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Robust Control of Base-Isolated Structures under Earthquake Excitation[1]

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Abstract. We propose the use of robust control in conjunction with base isolation in order to assure arbitrarily small motion of a seismically excited structure. The proposed method requires control force application only at the base (first) floor. The efficacy of the scheme is illustrated by extensive simulations for a prototype six-story building.

Key Words. Robust control, aseismic control, base isolation.

1. Introduction

Conventional methods of earthquake protection for buildings and other structures rely on the strength of the structure and its capacity to dissipate earthquake-produced energy. This energy absorbing capacity is produced by inelastic action in the structural frame and can, if many cycles of loading are involved, result in damage to the frame of the building, and could also lead to damage to nonstructural components and internal equipment.

It is the horizontal components of the ground motion that are the most damaging to a building and to its contents. In a new approach to earthquake protection, called base isolation, the building rests on a system of isolators, which act to uncouple the building from the horizontal ground motion. The building is isolated at the base, and not only are the loads on the structural system reduced but occupants and contents are also protected. The concept of base isolation is not new, but it has become a practical reality in recent years through developments in rubber technology.

Rubber bearings offer the simplest method of isolation and are relatively easy to manufacture. Long experience with bridge bearings which are very

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similar has provided confidence in their longevity and reliability. The bearings used in seismic isolation systems are made by vulcanization bonding of the thin sheets of natural or artificial rubber (usually neoprene) to thin reinforcing steel plates. The bearings have the mechanical characteristic of being very flexible in the horizontal direction and very stiff in the vertical direction, and their action under seismic loading is to isolate the building or structure from the horizontal components of the ground motion. The vertical components of the earthquake ground motion are transmitted unchanged into the structure, although the bearings will provide isolation against higher frequencies of ground motion such as are caused by traffic and underground transit systems. These bearings are suitable for buildings which are rigid and low-rise, up to about seven stories, for which uplift on the bearings will not occur and for which wind loading will be relatively unimportant.

Buildings have been built on base isolation systems in France and New Zealand. The first base-isolated building in the United States has been built in California, and there are plans to start construction in 1986 of a second base-isolated building in California. The nuclear industry is conducting studies of the potential application of base isolation to liquid-metal fast breeder reactors in the United States and to conventional reactors in Japan.

The simple elastomeric base isolation system is limited to low-rise structures, because of the possibility of uplift forces being generated in the isolator when the building is tall. Although the isolation system has the effect of reducing the *absolute horizontal acceleration* of the structure below that for the conventionally based structure, there will be some acceleration at each level of the building and these accelerations will, if the building is tall enough, produce an overturning moment which might produce tension in corner bearings. Furthermore, there have been proposals to use base isolation for large individual components in nuclear power plants to reduce the seismic hazard of these components, but many of these components are tall and slender and have thus the same problems for isolation as tall buildings.

These problems can be overcome by combining base isolation and active control.

The use of active control to reduce the damage to buildings and other structures caused by earthquakes has become an area of considerable theoretical interest in recent years. Control techniques applicable to the seismic problem have been developed although most of the research was directed to other environmental disturbances. The seismic problem has features which make it signicantly different from control of vibrations caused by, for example, wind loading. The loading may not occur for many years following a long period of quiescence. The nature of the disturbance (its intensity and frequency content for example) cannot be accurately predicted at the design stage.

In addition, the loading is applied to the structure at the base and for this reason many of the methods suggested or developed to provide control forces for wind and other types of excitation cannot be used. For example, the use of an active mass damper at the top of a building is not appropriate. Several of the control force methods which might be used are:

(a) the generation of internal control forces by the use of tendons;

(b) the generation of external control forces by, for example, jets at each floor level;

(c) the modification of structural stiffness by, for example, changing the stiffness of internal bracing.

At best all of these methods can be used to control the *relative acceleration* of the structure with respect to the ground. If they reduce the relative acceleration of each floor level to zero, then relative velocity and relative displacement will also be so reduced. Thus the interstory stress will be reduced to zero. However, the building will then follow the motion of the ground exactly and each floor will experience from the controller a force equal to its mass times the acceleration of the ground. This acceleration will of course be much less than the acceleration of each floor without control, for typical high-rise structures significantly amplify at the higher levels the acceleration at the base.

The peak ground acceleration will still have to be sustained by sensitive equipment in the buildings and by the occupants. The control forces needed will be of the order of the mass of the floor and the peak ground acceleration, and these could be unattainable for a practical system.

When the concept of active control is combined with seismic isolation where the structure is decoupled at the base from the ground acceleration, then the goal of the control system is to minimize the *absolute displacement and velocity*, and the control forces are needed to overcome only the forces which would be generated by the isolation system at the base of the structure. These forces are an order of magnitude less than those required by the schemes proposed heretofore.

2. Active Isolation Concept

It is the combination of base isolation (e.g., Refs. 1-4) with active control (e.g., Refs. 5-10) which makes this scheme so attractive, as we hope to show here.

The method for achieving earthquake-induced damage reduction discussed here differs from previously proposed ones in some important respects.

