JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 56, No. 3, MARCH 1988

Recursive Solution of Linear–Quadratic Nash Games for Weakly Interconnected Systems[1]

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Communicated by J. B. Cruz, Jr.

Abstract. A recursive method is developed for the solution of coupled algebraic Riccati equations and corresponding linear Nash strategies of weakly interconnected systems. It is shown that the given algorithm converges to the exact solution with the rate of convergence of $O(\epsilon^2)$, where ϵ is a small coupling parameter. In addition, only low-order systems are involved in algebraic computations; the amount of computations required does not grow per iteration and no analyticity assumption is imposed on the system coefficients.

Key Words. Nash differential games, weak coupling, coupled Riccati equations, recursive algorithm.

1. Introduction

The linear quadratic Nash game strategies of large-scale weakly interconnected systems were studied in Ref. 1 by means of a power series expansion method with respect to a small coupling parameter ϵ. This approach, originated in Ref. 2, is not recursive in its application and can be inferior compared to the hierarchical type decentralized control method (especially when ϵ is not very small), as was pointed out in Ref. 3. In this paper, we develop a new recursive technique which will recover the importance of ideas presented in Ref. 2. Motivated by previous results for singularly perturbed systems (Ref. 4), we have shown that weak coupling produces algebraic problems similar to those of Ref. 4 and the fixed-point method used in Ref. 4 is very efficient in this case also.

[1] This work was supported by Rutgers University Research Council under Grant No. 2-02188.

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As a matter of fact, we have developed an algorithm which converges very rapidly to the exact, nonnegative definite stabilizing solution of the coupled algebraic Riccati equations, and thus to the optimal linear Nash strategies, even in the case when ϵ is not very small.

2. Problem Formulation

A controlled linear dynamic system under consideration is given by

$$\dot{x} = A(\epsilon)x + B_1(\epsilon)u_1 + B_2(\epsilon)u_2, \tag{1}$$

where $x \in R^n$ is a state vector, $u_1 \in R^{m_1}$ and $u_2 \in R^{m_2}$ are control inputs, $A(\epsilon)$, $B_1(\epsilon)$, $B_2(\epsilon)$ are bounded matrix functions of a small parameter ϵ with compatible dimensions.

A quadratic type functional is associated with each control agent,

$$J_1 = \int_0^\infty [x^T Q_1(\epsilon)x + u_1^T R_1(\epsilon)u_1 + u_2^T R_{12}(\epsilon)u_2]\, dt, \tag{2a}$$

$$J_2 = \int_0^\infty [x^T Q_2(\epsilon)x + u_1^T R_{21}(\epsilon)u_1 + u_2^T R_2(\epsilon)u_2]\, dt, \tag{2b}$$

where the weighting matrices are symmetric satisfying

$$Q_i(\epsilon) \geq 0, \quad R_i(\epsilon) > 0, \qquad i = 1, 2,$$

$$R_{ij}(\epsilon) \geq 0, \qquad i \neq j,\ i = 1, 2,\ j = 1, 2.$$

The optimal solution to the given problem with the conflict of interest and simultaneous decision making (Ref. 5) leads to so-called Nash strategies u_1^* and u_2^* satisfying

$$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*), \tag{3a}$$

$$J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2). \tag{3b}$$

It was shown in Ref. 5 that the optimal closed loop strategies are given by

$$u_i^* = -R_i^{-1}(\epsilon)B_i^T(\epsilon)K_i(\epsilon)x, \qquad i = 1, 2, \tag{4}$$

where K_i's satisfy coupled algebraic Riccati equations

$$\begin{aligned} &K_1(\epsilon)A(\epsilon) + A^T(\epsilon)K_1(\epsilon) + Q_1(\epsilon) - K_1(\epsilon)S_1(\epsilon)K_1(\epsilon) \\ &- K_1(\epsilon)S_2(\epsilon)K_2(\epsilon) - K_2(\epsilon)S_2(\epsilon)K_1(\epsilon) + K_2(\epsilon)Z_2(\epsilon)K_2(\epsilon) \\ &= 0 = \mathcal{N}_1(K_1, K_2), \end{aligned} \tag{5a}$$

$$\begin{aligned} &K_2(\epsilon)A(\epsilon) + A^T(\epsilon)K_2(\epsilon) + Q_2(\epsilon) - K_2(\epsilon)S_2(\epsilon)K_2(\epsilon) \\ &- K_2(\epsilon)S_1(\epsilon)K_1(\epsilon) - K_1(\epsilon)S_1(\epsilon)K_2(\epsilon) + K_1(\epsilon)Z_1(\epsilon)K_1(\epsilon) \\ &= 0 = \mathcal{N}_2(K_1, K_2), \end{aligned} \tag{5b}$$

where

$$S_i(\epsilon) = B_i(\epsilon) R_i^{-1}(\epsilon) B_i^T(\epsilon), \qquad i = 1, 2,$$
$$Z_i(\epsilon) = B_i(\epsilon) R_i^{-1}(\epsilon) R_{ji}(\epsilon) R_i^{-1}(\epsilon) B_i^T(\epsilon), \qquad i = 1, 2, \; j = 1, 2, \; i \neq j.$$

The existence of the nonlinear optimal Nash strategies was established in Ref. 6, so that (4), in fact, are the best linear optimal strategies. Since a linear control law, from a practical point of view, is very desirable, the linear strategies (4) attract the attention of many researchers.

