellar (Ref. 2). Here, by stability we mean the qualitative analysis, that is, the study of various continuity properties of the perturbation (or marginal) function (or map) of a family of parametrized optimization problems. On the other hand, by sensitivity we mean the quantitative analysis, that is, the study of derivatives of the perturbation function.

[1] This research was made while the author stayed at the International Institute for Applied Systems Analysis, Laxenburg, Austria.

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[3] Associate Professor, Department of Mechanical Engineering II, Tohoku University, Sendai, Japan.

For multiobjective optimization, the optimal value of a problem is not unique, and hence we must consider not a function but a set-valued perturbation map. The author and Sawaragi investigated some sufficient conditions for the semicontinuity of the perturbation map (Ref. 3). However, their results are qualitative and therefore provide no quantitative information. In this paper, the behavior of the perturbation map will be studied quantitatively via the concept of contingent derivative introduced by Aubin (Ref. 4). Though several other concepts of derivatives of set-valued maps were proposed [see Aubin and Ekeland (Ref. 5, p. 493)], the concept of contingent derivative is the most adequate for our purpose. Because it depends on the point in the graph of a set-valued map, when we discuss the sensitivity of the perturbation map, we fix some point in its graph.

The contents of this paper are as follows. In Section 2, we introduce the concept of contingent derivatives of set-valued maps along with some basic properties which are necessary in the later sections. Section 3 is devoted to the analysis of the contingent derivative of the perturbation map, which is defined from a feasible set map by taking the set of minimal points with respect to a given closed convex cone. In Section 4, we analyze the sensitivity in general multiobjective optimization problems specified by feasible decision sets and objective functions which depend on a parameter vector. In Section 5, we concentrate on multiobjective programming problems in which only the right-hand side of the inequality constraints is perturbed. It is shown that the sensitivity of the perturbation map is closely related with the Lagrange multipliers for the nominal problem.

2. Contingent Derivatives of Set-Valued Maps

In this section, we introduce the concept of contingent derivatives of set-valued maps. Throughout this section, V and Z are two Banach spaces and F is a set-valued map from V to Z.

Definition 2.1. (*Aubin and Ekeland, Ref. 5*). Let C be a nonempty subset of a Banach space V and $\hat{v} \in V$. The set $T_C(\hat{v}) \subset V$, defined by

$$T_C(\hat{v}) = \bigcap_{\epsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h\leqq\alpha} \left(\frac{1}{h}(C-\hat{v})+\epsilon B\right), \tag{1}$$

is called the contingent cone to C at $\hat{v}$, where B is the unit ball in V. In other words, $v \in T_C(\hat{v})$ if and only if there exist sequences $\{h_k\} \subset \mathring{R}_+$ and $\{v^k\} \subset V$ such that $h_k \to 0$, $v^k \to v$, and

$$\hat{v} + h_k v^k \in C, \qquad \text{for any } k,$$

where $\mathring{R}_+$ is the set of positive real numbers.

It is well known that $T_C(\hat{v})$ is a closed (but not always convex) cone.

The graph of a set-valued map F from V to Z is defined and denoted by

$$\text{graph } F = \{(v, z) | z \in F(v)\} \subset V \times Z. \tag{2}$$

The contingent derivative of F is defined by considering the contingent cone to graph F.

Definition 2.2. (*Aubin and Ekeland, Ref. 5*). Let $(\hat{v}, \hat{z})$ be a point in graph F. We denote by $DF(\hat{v}, \hat{z})$ the set-valued map from V to Z whose graph is the contingent cone $T_{\text{graph } F}(\hat{v}, \hat{z})$ to the graph of F at $(\hat{v}, \hat{z})$ and call it the contingent derivative of F at $(\hat{v}, \hat{z})$. In other words, $z \in DF(\hat{v}, \hat{z})(v)$ if and only if $(v, z) \in T_{\text{graph } F}(\hat{v}, \hat{z})$.

$DF(\hat{v}, \hat{z})$ is a positively homogeneous set-valued map with closed graph. Due to Definition 2.1, $z \in DF(\hat{v}, \hat{z})(v)$ if and only if there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{v^k\} \subset V$, and $\{z^k\} \subset Z$ such that $h_k \to 0$, $v^k \to v$, $z^k \to z$, and

$$\hat{z} + h_k z^k \in F(\hat{v} + h_k v^k), \qquad \text{for any } k.$$

Now, we consider a nonempty pointed[4] closed convex cone P in Z. This cone P introduces a partial order on Z. We use the following notations: For $z, z' \in Z$,

$$z \leqq_P z', \qquad \text{iff } z' - z \in P, \tag{3}$$

$$z \leq_P z', \qquad \text{iff } z' - z \in P \backslash \{0\}. \tag{4}$$

We consider the set-valued map $F + P$ from V to Z defined by

$$(F + P)(v) = F(v) + P, \qquad \text{for all } v \in V.$$

The graph of $F + P$ is often called the P-epigraph of F[Sawaragi *et al.* (Ref. 6, p. 23)]. The following result, which shows a relationship between the contingent derivatives of $F + P$ and F, is useful.

Proposition 2.1. Let $(\hat{v}, \hat{z})$ belongs to graph F. Then,

$$DF(\hat{v}, \hat{z})(v) + P \subset D(F + P)(\hat{v}, \hat{z})(v), \qquad \text{for any } v \in V. \tag{5}$$

Proof. Let $z \in DF(\hat{v}, \hat{z})(v)$ and $d \in P$. Then, there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{v^k\} \subset V$, and $\{z^k\} \subset Z$ such that $h_k \to 0$, $v^k \to v$, $z^k \to z$, and

$$\hat{z} + h_k z^k \in F(\hat{v} + h_k v^k), \qquad \text{for any } k.$$

[4] A cone P is said to be pointed if $P \cap (-P) = \{0\}$.

Let $\bar{z}^k = z^k + d$, for all k. Then, $\bar{z}^k \to z + d$ and

$$\hat{z} + h_k \bar{z}^k = \hat{z} + h_k z^k + h_k z^k + h_k d^k \in F(\hat{v} + h_k v^k) + P, \qquad \text{for all } k.$$

Hence,

$$z + d \in D(F+P)(\hat{v}, \hat{z})(v),$$

and the proof is complete. □

The converse inclusion relation of this proposition,

$$D(F+P)(\hat{v}, \hat{z})(v) \subset DF(\hat{v}, \hat{z})(v) + P,$$

does not generally hold. See Proposition 2.2 below and Examples 3.3 and 3.4.