It is quite well known that base isolation can be efficacious in greatly reducing disturbances transmitted from the ground to the base (first) floor of a building and by reducing the natural frequency of the base floor. Indeed, essentially complete decoupling can be achieved in principle (say, by supporting the structure on ball bearings); see Appendix A. Clearly, such a scheme is not practical, since even small disturbances would result in motion of the structure; it would simply slide off the foundation. This, then, is where *active control* enters the picture, for it can be designed to employ information about the motion of the structure to activate forces to counteract this motion. Active control of structures has been proposed earlier (e.g., see Refs. 5–10); however, these proposals suffer from the drawback of requiring relatively large control forces at each floor of a building or for each mode of a structure. Such a requirement appears to be impractical. We may now ask: How does this requirement arise and how can it be obviated? The first part of the question can be answered readily; see Appendix B. It arises from the desire to keep the motion (displacement and velocity) of each floor relative to the ground small (and hence, of a given floor relative to those below and above it); if relative displacements and velocities are small, so are the internal stresses. Crudely speaking, in the previously proposed schemes, active control is employed to achieve that end by attempting to move the whole structure so as to follow the motion of the ground. Since relative coordinates are used in the system description, the *ground acceleration acts as a disturbance at each floor*; hence, *control forces must be applied at each floor* (see Appendix B). The second part of the question derives its answer from the realization that the philosophy of base isolation is one of keeping the whole structure stationary relative to its initial, undisturbed configuration (i.e., relative to an *inertial frame of reference*) and, again crudely speaking, letting the ground move under it. Thus, in this case, the appropriate description of the system is in terms of *absolute coordinates*, i.e., in inertial reference frame coordinates. Now, the ground-induced disturbance is in the form of ground displacement and velocity, and this *disturbance acts only at the base floor*; consequently, *control force need be applied only at the base floor* (see Appendix A). Furthermore, as will be shown, the maximum magnitude of the control force need not exceed the maximum magnitude of the disturbing force transmitted to the base floor in order to assure arbitrarily small motion (absolute displacement and velocity) of the base floor. Thus, base isolation is of great importance, since it permits one to make the maximum disturbance as small as desired. Finally, as will be seen, the possibility of reducing the natural frequency of the base floor by appropriate choice of the base isolation system's parameters (stiffness and damping), and hence the control force frequency, is vital in view of time delays engendered by the time constants of sensors and actuators, whose dynamics are usually neglected in the mathematical model of the system.

3. Controller Design

In order to illustrate the utility of active control in conjunction with base isolation, we consider a linear, lumped-parameter model of an N-story building in planar motion. We employ *absolute coordinates* in the description of the system. Thus, the governing equations are those of Appendix A.

Since the controller theory is based on a state space description of the system, it is convenient to introduce the state $x = (x_1, x_2, \ldots, x_{2N})^T$ of the system, where y_i denotes the absolute displacement of the ith floor, and y_0 that of the ground (see Appendix A, Fig. 15), and

$$x_i = \dot{y}_i,$$

$$x_{i+N} = y_i, \qquad i = 1, 2, \ldots, N.$$

The equations for the uncontrolled system (7) can then be written as

$$\dot{x}(t) = Ax(t) + Cv(t), \tag{1}$$

where

$$A = \left[\begin{array}{ccc|ccc} (-c_0 - c_1)/m_1 & c_1/m_1 & & (-k_0 - k_1)/m_1 & k_1/m_1 & \\ \ddots\, c_{i-1}/m_i & (-c_{i-1} - c_i)/m_i & c_i/m_i \ddots & \ddots\, k_{i-1}/m_i & (-k_{i-1} - k_i)/m_i & k_i/m_i \ddots \\ & c_{N-1}/m_N & -c_{N-1}/m_N & & k_{N-1}/m_N & -k_{N-1}/m_N \\ \hline & I_{N\times N} & & & O_{N\times N} & \end{array}\right],$$

$$C = \left[\begin{array}{cc} k_0/m_1 & c_0/m_1 \\ \multicolumn{2}{c}{O_{N-1\times 2}} \\ \hline \multicolumn{2}{c}{O_{N\times 2}} \end{array}\right],$$

$$v(t) = \begin{bmatrix} y_0(t) \\ \dot{y}_0(t) \end{bmatrix}.$$

Now, if a control force $u_1(t)$ is applied to the base (first) floor, the *controlled system* equation becomes

$$\dot{x}(t) = Ax(t) + Bu_1(t) + Cv(t), \tag{2}$$

with

$$B = \begin{bmatrix} 1/m_1 \\ O_{N-1\times 1} \\ \hdashline O_{N\times 1} \end{bmatrix}.$$

This equation is of the form (9a) of Appendix C, with

$$\Delta A(r) = \Delta B(s) \equiv 0.$$

Here, the *uncertain input* $v(t)$ is due to the unknown ground motion. In conformity with the controller theory summarized in Appendix C, we assume $v(t) \in \mathscr{V}$, a known compact set. This assumption is reasonable, since maximum values of ground displacement $y_0^{\max}$ and velocity $\dot{y}_0^{\max}$ are known for the worst earthquakes on record. Thus,

$$\mathscr{V} = \{v \in R^2 \,|\, |v_1| \leq y_0^{\max}, |v_2| \leq \dot{y}_0^{\max}\}. \tag{3}$$

It is readily verified that the other assumptions of Appendix C are satisfied; that is, the matching conditions are met with

$$F = [k_0, c_0],$$

and $\bar{A} = A$ is an appropriate choice, since A is stable.

Thus, the control that assures practical stability (see Appendix C) of system (2), for all disturbances $v(\cdot)$ whose values range in $\mathscr{V}$, given $\epsilon > 0$, is

$$u_1 = p_\epsilon(x) = \begin{cases} (-B^T Px/|B^T Px|)\rho, & \text{if } |B^T Px| \geq \epsilon, \\ (-B^T Px/\epsilon)\rho, & \text{if } |B^T Px| < \epsilon, \end{cases} \tag{4}$$

where P is the solution of

$$PA + A^T P + Q = 0, \tag{5}$$

for given $Q > 0$, and

$$\rho = \max_{v \in \mathscr{V}} \|Fv\| = \sqrt{(k_0 y_0^{\max})^2 + (c_0 \dot{y}_0^{\max})^2}. \tag{6}$$

Note that the *scalar control* given in (4) is a saturation control whose value cannot exceed ρ; indeed, if it is not saturated, i.e., if $|B^T Px| < \epsilon$, it is a linear feedback control. Note also that, given the disturbance bounds $y_0^{\max}$ and $\dot{y}_0^{\max}$, the maximum of the control ρ can be reduced by appropriate base isolation design, that is, by reducing k_0 and c_0. The control given by (4) can be readily implemented, since A and B are known *a priori*. Information about the state x can be obtained as a function of time during the earthquake excitation.