The existence of Nash strategies (4) and solutions of coupled Riccati equations (5) has been studied in Ref. 7, by means of Brower's fixed-point theorem and by imposing norm conditions on the given matrices. In a recent paper (Ref. 8), under control-oriented assumptions (Refs. 9 and 10), the existence of nonnegative-definite stabilizing solutions of (5) has been established.

It is important to point out that, at the present time, there is no published method for finding stabilizing solutions of coupled algebraic Riccati equations (5). Some attempts in that direction have been made in Refs. 15 and 16.

In this paper, the Nash game problem is considered for a special case of weakly interconnected systems characterized by

$$A(\epsilon) = \begin{bmatrix} A_1(\epsilon) & \epsilon A_{12}(\epsilon) \\ \epsilon A_{21}(\epsilon) & A_2(\epsilon) \end{bmatrix},$$
$$B_1(\epsilon) = \begin{bmatrix} B_{11}(\epsilon) \\ \epsilon B_{21}(\epsilon) \end{bmatrix}, \qquad B_2(\epsilon) = \begin{bmatrix} \epsilon B_{12}(\epsilon) \\ B_{22}(\epsilon) \end{bmatrix},$$
$$Q_1(\epsilon) = \begin{bmatrix} U_1(\epsilon) & \epsilon U_{12}(\epsilon) \\ \epsilon U_{12}^T(\epsilon) & \epsilon^2 U_2(\epsilon) \end{bmatrix}, \qquad Q_2(\epsilon) = \begin{bmatrix} \epsilon^2 V_1(\epsilon) & \epsilon V_{12}(\epsilon) \\ \epsilon V_{12}^T(\epsilon) & V_2(\epsilon) \end{bmatrix}.$$

This partition decomposes the state vector x into two vectors $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$, such that $n_1 + n_2 = n$. Since the small coupling parameter ϵ cannot change the basic structures of the subsystems by destroying their main properties (otherwise, we cannot talk about the weak coupling), it is very natural to adopt the following form for the subsystem matrices.

Assumption 2.1. *Weak Coupling Assumption.* For $i = 1, 2$,

$$A_i(\epsilon) = A_{i0} + \epsilon A_{0i}(\epsilon),$$
$$B_{ii}(\epsilon) = B_{i0} + \epsilon B_{0i}(\epsilon),$$
$$U_1(\epsilon) = U_{10} + \epsilon U_{01}(\epsilon),$$
$$V_2(\epsilon) = V_{20} + \epsilon V_{02}(\epsilon),$$
$$R_i(\epsilon) = R_{i0} + \epsilon R_{0i}(\epsilon),$$

where $A_{0i}(\epsilon)$, $B_{0i}(\epsilon)$, $R_{0i}(\epsilon)$, $i=1,2$, $U_{01}(\epsilon)$ and $V_{02}(\epsilon)$ are continuous functions of ϵ, whereas A_{i0}, B_{i0}, R_{i0}, $i=1,2$, and U_{10}, V_{20} are independent of ϵ.

In order to simplify the algebra, we will assume, without loss of generality (Ref. 19), that

$$U_{12}(\epsilon)=0, \quad V_{12}(\epsilon)=0, \quad R_{12}(\epsilon)=0, \quad R_{21}(\epsilon)=0,$$
$$U_2(\epsilon)=0, \quad V_1(\epsilon)=0, \quad B_{12}(\epsilon)=0, \quad B_{21}(\epsilon)=0.$$

Note that we are studying a more general case than the one studied in Ref. 1, because of the ϵ-dependence of the problem matrices. In addition, we do not need to impose the analyticity assumption with respect to ϵ, which must be done for the power series expansion method.

The following scaling of $K_1(\epsilon)$ and $K_2(\epsilon)$ is consistent with the nature of the solution of (5)

$$K_1(\epsilon)=\begin{bmatrix} M_1(\epsilon) & \epsilon M_{12}(\epsilon) \\ \epsilon M_{12}^T(\epsilon) & \epsilon^2 M_2(\epsilon)\end{bmatrix}, \qquad K_2(\epsilon)=\begin{bmatrix} \epsilon^2 N_1(\epsilon) & \epsilon N_{12}(\epsilon) \\ \epsilon N_{12}^T(\epsilon) & N_2(\epsilon)\end{bmatrix}. \tag{6}$$

The very well-known ϵ-decoupling method (Ref. 2), based on the power series expansion with respect to ϵ, will convert the given full-order problem (5) to a family of reduced-order problems (Ref. 1). However, the power series expansion method is not recursive in nature and, in the case when we are interested in high order of accuracy or when ϵ is not very small, the size of the required computations can be considerable. Moreover, when the problem matrices are functions of ϵ, the power series method demands the analyticity of all matrices. On the other hand, the expansion of quadratic terms [for example, $K_1(\epsilon)B_1(\epsilon)R_1^{-1}(\epsilon)B_1^T(\epsilon)K_1(\epsilon)$] will produce an enormous number of terms, so that the reduced-order advantage of the series expansion method becomes questionable. The presence of a small parameter ϵ will be exploited in the next section from a different point of view, leading to the recursive scheme for the solution of (5). Since the proposed method is of the fixed-point type, the boundness of all problem matrices over a compact set $\epsilon\in[0, \epsilon_1]$ has to be imposed. This is a much milder condition than the analyticity requirement of the power series expansion method.