Since we deal with multiobjective optimization, we must introduce the concepts of minimal points and properly minimal points with respect to the cone P.

Definition 2.3. Let S be a subset of Z.

(i) A point $\hat{z} \in S$ is said to be a P-minimal point of S if there exists no $z \in S$ such that $z \leq_P \hat{z}$. We denote the set of all P-minimal points of S by $\mathrm{Min}_P\, S$, i.e.,

$$\begin{aligned}\mathrm{Min}_P\, S &= \{\hat{z} \in S \mid \text{there exists no } z \in S \text{ such that } z \leq_P \hat{z}\} \\ &= \{\hat{z} \in S \mid (S - \hat{z}) \cap (-P) = \{0\}\}. \end{aligned} \tag{6}$$

(ii) A point $\hat{z} \in S$ is said to be a properly P-minimal point of S if

$$\left[\mathrm{cl} \bigcup_{\alpha > 0} \alpha(S - \hat{z})\right] \cap (-P) = \{0\}. \tag{7}$$

Of course, every properly P-minimal point of S is P-minimal, since

$$S - \hat{z} \subset \mathrm{cl} \bigcup_{\alpha > 0} \alpha(S - \hat{z}).$$

Now, we consider sufficient conditions for the converse inclusion of Proposition 2.1. We introduce the following property of set-valued maps.

Definition 2.4. (*Aubin and Ekeland, Ref. 5*). F is said to be upper locally Lipschitz at $\hat{v} \in V$ if there exist a neighborhood N of $\hat{v}$ and a positive constant M such that

$$F(v) \subset F(\hat{v}) + M\|v - \hat{v}\| B, \qquad \text{for any } v \in N. \tag{8}$$

Remark 2.1. If F is upper locally Lipschitz at $\hat{v}$, then it is upper semicontinuous at $\hat{v}$, i.e., for any $\epsilon > 0$, there exists a positive number δ such that

$$F(v) \subset F(\hat{v}) + \epsilon B, \qquad \text{for any } v \text{ such that } \|v - \hat{v}\| \leq \delta.$$

Definition 2.5. (*Holmes, Ref. 7*). A base for P is a nonempty convex subset Q of P with $0 \notin Q$ such that every $d \in P$, $d \notin 0$, has a unique representation of the form αb, where $b \in Q$ and $\alpha > 0$.

We can prove the converse inclusion of Proposition 2.1 under some assumptions. Examples 3.3 and 3.4 will illustrate the importance of those assumptions.

Proposition 2.2. If $\hat{z}$ is a properly P-minimal point of $F(\hat{v})$, F is upper locally Lipschitz at $\hat{v}$, and P has a compact base Q, then

$$D(F+P)(\hat{v}, \hat{z})(v) = DF(\hat{v}, \hat{z})(v) + P, \qquad \text{for any } v \in V. \tag{9}$$

Proof. In view of Proposition 2.1, it suffices to prove that

$$D(F+P)(\hat{v}, \hat{z})(v) \subset DF(\hat{v}, \hat{z})(v) + P, \qquad \text{for any } v \in V.$$

Let

$$z \in D(F+P)(\hat{v}, \hat{z})(v).$$

From the definition, there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{v^k\} \subset V$, $\{z^k\} \subset Z$, and $\{d^k\} \subset P$ such that $h_k \to 0$, $v^k \to v$, $z^k \to z$, and

$$\hat{z} + h_k z^k - d^k \in F(\hat{v} + h_k v^k), \qquad \text{for any } k,$$

i.e.,

$$\hat{z} + h_k(z^k - d^k/h_k) \in F(\hat{v} + h_k v^k), \qquad \text{for any } k.$$

Since $d^k \in P$, $d^k = \alpha_k b^k$, with $\alpha_k > 0$, and $b^k \in Q$ for each k. Since Q is compact, we may assume that $b^k \to b \in Q$. Suppose that $\{\alpha_k/h_k\}$ has a convergent subsequence. Then, we may assume that

$$\alpha_k/h_k \to \alpha \in R_+,$$

and therefore

$$d^k/h_k = \alpha_k b^k/h_k \to \alpha b \in P.$$

This implies that

$$z - \alpha b \in DF(\hat{v}, \hat{z})(v),$$

namely that

$$z \in DF(\hat{v}, \hat{z})(v) + P.$$

Hence, we have the conclusion of the proposition. Therefore, it completes the proof of the proposition to show that $\{\alpha_k/h_k\} \subset R$ has a convergent subsequence. If this were not the case, it is clear that $\alpha_k/h_k \to +\infty$. Since

F is assumed to be upper locally Lipschitz at $\hat{v}$, there exist a neighborhood N of $\hat{v}$ and a positive number M satisfying (8). Since $\hat{v}+h_k v^k \to \hat{v}$,

$$\hat{v}+h_k v^k \in N, \qquad \text{for all } k \text{ sufficiently large.}$$

Hence, there exists a sequence $\{\hat{z}^k\}$ in $F(\hat{v})$ such that

$$\|\hat{z}+h_k(z^k-d^k/h_k)-\hat{z}^k\| \leqq M\|\hat{v}+h_k v^k-\hat{v}\|,$$

i.e.,

$$\|(\hat{z}-\hat{z}^k)/h_k+z^k-d^k/h_k\| \leqq M\|v^k\|,$$

for all k sufficiently large. Since $v^k \to v$, the right-hand side of the above inequality converges to $M\|v\|$. Therefore, the sequence

$$\{(\hat{z}-\hat{z}^k)/h_k+z^k-d^k/h_k\}$$

is bounded. Since $\alpha_k/h_k \to +\infty$, the sequence

$$\begin{aligned}&\{(h_k/\alpha_k)((\hat{z}-\hat{z}^k)/h_k+z^k-d^k/h_k)\}\\&=\{-(\hat{z}^k-\hat{z})/\alpha_k+(h_k/\alpha_k)z^k-b^k\}\end{aligned}$$

converges to the zero vector in Z. Since $z^k \to z$, the second term of the right-hand side converges to the zero vector. Hence, $(\hat{z}^k-\hat{z})/\alpha_k \to -b$. This implies that

$$-b \in \left[\operatorname{cl} \bigcup_{\alpha>0} \alpha(F(\hat{v})-\hat{z})\right] \cap (-P),$$

which contradicts the assumption of the proper P-minimality of $\hat{z}$. This completes the proof of the proposition. □