4. Simulation

In order to investigate the efficacy of the proposed control, prior to the implementation of an experimental program, a simulation program was carried out. The mathematical model is a linear, lumped-parameter one for a laboratory prototype of a six-story building; see Refs. 11 and 12. The computation of the spring and damping coefficients, listed in Table 1, can be found in Ref. 13.

The base isolation ststem coefficients deduced in Ref. 13 and used here are

$$k_0 = 1200 \text{ kN/m}, \qquad c_0 = 2.4 \text{ kNs/m}.$$

The floor masses are

$$m_1 = 6800 \text{ kg}, \qquad \text{base floor},$$

$$m_i = 5897 \text{ kg}, \qquad i = 2, 3, \ldots, 6.$$

The simulated ground motion is that of the 1940 El Centro earthquake; Figures 1-3 show ground acceleration, velocity, and displacement, respectively. The assumed maximum values of ground velocity and displacement are

$$\dot{y}_0^{\max} = 0.35 \text{ m/s}, \qquad y_0^{\max} = 0.11 \text{ m},$$

determining the maximum value of the control (4). Furthermore, the P matrix in the control is based on $Q = I_{12\times 12}$ in Eq. (5).

Finally, simulation results are presented for two values of the design parameter ϵ, $\epsilon = 0.001$ and $\epsilon = 0.0001$.

Figures 4a-4f and 5a-5f show the velocity and displacement histories of the six floors of the *base-isolated but uncontrolled building* under the assumed earthquake excitation. We note that the responses of the floors are essentially identical. This is not unexpected in view of the relatively high stiffness of the building vis-a-vis that of the base isolation system. In other words, the base-isolated system moves essentially as a single unit.

Table 1. Model coefficients.

Spring coefficient (KN/m)	Damping coefficient (KNs/m)
$k_1 = 33{,}732$	$c_1 = 67$
$k_2 = 29{,}093$	$c_2 = 58$
$k_3 = 28{,}621$	$c_3 = 57$
$k_4 = 24{,}954$	$c_4 = 50$
$k_5 = 19{,}059$	$c_5 = 38$

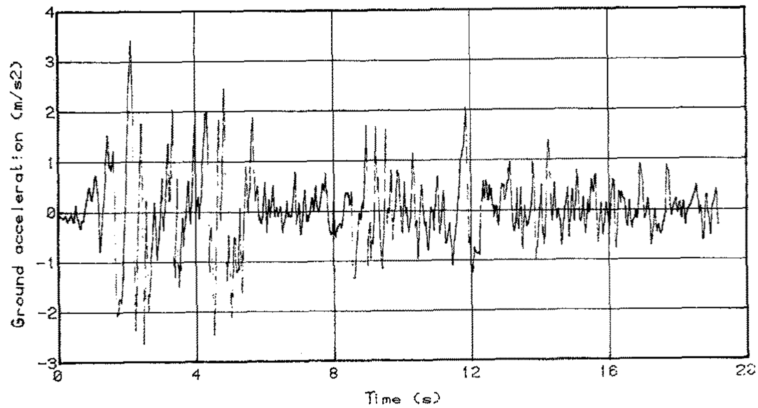

Fig. 1. El Centro earthquake, ground acceleration.

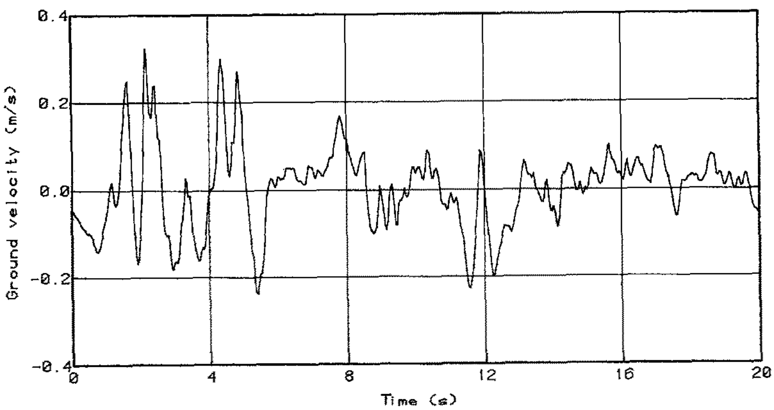

Fig. 2. El Centro earthquake, ground velocity.

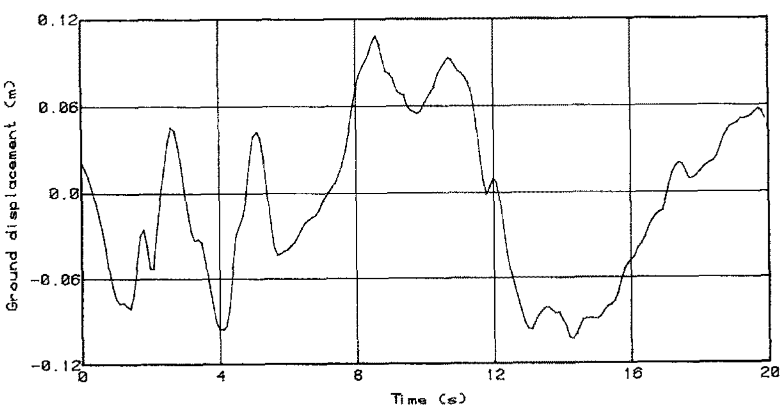

Fig. 3. El Centro earthquake, ground displacement.