3. Iterative Solution of Coupled Algebraic Riccati Equations

Partitioning (5) compatibly with (6), we get the following set of equations:

$$\begin{aligned}&M_1(\epsilon)A_1(\epsilon)+A_1^T(\epsilon)M_1(\epsilon)+U_1(\epsilon)-M_1(\epsilon)S_{11}(\epsilon)M_1(\epsilon)\\&+\epsilon^2\{M_{12}(\epsilon)A_{21}(\epsilon)+A_{21}^T(\epsilon)M_{12}^T(\epsilon)-M_{12}(\epsilon)S_{22}(\epsilon)N_{12}^T(\epsilon)\\&-N_{12}(\epsilon)S_{22}(\epsilon)M_{12}^T(\epsilon)\}=0,\end{aligned} \tag{7a}$$

$$M_1(\epsilon)A_{12}(\epsilon)+M_{12}(\epsilon)A_2(\epsilon)-M_1(\epsilon)S_{11}(\epsilon)M_{12}(\epsilon)$$
$$-M_{12}(\epsilon)S_{22}(\epsilon)N_2(\epsilon)-\epsilon^2\{N_{12}(\epsilon)S_{22}(\epsilon)M_2(\epsilon)$$
$$-A_{21}^T(\epsilon)M_2(\epsilon)\}+A_1^T(\epsilon)M_{12}(\epsilon)=0, \tag{7b}$$
$$M_2(\epsilon)A_2(\epsilon)+A_2^T(\epsilon)M_2(\epsilon)-M_2(\epsilon)S_{22}(\epsilon)N_2(\epsilon)$$
$$-N_2(\epsilon)S_{22}(\epsilon)M_2(\epsilon)+M_{12}^T(\varepsilon)A_{12}(\epsilon)+A_{12}^T(\epsilon)M_{12}(\epsilon)$$
$$-M_{12}^T(\epsilon)S_{11}(\epsilon)M_{12}(\epsilon)=0, \tag{7c}$$
$$N_1(\epsilon)A_1(\epsilon)+A_1^T(\epsilon)N_1(\epsilon)-N_1(\epsilon)S_{11}(\epsilon)M_1(\epsilon)$$
$$-M_1(\epsilon)S_{11}(\epsilon)N_1(\epsilon)+N_{12}(\epsilon)A_{21}(\epsilon)+A_{21}^T(\epsilon)N_{12}^T(\epsilon)$$
$$-N_{12}(\epsilon)S_{22}(\epsilon)N_{12}^T(\epsilon)=0, \tag{7d}$$
$$\epsilon^2N_1(\epsilon)A_{12}(\epsilon)+N_{12}(\epsilon)A_2(\epsilon)-N_{12}(\epsilon)S_{22}(\epsilon)N_2(\epsilon)$$
$$-\epsilon^2N_1(\epsilon)S_{11}(\epsilon)M_{12}(\epsilon)-M_1(\epsilon)S_{11}(\epsilon)N_{12}(\epsilon)+A_{21}^T(\epsilon)N_2(\epsilon)$$
$$+A_1^T(\epsilon)N_{12}(\epsilon)=0, \tag{7e}$$
$$N_2(\epsilon)A_2(\epsilon)+A_2^T(\epsilon)N_2(\epsilon)+V_2(\epsilon)-N_2(\epsilon)S_{22}(\epsilon)N_2(\epsilon)$$
$$+\epsilon^2\{N_{12}^T(\epsilon)A_{12}+A_{12}^T(\epsilon)N_{12}(\epsilon)-N_{12}^T(\epsilon)S_{11}(\epsilon)M_{12}(\epsilon)$$
$$-M_{12}^T(\epsilon)S_{11}(\epsilon)N_{12}(\epsilon)\}=0, \tag{7f}$$

where

$$S_{ii}(\epsilon)=B_{ii}(\epsilon)R_i^{-1}(\epsilon)B_{ii}^T(\epsilon), \qquad i=1,2.$$

3.1. Zeroth-Order Approximation. Let us define the $O(\epsilon^2)$ perturbation of (7) as

$$\underline{M}_1(\epsilon)A_1(\epsilon)+A_1^T(\epsilon)\underline{M}_1(\epsilon)+U_1(\epsilon)-\underline{M}_1(\epsilon)S_{11}(\epsilon)\underline{M}_1(\epsilon)=0, \tag{8a}$$
$$\underline{M}_{12}(\epsilon)D_2(\epsilon)+D_1(\epsilon)^T\underline{M}_{12}(\epsilon)=-\underline{M}_1(\epsilon)A_{12}(\epsilon), \tag{8b}$$
$$\underline{M}_2(\epsilon)D_2(\epsilon)+D_2(\epsilon)^T\underline{M}_2(\epsilon)$$
$$=\underline{M}_{12}^T(\epsilon)S_{11}(\epsilon)\underline{M}_{12}(\epsilon)-\underline{M}_{12}^T(\epsilon)A_{12}(\epsilon)-A_{12}^T(\epsilon)\underline{M}_{12}(\epsilon), \tag{8c}$$
$$\underline{N}_1(\epsilon)D_1(\epsilon)+D_1^T(\epsilon)\underline{N}_1(\epsilon)$$
$$=\underline{N}_{12}(\epsilon)S_{22}(\epsilon)\underline{N}_{12}^T(\epsilon)-\underline{N}_{12}(\epsilon)A_{21}(\epsilon)-A_{21}^T(\epsilon)\underline{N}_{12}^T(\epsilon), \tag{8d}$$
$$\underline{N}_{12}(\epsilon)D_2(\epsilon)+D_1^T(\epsilon)\underline{N}_{12}(\epsilon)=-A_{21}^T(\epsilon)\underline{N}_2(\epsilon), \tag{8e}$$
$$\underline{N}_2(\epsilon)A_2(\epsilon)+A_2^T(\epsilon)\underline{N}_2(\epsilon)+V_2(\epsilon)-\underline{N}_2(\epsilon)S_{22}(\epsilon)\underline{N}_2(\epsilon)=0, \tag{8f}$$