Corollary 2.1. If $\hat{z}$ is a properly P-minimal point of $F(\hat{v})$, F is upper locally Lipschitz at $\hat{v}$, and Z is finite dimensional, then

$$D(F+P)(\hat{v},\hat{z})(v) = DF(\hat{v},\hat{z})(v)+P, \qquad \text{for any } v \in V.$$

Proof. It is clear that the set $\{d \in P \mid \|d\|=1\}$ is a compact base for P when Z is finite dimensional. □

Corollary 2.2. If $\hat{z}$ is a properly P-minimal point of $F(\hat{v})$, F is upper locally Lipschitz at $\hat{v}$, and Z is finite dimensional, then

$$\operatorname{Min}_P DF(\hat{v},\hat{z})(v) = \operatorname{Min}_P D(F+P)(\hat{v},\hat{z})(v), \qquad \text{for any } v \in V. \tag{10}$$

Proof. In view of Corollary 2.1, by using Proposition 3.1.2 of Sawaragi *et al.* (Ref. 6), we can prove that, for any v,

$$\begin{aligned}\mathrm{Min}_P\, DF(\hat{v}, \hat{z})(v) &= \mathrm{Min}_P(DF(\hat{v}, \hat{z})(v) + P)\\ &= \mathrm{Min}_P\, D(F+P)(\hat{v}, \hat{z})(v).\end{aligned}$$ □

The following theorem is also fundamental.

Theorem 2.1. Let $(\hat{v}, \hat{z})$ belong to graph F, and suppose that P has a compact base Q. Then, for any $v \in V$,

$$\mathrm{Min}_P\, D(F+P)(\hat{v}, \hat{z})(v) \subset DF(\hat{v}, \hat{z})(v). \tag{11}$$

Proof. Let

$$z \in \mathrm{Min}_P\, D(F+P)(\hat{v}, \hat{z})(v).$$

Since

$$z \in D(F+P)(\hat{v}, \hat{z})(v),$$

there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{v^k\} \subset V$, $\{z^k\} \subset Z$, and $\{d^k\} \subset P$ such that $h^k \to 0$, $v^k \to v$, $z^k \to z$, and

$$\hat{z} + h_k z^k - d^k \in F(\hat{v} + h_k v^k), \qquad \text{for any } k.$$

We shall prove that $d^k/h_k \to 0$. Since $d^k \in P$, there exist some $\alpha_k > 0$ and $b^k \in Q$ such that $d^k = \alpha_k b^k$, for each k. Since Q is compact, we may assume without loss of generality that $b^k \to b \in Q$. Then, $d^k/h_k = \alpha_k b^k/h_k$ and $d^k/h_k \to 0$ when and only when $\alpha_k/h_k \to 0$. Suppose that $\{\alpha_k/h_k\}$ does not converge to 0. Then, for some $\epsilon > 0$, we may assume without loss of generality that $\alpha_k/h_k \geqq \epsilon$, for all k, by taking a subsequence if necessary. Let

$$\bar{d}^k = (\epsilon h_k/\alpha_k) d^k \in P.$$

Then,

$$\bar{d}^k \leqq_P d^k$$

and

$$\hat{z} + h_k z^k - \bar{d}^k \in F(\hat{v} + h_k v^k) + P.$$

Since $\bar{d}^k/h_k = \epsilon b^k$ for all k,

$$\bar{d}^k/h_k \to \epsilon b \neq 0.$$

Thus,

$$z^k - \bar{d}^k/h_k \to z - \epsilon b,$$

and hence

$$z-\epsilon b \in D(F+P)(\hat{v}, \hat{z})(v).$$

However, this contradicts the assumption

$$z \in \mathrm{Min}_P D(F+P)(\hat{v}, \hat{z})(v),$$

since $z-\epsilon b \leq_P z$. Therefore, we can conclude that $d^k/h_k \to 0$. This implies that

$$\hat{z}+h_k(z^k-d^k/h_k) \in F(\hat{v}+h_k v^k), \qquad \text{for any } k,$$

and

$$z^k-d^k/h_k \to z.$$

Therefore, $z \in DF(\hat{v}, \hat{z})(v)$, and this completes the proof of the theorem. □

Corollary 2.3. Let $(\hat{v}, \hat{z})$ belong to graph F, and suppose that Z is finite dimensional. Then, for any $v \in V$,

$$\mathrm{Min}_P D(F+P)(\hat{v}, \hat{z})(v) \subset DF(\hat{v}, \hat{z})(v).$$

3. Contingent Derivative of the Perturbation Map

In this section, we consider a family of parametrized multiobjective optimization problems. Let Y be a set-valued map from U to R^p, where U is the Banach space of a perturbation parameter vector, R^p is the objective space, and Y is considered as the feasible set map in the objective space. Let P be a nonempty pointed closed convex cone in R^p. In the optimization problem corresponding to each parameter value u, we aim to find the set of P-minimal points of the feasible objective value set $Y(u)$. Hence, we define another set-valued map W from U to R^p by

$$W(u)=\mathrm{Min}_P Y(u), \qquad \text{for every } u \in U, \tag{12}$$

and call it the perturbation map (or P-minimal map), since it is a generalization of the perturbation function (optimal value function) in scalar optimization. The purpose of this section is to investigate relationships between the contingent derivative of W and that of Y. Hereafter in this paper, we fix a nominal value of u as $\hat{u}$ and consider a point $\hat{y} \in W(\hat{u})$.

In view of Theorem 2.1, we have the following relationship:

$$\mathrm{Min}_P D(W+P)(\hat{u}, \hat{y})(u) \subset DW(\hat{u}, \hat{y})(u), \qquad \text{for any } u \in U. \tag{13}$$

Definition 3.1. We say that Y is P-minicomplete near $\hat{u}$ if

$$Y(u) \subset W(u) + P, \qquad \text{for any } u \in N, \tag{14}$$

where N is some neighborhood of $\hat{u}$.