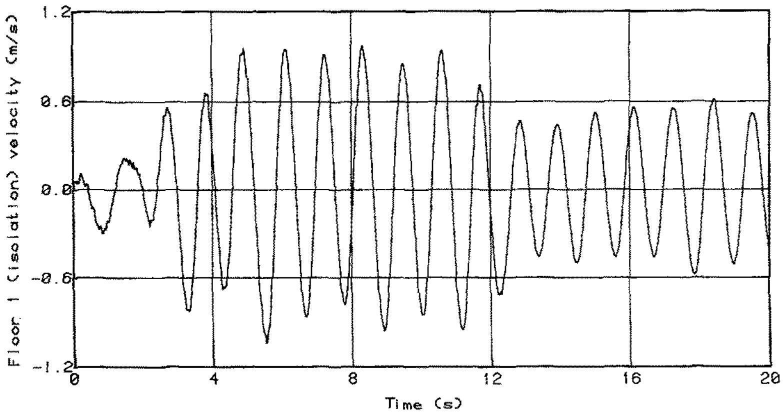

Fig. 4a. Uncontrolled first floor velocity.

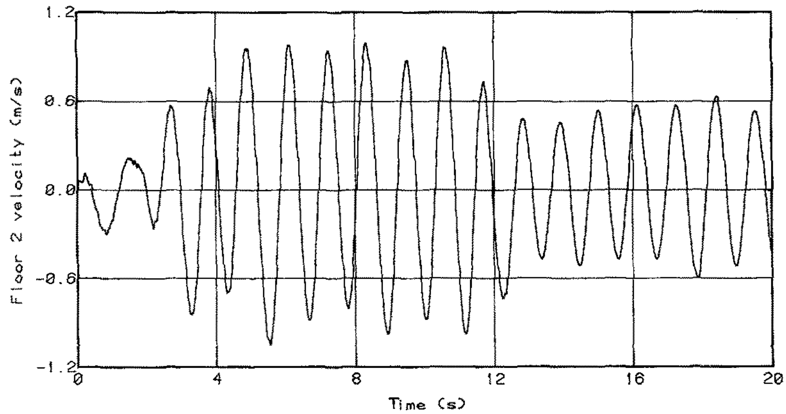

Fig. 4b. Uncontrolled second floor velocity.

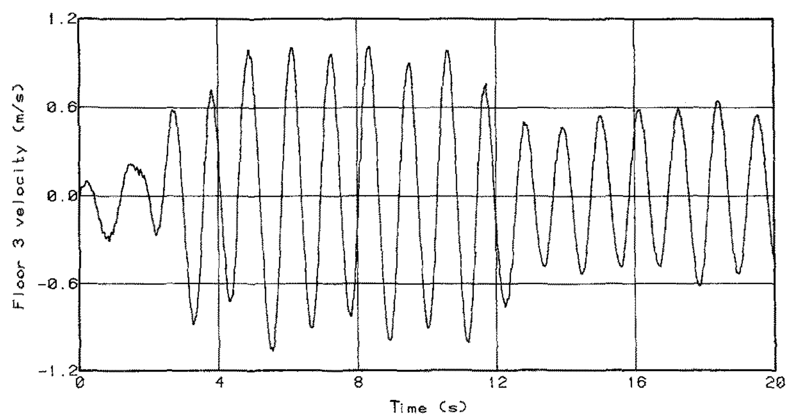

Fig. 4c. Uncontrolled third floor velocity.

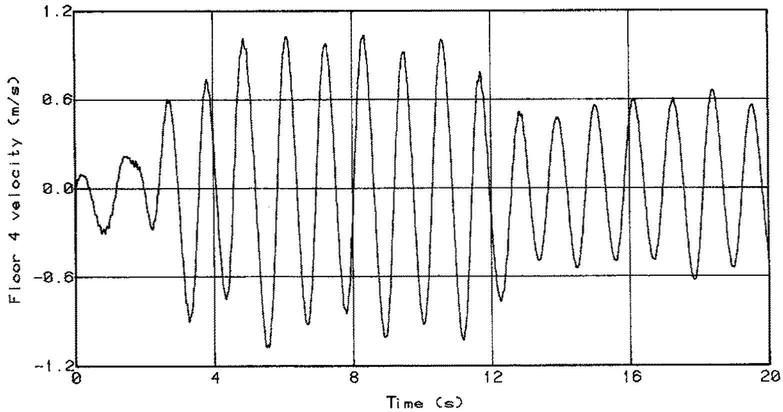

Fig. 4d. Uncontrolled fourth floor velocity.

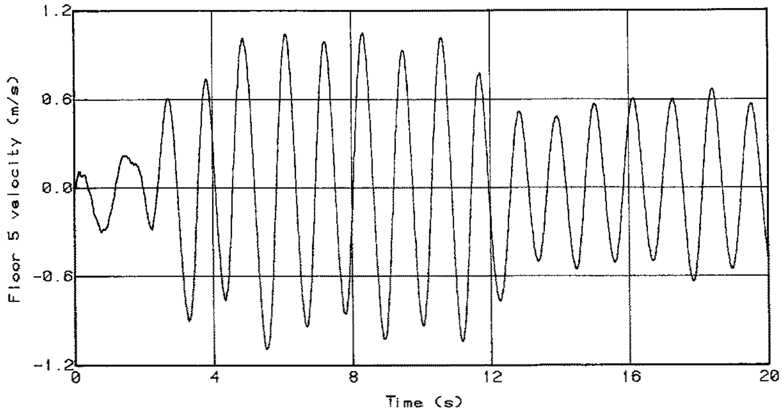

Fig. 4e. Uncontrolled fifth floor velocity.