where

$$D_1(\epsilon)=A_1(\epsilon)-S_{11}(\epsilon)\underline{M}_1(\epsilon),$$
$$D_2(\epsilon)=A_2(\epsilon)-S_{22}(\epsilon)\underline{N}_2(\epsilon).$$

This system of equations has decoupled form and can be solved like two lower-order Riccati equations (8a), (8f) and four low-order Lyapunov equations (8b)-(8e). The nonnegative-definite stabilizing solution of (8a) and (8f) exist under the well-known stabilizability-detectability assumption (Refs. 9 and 10).

Assumption 3.1. The triples $(A_1(0), B_1(0), \sqrt{U_1(0)})$ and $(A_2(0), B_2(0), \sqrt{V_2(0)})$ are stabilizable-detectable.

Under the same assumption, the unique solution of (8b)-(8e) exist, since $D_1(\epsilon)$ and $D_2(\epsilon)$ are stable matrices (Refs. 9 and 10).

3.2. Solution of Higher-Order Accuracy. The zeroth-order solutions $\underline{M}(\epsilon)$ and $\underline{N}(\epsilon)$ are $O(\epsilon^2)$ close to the exact ones. Then, the exact solutions can be sought in the form

$$K_1(\epsilon) = \begin{bmatrix} \underline{M}_1(\epsilon) + \epsilon^2 E_1(\epsilon) & \epsilon\{\underline{M}_{12}(\epsilon) + \epsilon^2 E_{12}(\epsilon)\} \\ \epsilon\{\underline{M}_{12}(\epsilon) + \epsilon^2 E_{12}(\epsilon)\}^T & \epsilon^2\{\underline{M}_2(\epsilon) + \epsilon^2 E_2(\epsilon)\} \end{bmatrix}, \tag{9a}$$

$$K_2(\epsilon) = \begin{bmatrix} \epsilon^2\{\underline{N}_1(\epsilon) + \epsilon^2 G_1(\epsilon)\} & \epsilon\{\underline{N}_{12}(\epsilon) + \epsilon^2 G_{12}(\epsilon)\} \\ \epsilon\{\underline{N}_{12}(\epsilon) + \epsilon^2 G_{12}(\epsilon)\}^T & \underline{N}_2(\epsilon) + \epsilon^2 G_2(\epsilon) \end{bmatrix}. \tag{9b}$$

Obviously, $O(\epsilon^k)$ approximations of $E(\epsilon)$'s and $G(\epsilon)$'s will produce $O(\epsilon^{k+2})$ approximations of required solutions, which is why we are interested in finding a convenient form for these error terms and the appropriate algorithm for their solution.

Subtracting Eqs. (8) from corresponding Eqs. (7), and after doing some algebra, we get the following expressions for the error equations:

$$E_1 D_1 + D_1^T E_1 = C_1 + \epsilon^2 F_1(E_1, E_{12}, G_{12}), \tag{10a}$$

$$E_1 D_{12} + E_{12} D_2 + D_1^T E_{12} - \underline{M}_{12} S_{22} G_2$$
$$= C_2 + \epsilon^2 F_2(E_1, E_{12}, E_2, G_{12}, G_2), \tag{10b}$$

$$E_{12}^T D_{12} + D_{12}^T E_{12} + E_2 D_2 + D_2^T E_2 - G_2 S_{22} \underline{M}_2 - \underline{M}_2 S_{22} G_2$$
$$= \epsilon^2 F_3(E_{12}, E_2, G_2), \tag{10c}$$

$$G_1 D_1 + D_1^T G_1 + G_{12} D_{21} + D_{21}^T G_{12}^T - E_1 S_{11} \underline{N}_1 - \underline{N}_1 S_{11} E_1$$
$$= \epsilon^2 F_4(E_1, G_{12}, G_2), \tag{10d}$$

$$G_{12} D_2 + D_1^T G_{12} + D_{21}^T G_2 - E_1 S_{11} \underline{N}_{12}$$
$$= C_5 + \epsilon^2 F_5(E_1, E_{12}, G_1, G_{12}, G_2), \tag{10e}$$

$$G_2 D_2 + D_2^T G_2 = C_6 + \epsilon^2 F_6(E_{12}, G_{12}, G_2), \tag{10f}$$

where

$$D_{12} = D_{12}(\epsilon) = A_{12}(\epsilon) - S_{11}(\epsilon)\underline{M}_{12}(\epsilon),$$
$$D_{21} = D_{21}(\epsilon) = A_{21}(\epsilon) - S_{22}(\epsilon)\underline{N}_{12}(\epsilon).$$

Matrices F_i, $i = 1, 2, \ldots, 6$, and constant matrices C_j are given in the Appendix (Section 7). In order to simplify notation, the ϵ-dependence of the problem matrices in Eq. (10) and in the remaining part of the paper is omitted.