Since $W(u) \subset Y(u)$, the P-minicompleteness of Y near $\hat{u}$ implies that

$$W(u) + P = Y(u) + P, \qquad \text{for any } u \in N. \tag{15}$$

Hence, if Y is P-minicomplete near $\hat{u}$, then

$$D(Y+P)(\hat{u}, \hat{y}) = D(W+P)(\hat{u}, \hat{y}), \qquad \text{for all } \hat{y} \in W(\hat{u}).$$

Thus, we obtain the following theorem from (13).

Theorem 3.1. If Y is P-minicomplete near $\hat{u}$, then

$$\text{Min}_P D(Y+P)(\hat{u}, \hat{y})(u) \subset DW(\hat{u}, \hat{y})(u), \qquad \text{for any } u \in U. \tag{16}$$

Some sufficient conditions for the P-minicompleteness can be seen in Sawaragi *et al.* (Ref. 6, Chapter 3). The following example illustrates that the P-minicompleteness is essential for the above theorem.

Example 3.1. (Y is not P-minicomplete near $\hat{u}$.) Let $U = R$, $p = 1$, $P = R_+$, and Y is defined by

$$Y(u) = \begin{cases} \{y \in R \mid y \geqq 0\}, & \text{if } u = 0, \\ \{y \in R \mid y > |u|\} & \text{if } u \neq 0. \end{cases}$$

Then,

$$W(u) = \begin{cases} \{0\}, & \text{if } u = 0, \\ \varnothing, & \text{if } u \neq 0. \end{cases}$$

Let $\hat{u} = 0$. Then, $\hat{y} = 0$ and

$$D(Y+P)(\hat{u}, \hat{y})(u) = DY(\hat{u}, \hat{y})(u) = \{y \mid y \geqq |u|\}, \qquad \text{for any } u \in R,$$

$$\text{Min}_P D(Y+P)(\hat{u}, \hat{y})(u) = \text{Min}_P DY(\hat{u}, \hat{y})(u) = \{|u|\}.$$

On the other hand,

$$DW(\hat{u}, \hat{y})(u) = \begin{cases} \{0\}, & \text{if } u = 0, \\ \varnothing, & \text{if } u \neq 0. \end{cases}$$

Hence,

$$\text{Min}_P D(Y+P)(\hat{u}, \hat{y})(u) \not\subset DW(\hat{u}, \hat{y})(u), \qquad \text{for } u \neq 0.$$

The converse inclusion of the theorem does not generally hold as is shown in the following example.

Example 3.2. Let $U = R$, $p = 2$, and Y be defined by

$$Y(u) = \begin{cases} \{(0,0)\}, & \text{if } u \leqq 0, \\ \{y \in R^2 | y_2 = -(y_1)^2, 0 \leqq y_1 \leqq u\}, & \text{if } u > 0. \end{cases}$$

Let

$$P = R^2_+, \qquad \hat{u} = 0, \qquad \hat{y} = (0, 0).$$

Then, $W(u) = Y(u)$, for every u, and

$$T_{\text{graph } Y}(\hat{u}, \hat{y}) = T_{\text{graph } W}(\hat{u}, \hat{y})$$
$$= \{(u, y) | u \leqq 0, y = 0\}$$
$$\cup \{(u, y) | u > 0, y_2 = 0, 0 \leqq y_1 \leqq u\},$$
$$T_{\text{graph } (Y+P)}(\hat{u}, \hat{y}) = T_{\text{graph } (W+P)}(\hat{u}, \hat{y}) = \{(u, y) | y \geqq 0\},$$
$$DW(\hat{u}, \hat{y})(u) = DY(\hat{u}, \hat{y})(u) = \begin{cases} \{(0,0)\}, & \text{if } u \leqq 0, \\ \{y | y_2 = 0, 0 \leqq y_1 \leqq u\}, & \text{if } u > 0, \end{cases}$$
$$D(Y+P)(\hat{u}, \hat{y})(u) + D(W+P)(\hat{u}, \hat{y})(u) = \{y | y \geqq 0\},$$
$$\text{Min}_P D(Y+P)(\hat{u}, \hat{y})(u) = \text{Min}_P D(W+P)(\hat{u}, \hat{y}) = \{(0, 0)\}, \quad \text{for any } u.$$

By combining Theorem 3.1 and Corollary 2.2, we have the following theorem.

Theorem 3.2. If Y is P-minicomplete near $\hat{u}$ and upper locally Lipschitz at $\hat{u}$, and if $\hat{y}$ is a properly P-minimal point of $Y(\hat{u})$, then

$$\text{Min}_P DY(\hat{u}, \hat{y})(u) \subset DW(\hat{u}, \hat{y})(u), \qquad \text{for any } u \in U.$$

Example 3.1 shows that the P-minicompleteness of Y is essential for the above theorem. The following two examples illustrate the importance of the other two conditions in Theorem 3.2, namely the Lipschitz property of Y and the proper P-minimality of $\hat{y}$. These examples illustrate Proposition 2.2 and Corollary 2.2, too.

Example 3.3. (Y is not upper locally Lipschitz at $\hat{u}$.) Let $U = R$, $p = 1$, $P = R_+$, and Y be defined by

$$Y(u) = \begin{cases} \{0\}, & \text{if } u \leqq 0, \\ \{0, -\sqrt{u}\}, & \text{if } u > 0. \end{cases}$$

Then,

$$W(u) = \begin{cases} \{0\}, & \text{if } u \leqq 0, \\ \{-\sqrt{u}\}, & \text{if } u > 0. \end{cases}$$

Let $\hat{u}=0$ and $\hat{y}=0$. Then,

$$DY(0,0)(u)=\begin{cases}\{0\}, & \text{if } u\neq 0,\\ \{y\,|\,y\leqq 0\}, & \text{if } u=0,\end{cases}$$

$$\mathrm{Min}_P\,DY(0,0)(u)=\begin{cases}\{0\}, & \text{if } u\neq 0,\\ \varnothing, & \text{if } u=0,\end{cases}$$

$$D(Y+P)(0,0)(u)=\begin{cases}\{y\,|\,y\geqq 0\}, & \text{if } u<0,\\ R, & \text{if } u\geqq 0,\end{cases}$$

$$\mathrm{Min}_P\,D(Y+P)(0,0)(u)=\begin{cases}\{0\}, & \text{if } u<0,\\ \varnothing, & \text{if } u\geqq 0,\end{cases}$$

$$DW(0,0)(u)=\begin{cases}\{0\}, & \text{if } u<0,\\ \{y\,|\,y\geqq 0\}, & \text{if } u=0,\\ \varnothing, & \text{if } u>0.\end{cases}$$