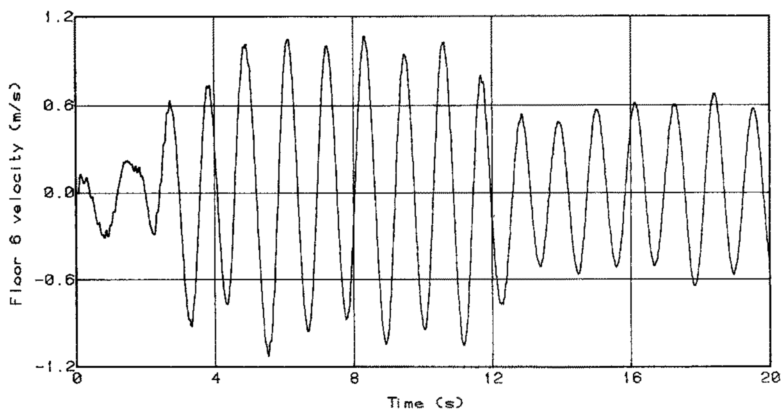

Fig. 4f. Uncontrolled sixth floor velocity.

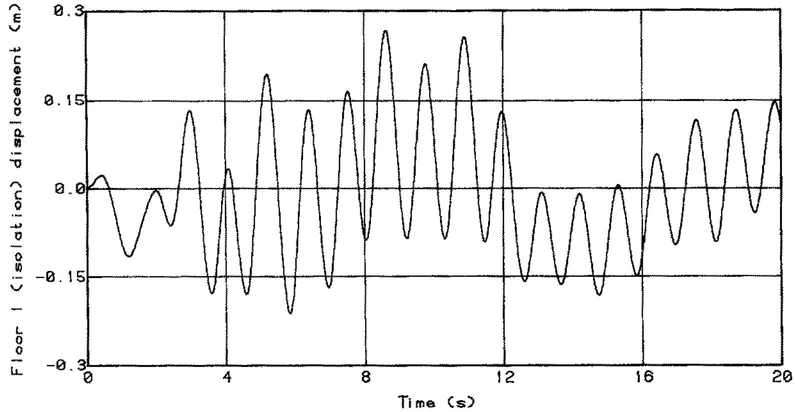

Fig. 5a Uncontrolled first floor displacement.

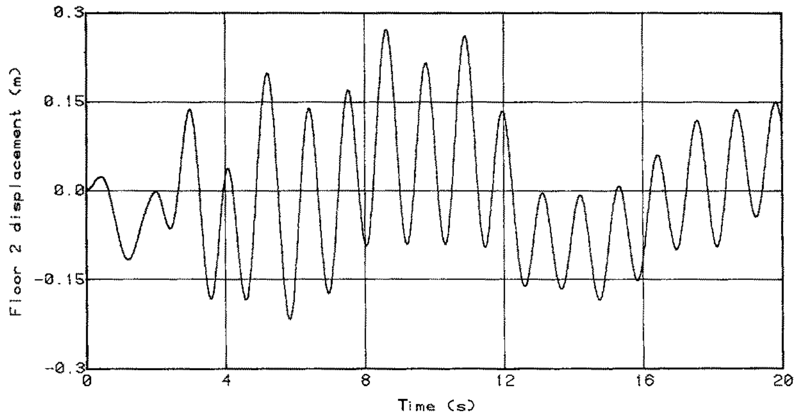

Fig. 5b. Uncontrolled second floor displacement.

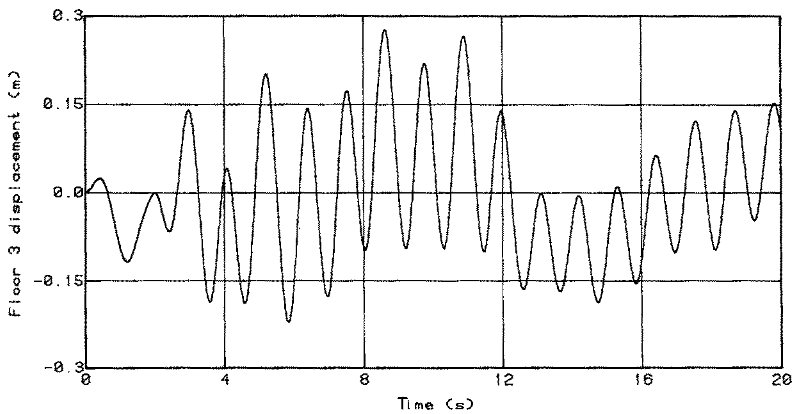

Fig. 5c. Uncontrolled third floor displacement.

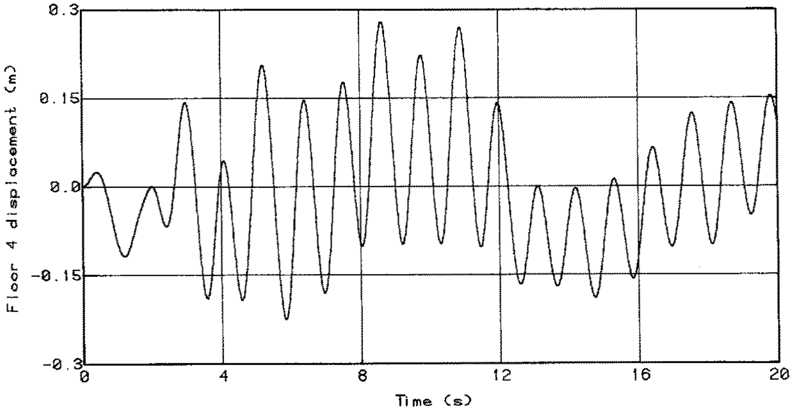

Fig. 5d. Uncontrolled fourth floor displacement.