The weakly coupled and hierarchical structure of (10) can be exploited by proposing the following recursive scheme, which leads, after some algebra, to the six low-order, completely decoupled Lyapunov equations

$$E_1^{(i+1)}D_1 + D_1^T E_1^{(i+1)} = \epsilon^2 E_1^{(i)} S_{11} E_1^{(i)} - M_{12}^{(i)} D_{21}^{(i)} - D_{21}^{(i)^T} M_{12}^{(i)^T}, \tag{11a}$$

$$E_{12}^{(i+1)}D_2 + D_1^T E_{12}^{(i+1)} = -E_1^{(i+1)} D_{12}^{(i)} + M_{12}^{(i)} S_{22} G_2^{(i+1)} - D_{21}^{(i)^T} M_2^{(i)}, \tag{11b}$$

$$E_2^{(i+1)}D_2 + D_2^T E_2^{(i+1)} = M_2^{(i)} S_{22} G_2^{(i+1)} + G_2^{(i+1)} S_{22} M_2^{(i)} - E_{12}^{(i+1)^T} D_{12} - D_{12}^T E_{12}^{(i+1)} + \epsilon^2 E_{12}^{(i+1)^T} S_{11} E_{12}^{(i+1)}, \tag{11c}$$

$$G_1^{(i+1)}D_1 + D_1^T G_1^{(i+1)} = E_1^{(i+1)} S_{11} N_1^{(i)} + N_1^{(i)} S_{11} E_1^{(i+1)} - G_{12}^{(i+1)} D_{21} - D_{21}^T G_{12}^{(i+1)^T} + \epsilon^2 G_{12}^{(i+1)} S_{22} G_{12}^{(i+1)^T}, \tag{11d}$$

$$G_{12}^{(i+1)}D_2 + D_1^T G_{12}^{(i+1)} = -D_{21}^{(i)^T} G_2^{(i+1)} + E_1^{(i+1)} S_{11} N_{12}^{(i)} - N_1^{(i)} D_{12}^{(i)}, \tag{11e}$$

$$G_2^{(i+1)}D_2 + D_2^T G_2^{(i+1)} = \epsilon^2 G_2^{(i)} S_{22} G_2^{(i)} - N_{12}^{(i)^T} D_{12}^{(i)} - D_{12}^{(i)^T} N_{12}^{(i)}, \tag{11f}$$

with $i = 0, 1, 2, 3, \ldots$, with initial conditions chosen as

$$E_1^{(0)} = E_{12}^{(0)} = E_2^{(0)} = G_1^{(0)} = G_{12}^{(0)} = G_2^{(0)} = 0,$$

where

$$M_{12}^{(i)} = \underline{M}_{12} + \epsilon^2 E_{12}^{(i)},$$
$$N_{12}^{(i)} = \underline{N}_{12} + \epsilon^2 G_{12}^{(i)},$$
$$N_1^{(i)} = \underline{N}_1 + \epsilon^2 G_1^{(i)},$$
$$M_2^{(i)} = \underline{M}_2 + \epsilon^2 E_2^{(i)},$$
$$D_{12}^{(i)} = A_{12} - S_{11} M_{12}^{(i)},$$
$$D_{21}^{(i)} = A_{21} - S_{22} N_{12}^{(i)^T},$$

with $i=1,2,3,\ldots$. These Lyapunov equations have to be solved in the given order, that is, first E_1 and G_2, then E_{12} and G_{12}, and finally E_2 and G_1.

The following theorem indicates the features of the proposed recursive scheme.

Theorem 3.1. Under imposed weak coupling and stabilizability and detectability assumptions the given algorithm (11) converges to the exact solution of the error terms, and thus of $K_1(\epsilon)$ and $K_2(\epsilon)$, with the rate of convergence of $O(\epsilon^2)$, that is,

$$\| E_j(\epsilon) - E_j^{(i)}(\epsilon) \| = O(\epsilon^{2i}), \tag{12a}$$

$$\| G_j(\epsilon) - G_j^{(i)}(\epsilon) \| = O(\epsilon^{2i}), \tag{12b}$$

$$\| E_{12}(\epsilon) - E_{12}^{(i)}(\epsilon) \| = O(\epsilon^{2i}), \tag{12c}$$

$$\| G_{12}(\epsilon) - G_{12}^{(i)}(\epsilon) \| = O(\epsilon^{2i}), \tag{12d}$$

with $j=1,2$, $i=1,2,3,\ldots$, and

$$\| K_j(\epsilon) - K_j^{(i)}(\epsilon) \| = O(\epsilon^{2i+2}), \qquad j=1,2,\ i=0,1,2.$$