Hence,

$$\{0\}=\mathrm{Min}_P\,DY(0,0)(u)\not\subset DW(0,0)(u)=\varnothing, \qquad \text{for } u>0.$$

Example 3.4. ($\hat{y}$ is not properly P-minimal.) Let $U=R$, $p=2$, $P=R^2_+$, and Y be defined by

$$Y(u)=\{y\,|\,y_1+y_2=0,\ y_1\leqq u\}\cup\{y\,|\,y_1+y_2+1=0,\ y_1>0\}.$$

Then,

$$W(u)=\{y\,|\,y_1+y_2=0,\ y_1\leqq \min(0,u)\}\cup\{y\,|\,y_1+y_2+1=0,\ y_1>0\}.$$

Let $\hat{u}=0$ and $\hat{y}=(0,0)$. Then,

$$DY(\hat{u},\hat{y})(u)=\mathrm{Min}_P\,DY(\hat{u},\hat{y})(u)=\{y\,|\,y_1+y_2=0,\ y_1\leqq u\},$$

$$D(Y+P)(\hat{u},\hat{y})(u)=\{y\,|\,y_1+y_2\geqq 0,\ y_2\geqq -u\}\cup\{y\,|\,y_1\geqq 0\},$$

$$\mathrm{Min}_P\,D(Y+P)(\hat{u},\hat{y})(u)=\begin{cases}\{y\,|\,y_1+y_2=0,\ y_1\leqq u\}, & \text{if } u<0,\\ \{y\,|\,y_1+y_2=0,\ y_1<0\}, & \text{if } u\geqq 0,\end{cases}$$

$$DW(\hat{u},\hat{y})(u)=\{y\,|\,y_1+y_2=0,\ y_1\leqq \min(0,u)\}.$$

Hence,

$$(1,-1)\notin DW(\hat{u},\hat{y})(1), \qquad \text{while } (1,-1)\in \mathrm{Min}_P\,DY(\hat{u},\hat{y})(1).$$

4. Sensitivity Analysis in General Multiobjective Optimization

In this section, we deal with a general multiobjective optimization problem in which the feasible set in the objective space is given by the composition of the feasible decision set $X(u)$ and the objective function $f(x, u)$. Namely, let X be a set-valued map from R^m to R^n, f be a continuously differentiable function from $R^n \times R^m$ into R^p, and Y defined by

$$Y(u) = f(X(u), u) = \{y \mid y = f(x, u),\ x \in X(u)\}, \qquad \text{for each } u \in R^m. \tag{17}$$

First, we investigate a relationship between the contingent derivatives of X and Y. Let

$$\hat{u} \in R^m,\ \hat{x} \in X(\hat{u}), \qquad \hat{y} = f(\hat{x}, \hat{u}) \in Y(\hat{u}).$$

Proposition 4.1. For any $u \in R^m$,

$$\nabla_x f(\hat{x}, \hat{u}) DX(\hat{u}, \hat{x})(u) + \nabla_u f(\hat{x}, \hat{u}) u \subset DY(\hat{u}, \hat{y})(u), \tag{18}$$

where $\nabla_x f(\hat{x}, \hat{u})$ [or $\nabla_u f(\hat{x}, \hat{u})$] is the $p \times n$ [or $p \times m$] matrix whose (i, j) component is $\partial f_i(\hat{x}, \hat{u})/\partial x_j$ [or $\partial f_i(\hat{x}, \hat{u})/\partial u_j$]. Moreover, let

$$\tilde{X}(u, y) = \{x \in R^n \mid x \in X(u), f(x, u) = y\}. \tag{19}$$

If $\tilde{X}$ is upper locally Lipschitz at $(\hat{u}, \hat{y})$ and $\tilde{X}(\hat{u}, \hat{y}) = \{\hat{x}\}$, then the converse inclusion of (18) is also valid, i.e.,

$$\nabla_x f(\hat{x}, \hat{u}) DX(\hat{u}, \hat{x})(u) + \nabla_u f(\hat{x}, \hat{u}) u = DY(\hat{u}, \hat{y})(u), \qquad \text{for any } u \in R^m. \tag{20}$$

Proof. First, we prove (18). Let $x \in DX(\hat{u}, \hat{x})(u)$. Then, there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{u^k\} \subset R^m$, and $\{x^k\} \subset R^n$ such that $h_k \to 0$, $u^k \to u$, $x^k \to x$, and

$$\hat{x} + h_k x^k \in X(\hat{u} + h_k u^k), \qquad \text{for any } k.$$

Then,

$$f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) \in Y(\hat{u} + h_k u^k), \qquad \text{for any } k,$$

i.e.,

$$\hat{y} + h_k \frac{f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) - f(\hat{x}, \hat{u})}{h_k} \in Y(\hat{u} + h_k u^k), \qquad \text{for any } k.$$

Since $h_k \to 0$, $u^k \to u$, and $x^k \to x$,

$$\lim_{k \to \infty} \frac{f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) - f(\hat{x}, \hat{u})}{h_k} = \nabla_x f(\hat{x}, \hat{u}) x + \nabla_u f(\hat{x}, \hat{u}) u.$$

Hence,

$$\nabla_x f(\hat{x}, \hat{u})x + \nabla_u f(\hat{x}, \hat{u})u \in DY(\hat{u}, \hat{y})(u).$$

Thus, (18) has been established. Next, we prove (20). Let $y \in DY(\hat{u}, \hat{y})(u)$ along with sequences $\{h_k\} \subset \mathring{R}_+$, $\{u^k\} \subset R^m$, and $\{y^k\} \subset R^p$ such that $h_k \to 0$, $u^k \to u$, $y^k \to y$, and $\hat{y} + h_k y^k \in Y(\hat{u} + h_k u^k)$. Then, there exists another sequence $\{x^k\} \subset R^n$ such that

$$\hat{x} + h_k x^k \in \tilde{X}(\hat{u} + h_k u^k, \hat{y} + h_k y^k), \qquad \text{for any } k.$$