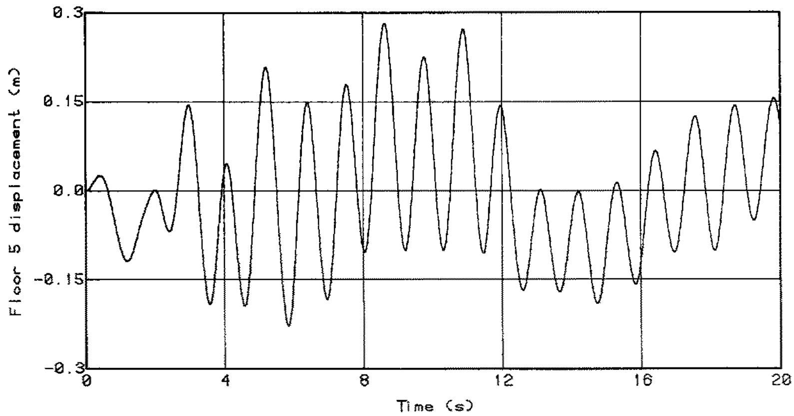

Fig. 5e. Uncontrolled fifth floor displacement.

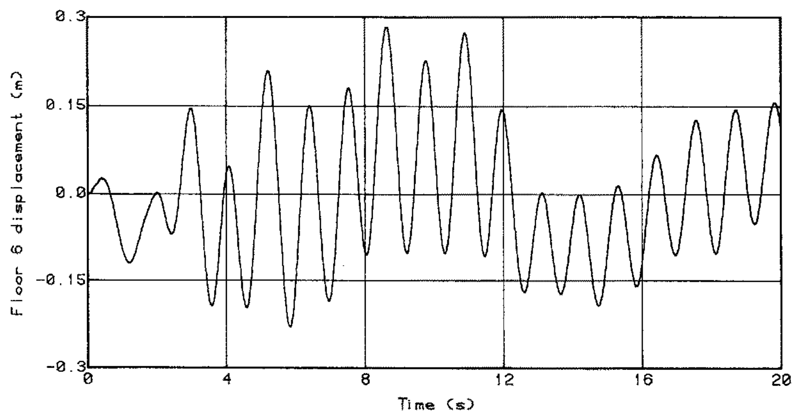

Fig. 5f. Uncontrolled sixth floor displacement.

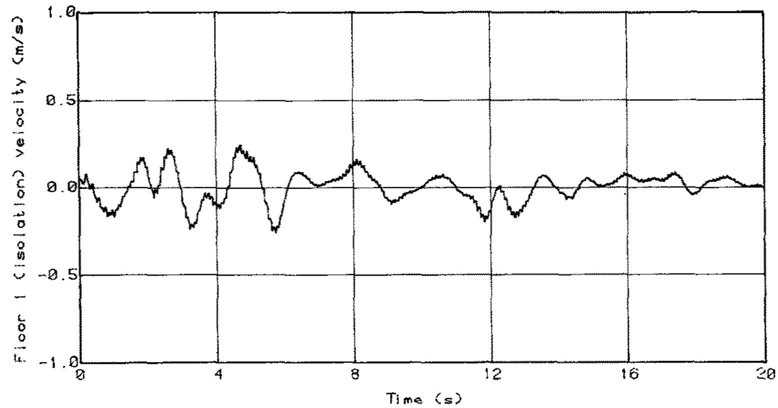

Fig. 6. Controlled ($\epsilon = 0.001$) first floor velocity.

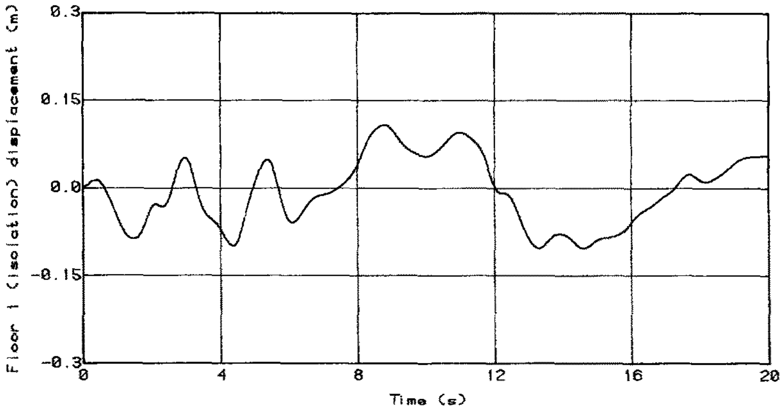

Fig. 7. Controlled ($\epsilon = 0.001$) first floor displacement.

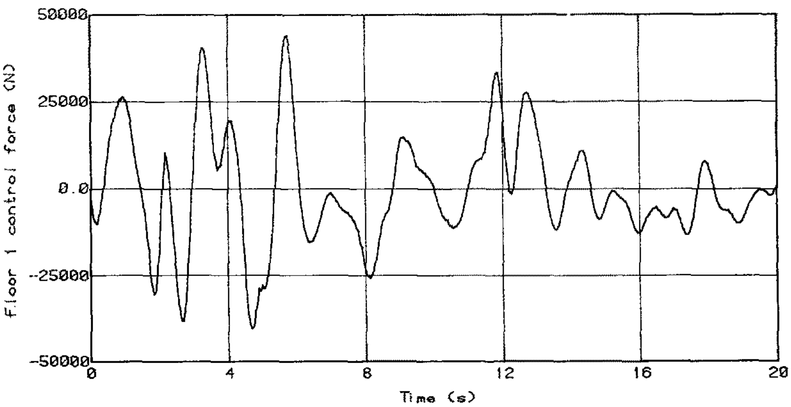

Fig. 8. Control force ($\epsilon = 0.001$).