Proof. As a starting point, we need to show the existence of a bounded solution of (10) in the neighborhood of $\epsilon=0$. By the implicit function theorem, it is enough to show that the corresponding Jacobian is nonsingular at $\epsilon=0$. The Jacobian is given by

$$J(\epsilon)\big|_{\epsilon=0} = \begin{bmatrix} \Gamma_1 & 0 & 0 & 0 & 0 & 0 \\ * & \Gamma_2 & 0 & 0 & 0 & * \\ 0 & * & \Gamma_3 & 0 & 0 & * \\ * & 0 & 0 & \Gamma_1 & * & 0 \\ * & 0 & 0 & 0 & \Gamma_2 & * \\ 0 & 0 & 0 & 0 & 0 & \Gamma_3 \end{bmatrix}, \tag{13}$$

where the asterisk denotes terms which are not important for a nonsingularity of the Jacobian. Γ's are given by the Kronecker product representation

$$\Gamma_i = I_{n_i} \times D_i^T(0) + D_i^T(0) \times I_{n_i}, \qquad i=1,3,$$

$$\Gamma_2 = I_{n_2} \times D_2^T(0) + D_1^T(0) \times I_{n_1},$$

where I_{n_1} and I_{n_2} are identity matrices. Under Assumptions 2.1 and 3.1, $D_1(0)$ and $D_2(0)$ are stable matrices for any sufficiently small $\epsilon \in [0, \epsilon_2]$ and, by the well-known properties of the Kronecker product (Ref. 12), so are matrices $\Gamma_1, \Gamma_2, \Gamma_3$. It is easy to see that the nonsingularity of the Jacobian is guaranteed by the nonsingularity of $\Gamma_1, \Gamma_2, \Gamma_3$.

The second step in the proof of the given theorem is to give an estimate of the rate of convergence.

For $i=0$, (10a) and (11a) imply that

$$(E_1 - E_1^{(1)})D_1 + D_1^T(E_1 - E_1^{(1)}) = \epsilon^2 F_1(E_1, E_{12}, G_{12}),$$

which by stability of D_1 and the existence of the bounded solution of (10) gives

$$\| E_1 - E_1^{(1)} \| = O(\epsilon^2). \tag{14a}$$

By the same arguments, from (10f) and (11f) we have

$$\| G_2 - G_2^{(1)} \| = O(\epsilon^2). \tag{14f}$$

Subtracting (11b) from (10b) and using (14a) and (14f) and the expression for F_3 (from the Appendix) lead to

$$(E_{12} - E_{12}^{(1)})D_2 + D_1^T(E_{12} - E_{12}^{(1)}) = O(\epsilon^2),$$

which implies that

$$\| E_{12} - E_{12}^{(1)} \| = O(\epsilon^2). \tag{14b}$$

By analogy [Eqs. (10b) and (10e) have similar form], (10e) and (11e) will produce

$$\| G_{12} - G_{12}^{(1)} \| = O(\epsilon^2). \tag{14e}$$

Also, from (10c), (11c), (14a,b,c,d,e,f) and the Appendix, we have

$$(E_2 - E_2^{(1)})D_2 + D_2^T(E_2 - E_2^{(1)}) = O(\epsilon^2),$$

that is,

$$\| E_2 - E_2^{(1)} \| = O(\epsilon^2); \tag{14c}$$

and, by analogy, from (10d) and (11d) we get

$$\| G_1 - G_1^{(1)} \| = O(\epsilon^2). \tag{14d}$$

Using these starting observations and the forms of F_j's and C_j's it can be shown that

$$\| F_j - F_j^{(i)} \| = O(\epsilon^{2i}), \qquad j = 1, 2,\ i = 1, 2, 3, \ldots \tag{15}$$

For example, for $j=1$,

$$F_1 - F_1^{(i)} = (E_1 - E_1^{(i)})S_{11}E_1^{(i)} + E_1 S_{11}(E_1 - E_1^{(i)})$$
$$- (E_{12} - E_{12}^{(i)})D_{21} - D_{21}^T(E_{12} - E_{12}^{(i)})^T$$
$$+ (G_{12} - G_{12}^{(i)})S_{22}M_{12}^{(i)^T} + M_{12}S_{22}(G_{12} - G_{12}^{(i)})^T,$$

so that, for $i=1$, from (14) we have

$$F_1 - F_1^{(1)} = O(\epsilon^2),$$

that is,

$$(E_1 - E_1^{(2)})D_1 + D_1^T(E_1 - E_1^{(2)}) = \epsilon^2(F_1 - F_1^{(1)}) = O(\epsilon^4),$$

which implies that

$$(E_1 - E_1^{(2)}) = O(\epsilon^4).$$

Continuing the same procedure, we can verify (15), which by the existence of the bounded solutions of E's and G's will imply (12). Note that the solution of (11) exists at each iteration, since the corresponding Jacobian is always given by (13), and thus nonsingular at $\epsilon = 0$ for every i. □

We would like to point out that the imposed form of the solution (9) is an additional limiting factor for a small parameter ϵ. It was shown in Ref. 8 that under Assumption 3.1, the solution of (5) is nonnegative definite. Since the solution of (10) is only symmetric (which can be easily seen from the form of corresponding equations), the small parameter ϵ has to be constrained to the set $\epsilon \in [0, \epsilon_3]$ such that, $\forall \epsilon$, $K_1(\epsilon)$ and $K_2(\epsilon)$ preserve the required nonnegative definiteness. Thus, the present method is applicable for $\epsilon \in [0, \epsilon^*]$, where

$$\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}.$$

However, the limiting condition

$$\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \ldots\}$$

is present in the entire theory of small parameters (weak coupling and singular perturbations); it is both method-dependent and problem-dependent, and it is not a direct consequence of the procedure studied in this paper.