Since $\tilde{X}$ is upper locally Lipschitz at $(\hat{u}, \hat{y})$ and $\tilde{X}(\hat{u}, \hat{y}) = \{\hat{x}\}$, there exists a positive number M such that

$$\|\hat{x} + h_k x^k - \hat{x}\| \leqq M \|(\hat{u} + h_k u^k, \hat{y} + h_k y^k) - (\hat{u}, \hat{y})\|,$$

i.e.,

$$\|x^k\| \leqq M \|(u^k, y^k)\|,$$

for all k sufficiently large. Since the right-hand side of the above inequality converges to $M\|(u, y)\|$ as $k \to \infty$, we may assume without loss of generality that x^k converges to some x. Then, clearly $x \in DX(\hat{u}, \hat{x})(u)$. Moreover,

$$\begin{aligned} y &= \lim_{k \to \infty} y^k = \lim_{k \to \infty} \frac{f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) - f(\hat{x}, \hat{u})}{h_k} \\ &= \nabla_x f(\hat{x}, \hat{u})x + \nabla_u f(\hat{x}, \hat{u})u. \end{aligned}$$

Therefore,

$$y \in \nabla_x f(\hat{x}, \hat{u}) DX(\hat{u}, \hat{x})(u) + \nabla_u f(\hat{x}, \hat{u})u,$$

and the proof of the proposition is completed. □

The following two examples show that the additional conditions in Proposition 4.1 are essential for (20).

Example 4.1. $(\tilde{X}(\hat{u}, \hat{y}) \neq \{\hat{x}\})$. Let

$$X(u) = \{x \in R | 0 \leqq x \leqq \max(1, 1 + u)\}, \qquad \text{for } u \in R,$$

$$f(x, u) = x(x - 1), \qquad \hat{u} = 0, \qquad \hat{x} = 0, \qquad \hat{y} = f(\hat{x}, \hat{u}) = 0.$$

Then,

$$\tilde{X}(\hat{u}, \hat{y}) = \{0, 1\},$$

and

$$Y(u) = \begin{cases} \{y | -1/4 \leqq y \leqq 0\}, & \text{if } u \leqq 0, \\ \{y | -1/4 \leqq y \leqq u(1 + u)\}, & \text{if } u > 0. \end{cases}$$

Hence, by taking

$$h_k = 1/k, \qquad u^k = 1, \qquad y^k = 1,$$

we can prove that

$$1 \in DY(\hat{u}, \hat{y})(1).$$

On the other hand,

$$DX(\hat{u}, \hat{x})(1) = R_+, \qquad \nabla_x f(\hat{x}, \hat{y}) = -1, \qquad \nabla_u f(\hat{x}, \hat{u}) = 0.$$

Therefore,

$$1 \notin \nabla_x f(\hat{x}, \hat{u}) DX(\hat{u}, \hat{x})(1) + \nabla_u f(\hat{x}, \hat{u}) 1,$$

and (20) does not hold.

Example 4.2. ($\tilde{X}$ is not upper locally Lipschitz at $(\hat{u}, \hat{y})$.) Replace $X(u)$ by

$$X(u) = \{x \in R \mid 0 \leqq x < \max(1, 1+u)\}$$

in Example 4.1. In this case, $\tilde{X}(\hat{u}, \hat{y}) = \{0\}$, but $\tilde{X}$ is not upper locally Lipschitz at $(\hat{u}, \hat{y})$. We can analogously prove that

$$1 \in DY(\hat{u}, \hat{y})(1),$$

but

$$1 \notin \nabla_x f(\hat{x}, \hat{u}) DX(\hat{u}, \hat{x})(1) + \nabla_u f(\hat{x}, \hat{u}) 1.$$

Example 4.3. ($\tilde{X}$ is not upper locally Lipschitz at $(\hat{u}, \hat{y})$.) Let

$$X(u) = \{x \in R \mid 0 \leqq x \leqq 1\}, \qquad \text{for every } u \in R,$$

$$f(x, u) = x^2, \qquad \hat{u} = 0, \qquad \hat{x} = 0, \qquad \hat{y} = 0.$$

Then,

$$Y(u) = \{y \in R \mid 0 \leqq y \leqq 1\}$$

and

$$DY(0, 0)(u) = R_+, \qquad \text{for any } u \in R.$$

However,

$$\nabla_x f(0, 0) DX(0, 0)(u) + \nabla_u f(0, 0) u = \{0\}.$$

In this case,

$$\tilde{X}(u, y) = \sqrt{y}, \qquad \text{for } 0 \leqq y \leqq 1 \text{ and any } u,$$

which is not upper locally Lipschitz at $(0, 0)$.

Finally, we should note sufficient conditions for the upper local Lipschitz continuity of Y at $\hat{u}$.

Lemma 4.1. If X is upper locally Lipschitz at $\hat{u}$ and if $X(\hat{u})$ is bounded, then Y is upper locally Lipschitz at $\hat{u}$.

Proof. Since X is upper locally Lipschitz at $\hat{u}$, there exist some ϵ-neighborhood N of $\hat{u}$ and a positive number M_1 such that

$$X(u) \subset X(\hat{u}) + M_1\|u - \hat{u}\|B, \qquad \text{for any } u \in N.$$

Since f is continuously differentiable and $\bigcup_{u \in N} X(u)$ is bounded from the boundedness of $X(\hat{u})$ and the above relation, there exists a positive number M_2 such that

$$\|f(x, u) - f(\tilde{x}, \hat{u})\| \leqq M_2\|(x, u) - (\tilde{x}, \tilde{u})\|,$$

for any $u \in N$, $x \in X(u)$, and $\tilde{x} \in X(\hat{u})$. For any $u \in N$ and $y \in Y(u)$, there exists $x \in X(u)$ such that $f(x, u) = y$. Then, there exists $\bar{x} \in X(\hat{u})$ such that

$$\|x - \bar{x}\| \leqq M_1\|u - \hat{u}\|.$$

Hence,

$$\begin{aligned}\|f(x, u) - f(\bar{x}, \hat{u})\| &\leqq M_2\|(x, u) - (\bar{x}, \hat{u})\| \\ &\leqq M_2(\|x - \bar{x}\| + \|u - \hat{u}\|) \\ &\leqq M_2(1 + M_1)\|u - \hat{u}\|.\end{aligned}$$

Putting

$$M = (1 + M_1)M_2,$$

we have

$$y \in Y(\hat{u}) + M\|u - \hat{u}\|B.$$

This completes the proof of the lemma. □

As can be seen from the proof, f need not be continuously differentiable, but locally Lipschitz continuous in the above lemma. The following example shows that the boundedness of $X(\hat{u})$ is essential.