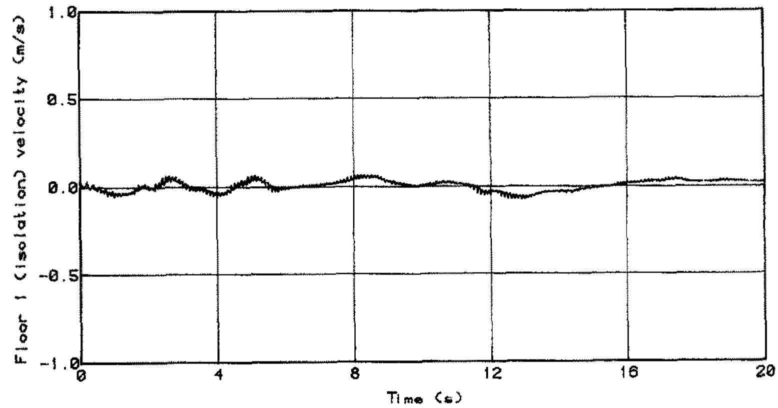

Fig. 9. Controlled ($\epsilon = 0.0001$) first floor velocity.

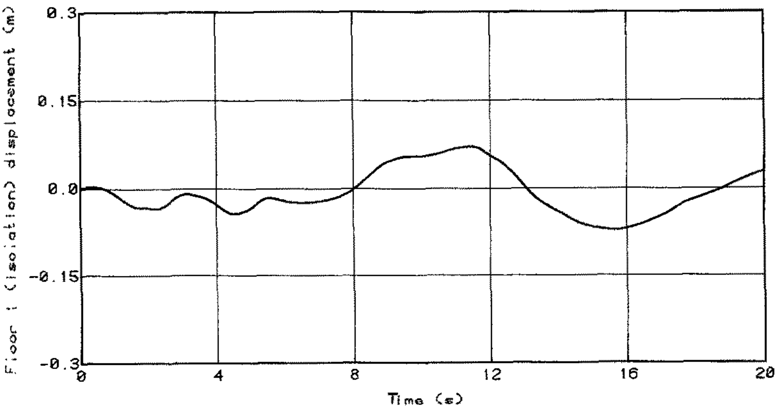

Fig. 10. Controlled ($\epsilon = 0.0001$) first floor displacement.

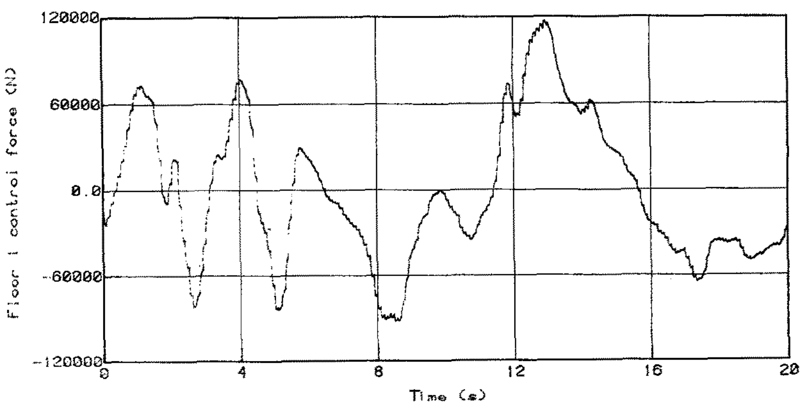

Fig. 11. Control force ($\epsilon = 0.0001$).

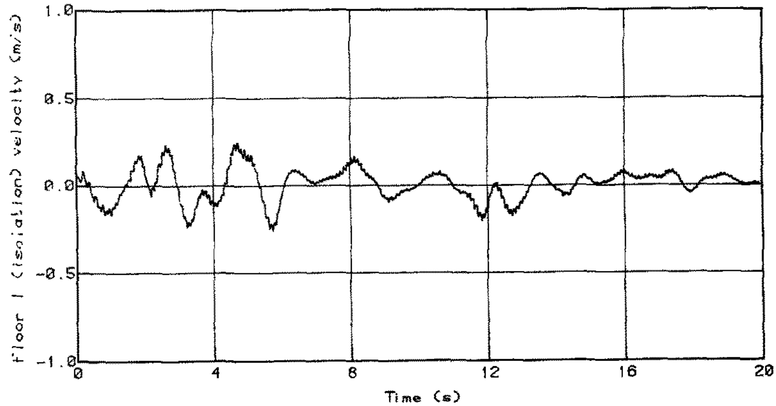

Fig. 12. Delayed state ($\Delta = 0.1$ sec), controlled ($\epsilon = 0.001$) first floor velocity.

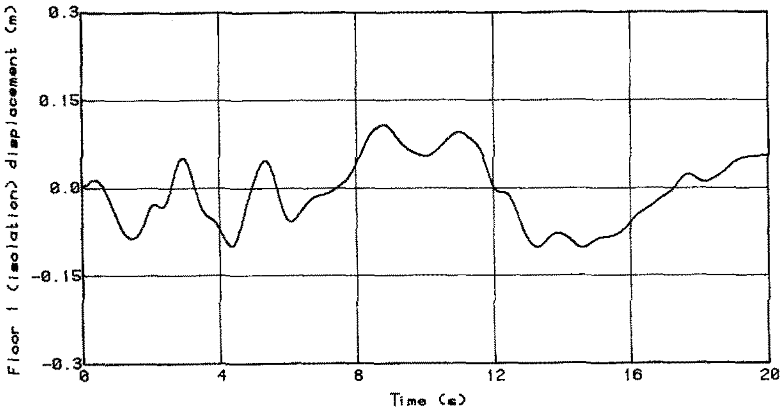

Fig. 13. Delayed state ($\Delta = 0.1$ sec), controlled ($\epsilon = 0.001$) first floor displacement.