Let us compare the proposed algorithm (11), based on the fixed-point iterations, for weakly coupled systems, and the power series expansion algorithm for the same type of systems. The comparison is done for the case where the problem matrices are not functions of ϵ [which is in favor of the power series expansion algorithm (see page 466)]. The equations corresponding to (11) are given by (Ref. 1)

$$M_1^{(i+1)}D_1+D_1^T M_1^{(i+1)}=Z_1^{(0,1,2,\ldots,i)}, \tag{16a}$$

$$M_{12}^{(i+1)}D_2+D_1^T M_{12}^{(i+1)}=Z_2^{(0,1,2,\ldots,i)}, \tag{16b}$$

$$M_2^{(i+1)}D_2+D_2^T M_2^{(i+1)}=Z_3^{(0,1,2,\ldots,i)}, \tag{16c}$$

$$N_1^{(i+1)}D_1+D_1^T N_1^{(i+1)}=Z_4^{(0,1,2,\ldots,i)}, \tag{16d}$$

$$N_{12}^{(i+1)}D_2+D_1^T G_{12}^{(i+1)}=Z_5^{(0,1,2,\ldots,i)}, \tag{16e}$$

$$N_2^{(i+1)}D_2+D_2^T N_2^{(i+1)}=Z_6^{(0,1,2,\ldots,i)}, \tag{16f}$$

where Z_j, $j=1,2,\ldots,6$, depend on the all previously obtained terms. For example,

$$\begin{aligned}Z_1^{(0,1,2,\ldots,i)}&=-(i+1)\{M_{12}^{(i)}A_{21}+A_{21}^T M_{12}^{(i)^T}\}\\&+\sum_{\substack{k=2\\k\text{ even}}}^{i-1}\binom{i+1}{k}M_1^{(i+1-k)}S_{11}M_1^{(k)}\\&+\sum_{\substack{k=1\\k\text{ odd}}}^{i}\left\{\binom{i+1}{k}M_{12}^{(i+1-k)}S_{22}N_{12}^{(k)^T}+N_{12}^{(i+1-k)}S_{22}M_{12}^{(k)}\right\}.\end{aligned} \tag{17}$$

Both approaches produce the same type of equations (Lyapunov ones); but, in order to form the right-hand side, for example of (11a), we have to perform only three matrix multiplications for every i, whereas for the corresponding equation of the power series expansion the number of required matrix multiplications grows very quickly as i increases in (17). Thus, the obvious advantages of the fixed-point iteration approach are: (i) the size of the required computations is considerably less, and since it does not grow per iteration, the proposed method is extremally efficient for obtaining the exact solution or the solution of very high accuracy; (ii) the fixed-point method is recursive in nature (the power series expansion method is not), and thus much easier to implement.

4. Suboptimal Linear Nash Strategies

The approximations of the suboptimal Nash strategies (4) can be defined by

$$u_j^{(i)}(t)=-R_j^{-1}(\epsilon)B_j^T(\epsilon)K_j^{(i)}(\epsilon)x(t), \qquad j=1,2,\ i=0,1,2,3,\ldots, \tag{18}$$

where

$$K_1^{(i)} = \begin{bmatrix} \underline{M}_1(\epsilon) + \epsilon^2 E_1^{(i)}(\epsilon) & \epsilon\{\underline{M}_{12}(\epsilon) + \epsilon^2 E_{12}^{(i)}(\epsilon)\} \\ \epsilon\{\underline{M}_{12}(\epsilon) + \epsilon^2 E_{12}^{(i)}(\epsilon)\}^T & \epsilon^2\{\underline{M}_2(\epsilon) + \epsilon^2 E_2^{(i)}(\epsilon)\} \end{bmatrix}, \tag{19a}$$

$$K_2^{(i)} = \begin{bmatrix} \epsilon^2\{\underline{N}_1(\epsilon) + \epsilon^2 G_1^{(i)}(\epsilon)\} & \epsilon\{\underline{N}_{12}(\epsilon) + \epsilon^2 G_{12}^{(i)}(\epsilon)\} \\ \epsilon\{\underline{N}_{12}(\epsilon) + \epsilon^2 G_{12}^{(i)}(\epsilon)\} & \underline{N}_2(\epsilon) + \epsilon^2 G_2^{(i)}(\epsilon) \end{bmatrix}. \tag{19b}$$

Then, by following the arguments of Ref. 13, the cost approximations produce

$$J_j^{(i)}(u_1^{(i)}, u_2^{(i)}) = J_j(u_1^*, u_2^*) + O(\epsilon^{2i+2}), \qquad j = 1, 2,\ i = 0, 1, 2, \ldots. \tag{20}$$

The approximate cost functions for the other cases, when the control agents use the approximate strategies of different order of accuracy [for example $u_1^{(p)}$ and $u_2^{(q)}$, $p \neq q$] can be obtained by using results of Ref. 13 also. But, since the proposed method is recursive in its nature, and thus very easy to implement, and since the amount of required computations is constant per iteration (does not grow with i), accuracy of very high order can be achieved at a very low cost, so that the proposed method can be efficient for finding the exact solution as well.