Example 4.4. ($X(\hat{u})$ is not bounded.) Let

$$X(u) = \{x \in R^2 | x_1 = u\}, \qquad \text{for } u \in R,$$

$$f(x, u) = x_1 x_2.$$

Then,

$$Y(u) = \begin{cases} \{0\}, & \text{if } u = 0, \\ R, & \text{if } u \neq 0. \end{cases}$$

Clearly, Y is not upper locally Lipschitz at $\hat{u} = 0$.

Finally, we have the following theorem. Note that Y is P-minicomplete near $\hat{u}$ if $X(u)$ is compact for each u near $\hat{u}$.

Theorem 4.1. Assume the following conditions:

(i) X is upper locally Lipschitz at $\hat{u}$;
(ii) $X(u)$ is compact for each u near $\hat{u}$;
(iii) $\hat{y}$ is a properly P-minimal point of $Y(\hat{u})$;
(iv) $\tilde{X}(\hat{u}, \hat{y}) = \{\hat{x}\}$;
(v) $\tilde{X}$ is upper locally Lipschitz at $(\hat{u}, \hat{y})$.

Then, for any $u \in R^m$,

$$\text{Min}_P\{\nabla_x f(\hat{x}, \hat{u})x + \nabla_u f(\hat{x}, \hat{u})u | x \in DX(\hat{u}, \hat{x})(u)\} \subset DW(\hat{u}, \hat{y})(u). \tag{21}$$

5. Sensitivity Analysis in Multiobjective Programming

In this section we apply the results obtained in the preceding section to a usual multiobjective programming problem:

$$P\text{-minimize} \quad f(x) = (f_1(x), f_2(x), \ldots, f_p(x)),$$
$$\text{subject to} \quad g(x) = (g_1(x), g_2(x), \ldots, g_m(x)) \leqq 0, \quad x \in R^n, \tag{22}$$

and discuss the sensitivity in connection with the Lagrange multipliers. Recall that, in usual nonlinear programming, the sensitivity of the perturbation function with respect to the parameter on the right-hand side of each inequality constraint is given by $-\lambda_j$, $j = 1, \ldots, m$, where λ_j is the corresponding Lagrange multiplier. Our final result will be an extension of this fact. Throughout this section, each function f_i, $i = 1, \ldots, p$, and g_j, $j = 1, \ldots, m$, is assumed to be continuously differentiable.

Let X be the set-valued map from R^m to R^n defined by

$$X(u) = \{x \in R^n | g(x) \leqq u\}, \qquad \text{for } u \in R^m. \tag{23}$$

Hence, in this case, the feasible set map Y from R^m to R^p, the objective space, is defined by

$$Y(u) = f(X(u)) = \{y \in R^p | y = f(x), x \in X(u)\}$$
$$= \{y \in R^p | y = f(x), g(x) \leqq u\}. \tag{24}$$

Of course, the nominal value of the parameter vector u is 0 in R^m. Take a point $\hat{x} \in X(0)$, and denote the index set of the active constraints at $\hat{x}$ by $J(\hat{x})$, i.e.,

$$J(\hat{x}) = \{j | g_j(\hat{x}) = 0\}. \tag{25}$$

First, we consider the contingent derivative of the set-valued map X.

Lemma 5.1. The contingent derivative of X at $(0, \hat{x})$ is given as follows:

$$DX(0, \hat{x})(u) = \{x \mid \langle \nabla g_j(\hat{x}), x \rangle \leqq u_j, \text{ for all } j \in J(\hat{x})\}, \tag{26}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the Euclidean space.

Proof. Note that

$$\text{graph } X = \{(u, x) \mid g_j(x) - u_j \leqq 0, \ j = 1, \ldots, m\}$$

is specified by m inequality constraints. The gradient vector of the jth constraint at $(0, \hat{x})$ with respect to (u, x) is $(-e^j, \nabla g_j(\hat{x}))$, where e^j is the jth unit vector in R^m, i.e.,

$$e^j_k = 0, \qquad \text{if } k \neq j,$$

$$e^j_j = 1.$$

Hence, these gradient vectors are linearly independent, and so the tangent cone to graph X is given by

$$\begin{aligned} T_{\text{graph } X}(0, \hat{x}) &= \{(u, x) \mid \langle (-e^j, \nabla g_j(\hat{x})), (u, x) \rangle \leqq 0, \text{ for } j \in J(\hat{x})\} \\ &= \{(u, x) \mid \langle \nabla g_j(\hat{x}), x \rangle \leqq u_j, \text{ for } j \in J(\hat{x})\}. \end{aligned}$$

This completes the proof of the lemma. □

In this case, $X(u)$ is a closed set for every u, since g is continuous. The next lemma provides sufficient conditions for the Lipschitz continuity of X around $\hat{u} = 0$. Here, X is said to be Lipschitz around $\hat{u}$ if there exist a neighborhood N of $\hat{u}$ and a positive number M such that

$$\|x - x'\| \leqq M \|u - u'\|, \quad \text{for any } u, u' \in N, \text{and } x \in X(u), x' \in X(u').$$

Of course, if X is Lipschitz around $\hat{u}$, then it is upper locally Lipschitz at $\hat{u}$.

Lemma 5.2. Assume that there exists a vector $\bar{u} > 0$ such that $X(\bar{u})$ is bounded, $X(0) \neq \varnothing$, and that the Cottle constraint qualification is satisfied at every $\bar{x} \in X(0)$, i.e.,

$$\sum_{j \in J(\bar{x})} \lambda_j \nabla g_j(\bar{x}) = 0 \text{ and } \lambda_j \geqq 0, \text{ for } j \in J(\bar{x}),$$

$$\text{imply that } \lambda_j = 0, \text{ for all } j \in J(\bar{x}). \tag{27}$$

Then, X is compact-valued and Lipschitz around $\hat{u} = 0$.

Proof. This lemma is due to Rockafellar (Ref. 8). Combine Theorem 2.1 and Corollary 3.3 in Ref. 8. □

Analogously, we have the following lemma concerning the set-valued map

$$\tilde{X}(u, y) = \{x \mid f(x) = y, \ g(x) \leqq u\}. \tag{28}$$

Lemma 5.3. Assume that $\tilde{X}$ is locally bounded around $(0, \hat{y})$, $\tilde{X}(0, \hat{y}) \neq \emptyset$ and that the Mangasarian–Fromovitz constraint qualification is satisfied at every $\hat{x} \in \tilde{X}(0, \hat{y})$, i.e.,

$$\sum_{i=1}^{p} \mu_i \nabla f_i(\hat{x}) + \sum_{j \in J(\hat{x})} \lambda_j \nabla g_j(\hat{x}) = 0 \text{ and } \lambda_j \geqq 0, \text{ for } j \in J(\hat{x}),$$

$$\text{imply that } \mu_i = 0, \text{ for all } i = 1, \ldots, p, \text{ and } \lambda_j = 0, \text{ for all } j \in J(\hat{x}). \tag{29}$$

Then, $\tilde{X}$ is compact-valued and Lipschitz around $(0, \hat{y})$.

We will proceed with the discussion under the following assumptions.

Assumption 5.1

(i) There exists $\bar{u} > 0$ such that $X(\bar{u})$ is bounded.

(ii) The Cottle constraint qualification (27) is satisfied at each $\hat{x} \in X(0)$.

(iii) $\hat{y} = f(\hat{x})$ is a properly P-minimal point[5] of $Y(0)$, where $\hat{x} \in X(0)$.

(iv) $\tilde{X}(0, \hat{y}) = \{\hat{x}\}$.

(v) The Mangasarian–Fromovitz constraint qualification (29) is satisfied at $\hat{x}$.

Then, we can apply Theorem 4.1 to obtain the relationship

$$\text{Min}_P \nabla f(\hat{x}) DX(0, \hat{x})(u) \subset DW(0, \hat{y})(u), \qquad \text{for any } u \in R^m. \tag{30}$$

In view of (26) of Lemma 5.1,

$$\nabla f(\hat{x}) DX(0, \hat{x})(u) = \{ y \mid y_i = \langle \nabla f_i(\hat{x}), x \rangle, \text{ for } i = 1, \ldots, p;$$
$$\langle \nabla g_j(\hat{x}), x \rangle \leqq u_j, \text{ for } j \in J(\hat{x}) \}.$$

Hence, the left-hand side of (30) consists of all the P-minimal values of the linear multiobjective programming problem:

$$P\text{-minimize} \quad \langle \nabla f_i(\hat{x}), x \rangle, \qquad i = 1, \ldots, P,$$
$$\text{subject to} \quad \langle \nabla g_j(\hat{x}), x \rangle \leqq u_j, \qquad j \in J(\hat{x}).$$

In view of Theorem 3.4.7 in Sawaragi *et al.* (Ref. 6) and the ordinary Kuhn–Tucker theorem, the necessary and sufficient P-minimality condition

[5] In this case, we call $\hat{x}$ a properly P-minimal solution to the problem (22).

for the above linear problem are that there exists a multiplier vector $(\mu,\lambda)\in R^p\times R^m$ such that

$$\sum_{i=1}^{p}\mu_i\nabla f_i(\hat{x})+\sum_{j\in J(\hat{x})}\lambda_j\nabla g_j(\hat{x})=0, \tag{31}$$

$$\mu\in\operatorname{int}P^+=\{\nu\in R^p|\langle\nu, d\rangle>0,\text{ for all }d\in P,\ d\neq 0\}, \tag{32}$$

$$\lambda_j\geqq 0,\qquad \text{for } j\in J(\hat{x}), \tag{33}$$

$$\lambda_j(\langle\nabla g_j(\hat{x}), x\rangle-u_j)=0,\qquad \text{for } j\in J(\hat{x}). \tag{34}$$

Since $\hat{x}$ is a properly P-minimal solution to problem (22), there exists a vector $(\mu,\lambda)\in R^p\times R^m$ satisfying (31)-(34). In fact, let

$$K=\{x\in R^n|\langle\nabla g_j(\hat{x}), x\rangle\leqq 0,\ j\in J(\hat{x})\},$$

and $G=R^n$ in Theorem 4 of Borwein (Ref. 9), and note that we have assumed the Cottle constraint qualification. Hence, if $x\in R^n$ satisfies

$$\langle\nabla g_j(\hat{x}), x\rangle\leqq u_j,\qquad \text{for all } j\in J(\hat{x}) \text{ such that } \lambda_j=0, \tag{35a}$$

$$\langle\nabla g_j(\hat{x}), x\rangle=u_j,\qquad \text{for all } j\in J(\hat{x}) \text{ such that } \lambda_j>0, \tag{35b}$$

then

$$\nabla f(\hat{x})x\in\operatorname{Min}_P DY(0,\hat{y})(u).$$

Moreover,

$$\sum_{i=1}^{p}\mu_i\langle\nabla f_i(\hat{x}), x\rangle+\sum_{j=1}^{m}\lambda_j u_j=0.$$

Thus, we have proved the following theorem.

Theorem 5.1. Suppose that Assumption 5.1 is satisfied, and let (μ,λ) be the multiplier vector corresponding to $\hat{x}$. Then, for each $x\in R^n$ satisfying (35),

$$\nabla f(\hat{x})x\in DW(0,\hat{y})(u).$$

Moreover,

$$\sum_{i=1}^{p}\mu_i\langle\nabla f_i(\hat{x}), x\rangle+\sum_{j=1}^{m}\lambda_j u_j=0.$$

Remark 5.1. The equations and inequalities in (35) characterize the set of admissible changes of x corresponding to a perturbation u. They can be also seen, say, in Malanowski (Ref. 10).

6. Conclusions

In this paper, we have studied sensitivity analysis in multiobjective optimization. The essential result that we have proved is that every cone minimal vector of the contingent derivative of the feasible set map in a direction is also an element of the contingent derivative of the perturbation map in that direction under some conditions (Theorem 3.2). We have also obtained the relationship between the contingent derivative of the perturbation map and the Lagrange multipliers for multiobjective programming problems (Theorem 5.1).

However, there remain several open problems which should be solved in the future. Some of them are the following. First, the contingent derivative of the perturbation map is not completely characterized. In other words, sufficient conditions for the converse inclusion of Theorem 3.2 have not been obtained yet. Secondly, the Lipschitz continuity of the perturbation map is not studied here. Thirdly, some more refined results may be obtained in the case of multiobjective programming. We should like to mention that the effects of the convexity assumptions will be made clear in another paper.

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