Since the proposed algorithm defines the error of approximation similarly to the power series expansion, it can be easily seen that the approximate Nash strategies (18) are also well posed in the sense of Khalil (Ref. 17).

5. Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, we have run a fourth-order example. Matrices A_1, A_{12}, A_{21}, A_2, B_{11}, B_{22} have been chosen randomly (standard deviation = 1, mean value = 0), and the matrices $R_1 = R_2 = U_1 = V_2 = I$ are chosen such that the required stabilizability-detectability assumptions are satisfied:

$$A_1 = \begin{bmatrix} -1.035 & -0.192 \\ 1.648 & -0.421 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} -1.084 & 0.597 \\ 1.327 & -0.841 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -1.370 & -0.533 \\ 1.069 & 0.835 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1.510 & -0.139 \\ 0.410 & 1.238 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -1.019 & 0.602 \\ -0.912 & 1.329 \end{bmatrix}, \qquad B_{22} = \begin{bmatrix} -1.641 & 0.330 \\ 1.068 & 0.243 \end{bmatrix},$$

$$U_1 = V_2 = R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Table 1. Dependence of number of iterations on ϵ.

ϵ	i = Number of required iterations such that $e^{(i)} < 10^{-10}$
0.8	16
0.6	11
0.4	8
0.2	5
0.1	4
0.05	3
0.01	2
0.001	1

The simulation results for different values of the coupling parameter ϵ are given in Table 1. Since we do not know the exact solution of (5) [no method is available in the literature at the present time], the error is defined as

$$e^{(i)} = \max\{\| \mathcal{N}_1(K_1^{(i)}, K_2^{(i)}) \|_\infty, \| \mathcal{N}_2(K_1^{(i)}, K_2^{(i)}) \|_\infty\}.$$

In Table 2, we show the propagation of the error per iteration when $\epsilon = 0.1$.

The results from Table 1 strongly support the necessity of the existence of the recursive scheme for the solution of weakly-coupled linear-quadratic Nash game problem, since, unless ϵ is very small, the zeroth and first order approximations are far from the optimal solution.

The results from Table 2 verify, for this particular example, the conclusions of Theorem 3.1, that is, the rate of convergence of the proposed algorithm is $O(\epsilon^2) = O(10^{-2})$. The simulation results have been obtained by using the software package LAS for computer-aided control system design (Ref. 18).

Table 2. Propagation of the error per iteration for a constant value of ϵ ($\epsilon = 0.1$).

i	Error $e^{(i)}$
0	0.89662×10^{-2}
1	0.65481×10^{-4}
2	0.10349×10^{-6}
3	0.40663×10^{-9}
4	0.92572×10^{-11}

6. Conclusions

The solution to the Nash strategies of weakly interconnected systems can be obtained up to an arbitrary accuracy by performing iterations on the Lyapunov equations corresponding to the local subsystem problems. Hopefully, this idea can be extended to generalized weak coupling problem (Ref. 11) and to weakly coupled nonlinear systems (Ref. 14).

7. Appendix

We give the expressions of the matrices F_i and C_j introduced in Section 3:

$$F_1 = E_1 S_{11} E_1 + \underline{M}_{12} S_{22} G_{12}^T + G_{12} S_{22} \underline{M}_{12}^T - E_{12} D_{21} - D_{21}^T E_{12}^T + \epsilon^2 (E_{12} S_{22} G_{12}^T + G_{12} S_{22} E_{12}^T),$$

$$F_2 = E_{12} S_{22} G_2 + E_1 S_{11} E_{12} + G_{12} S_{22} \underline{M}_2 - D_{21}^T E_2 + \epsilon^2 G_{12} S_{22} E_2,$$

$$F_3 = E_{12} S_{11} E_{12} + E_2 S_{22} G_2 + G_2 S_{22} E_2,$$

$$F_4 = G_{12} S_{22} G_{12}^T + E_1 S_{11} G_1 + G_1 S_{11} E_1,$$

$$F_5 = E_1 S_{11} G_{12} + G_{12} S_{22} G_2 + \underline{N}_1 S_{11} G_{12} - G_1 D_{12} + \epsilon^2 G_1 S_{11} E_{12},$$

$$F_6 = G_2 S_{22} G_2 + E_{12}^T S_{11} \underline{N}_{12} + \underline{N}_{12}^T S_{11} E_{12} - G_{12}^T D_{12} - D_{12}^T G_{12} + \epsilon^2 (E_{12}^T S_{11} G_{12} + G_{12}^T S_{11} E_{12}),$$

$$C_1 = -\underline{M}_{12} A_{12} - A_{21}^T \underline{M}_{12}^T + \underline{M}_{12} S_{22} \underline{N}_{12}^T + \underline{N}_{12} S_{22} \underline{M}_{12}^T,$$

$$C_2 = -D_{21}^T \underline{M}_2,$$

$$C_5 = -\underline{N}_1 D_{12},$$

$$C_6 = -\underline{N}_{12}^T A_{12} - A_{12}^T \underline{N}_{12} + \underline{M}_{12}^T S_{11} \underline{N}_{12} + \underline{N}_{12}^T S_{11} \underline{M}_{12}.$$

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