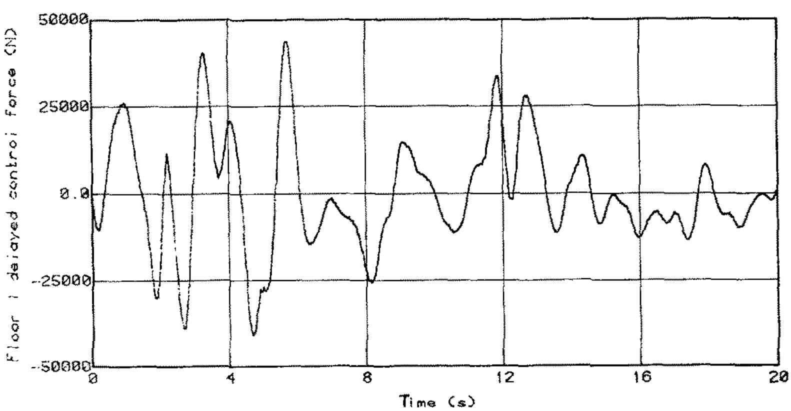

Fig. 14. Delayed state ($\Delta = 0.1$ sec), control ($\epsilon = 0.001$).

Figures 6 and 7 show the velocity and displacement histories of the first floor of the *base-isolated and controlled building* under the assumed earthquake excitation, for $\epsilon = 0.001$. Figure 8 shows the corresponding control force $u_1(t)$. There is considerable improvement over the uncontrolled situation, both in amplitudes and frequencies of the responses. The property of the floors moving in unison is preserved. This latter property is very important, since it obviates having to determine the state of the total system (twelve variables); it suffices to determine the velocity x_1 and displacement x_7 of the base floor.

Figures 9, 10, and 11 show the velocity, displacement (again only for the first floor), and force histories for $\epsilon = 0.0001$. As expected, further improvement in the responses results, albeit at the expense of a larger control force.

In view of the property of the floors moving in unison, setting $x_i = x_1$, $i = 2, \ldots, 6$, and $x_i = x_7$, $i = 8, \ldots, 12$, in control (4) should have no effect; simulation results (not presented here, but in Ref. 13) bear this out.

Finally, as already mentioned, the sensors and actuators (to determine the system's state and to implement the control) possess their own dynamics, neglected in the mathematical model. In effect, the state $x(t)$ required for the control $u_1(t) = p_\epsilon(x(t))$ is not available; rather, what is available at time t is a retarded state $x(t-\Delta)$, where $\Delta > 0$ is a delay due to the neglected system dynamics. Provided this delay is sufficiently small, the assured practical stability is not vitiated, except for a possible deterioration in the response bounds; e.g., see Ref. 14. To illustrate this, a delay of $\Delta = 0.1$ sec was introduced in the state upon which the control is based. As expected, in view of the relatively low response frequencies, no discernible effect on the response is detected. This is shown in Figs. 12, 13, and 14 for the case of $\epsilon = 0.001$.

Other simulation results may be found in Ref. 13.

5. Conclusions

We have proposed the combination of base isolation with active control of structures under earthquake excitation. In contradistinction to earlier proposals for active control of earthquake-excited structures, the scheme proposed here makes use of the advantages introduced by base isolation by seeking to keep the building stationary relative to its undisturbed configuration, rather than attempting to move it with the ground in order to keep relative motion small, which requires comparatively large control forces on each floor. Thus, the method of this paper permits one to utilize base isolation to reduce the control force which is now required only at the base floor.

A few words are in order concerning the determination of the required information for the implementation of the control, namely, the state of the system; in essence, that means the determination of the *absolute velocity and position* of the base floor. The absolute acceleration of the base floor is measured readily by means of an accelerometer. Velocity and displacement are then obtainable by integration, provided their initial values are known. In the simulation, we have taken these to be zero, $x(0) = 0$, supposing that the initial earthquake shock is used to trigger the measurement and integration process before the building responds to the excitation. This appears to be reasonable.

Of course, the illustrative example is confined to planar motion, and this suffices for laboratory tests. Before a scheme, such as the one proposed here, can be implemented in the field, mathematical models incorporating the effects of nonplanar motion must be employed. Finally, the possibility of uncertainty in the parameter values must be explored. While the control scheme of Appendix C allows for uncertainty in the system parameters [that is, $\Delta A(r) \neq 0$, $\Delta B(s) \neq 0$], it does so at the expense of more control requirements; in particular, controlling for uncertainty in k_i and c_i, $i = 1, 2, \ldots, N$, requires control forces at more than just the base floor.

6. Appendix A

Consider a linear, lumped-parameter model of an N-story building in planar motion; see Fig. 15. Let y_0 be the displacement of the ground, y_i be the displacement of the ith floor, relative to an *inertial reference frame*; let m_i be the mass of the ith floor, c_{i-1} be the damping coefficient, and k_{i-1} be the spring constant in the connection of the ith floor to the floor, respectively ground, below it. Finally, let a dot denote differentiation with respect to time t. Here, we shall delete the argument t, for the sake of brevity.

Then,

$$
\begin{aligned}
m_1\ddot{y}_1 &= -c_0(\dot{y}_1 - \dot{y}_0) - k_0(y_1 - y_0) + c_1(\dot{y}_2 - \dot{y}_1) + k_1(y_2 - y_1), \\
m_i\ddot{y}_i &= -c_{i-1}(\dot{y}_i - \dot{y}_{i-1}) - k_{i-1}(y_i - y_{i-1}) + c_i(\dot{y}_{i+1} - \dot{y}_i) + k_i(y_{i+1} - y_i), \qquad (7) \\
m_N\ddot{y}_N &= -c_{N-1}(\dot{y}_N - \dot{y}_{N-1}) - k_{N-1}(y_N - y_{N-1}).
\end{aligned}
$$

Thus, we see that, in this description (i.e., in terms of *absolute coordinates* y_i), the ground-motion-induced disturbance $c_0\dot{y}_0 + k_0y_0$ affects only the first (base) floor. This disturbance involves the ground displacement y_0 and velocity $\dot{y}_0$. It can be reduced by the design of the base isolation system, i.e., by reducing c_0 and k_0.

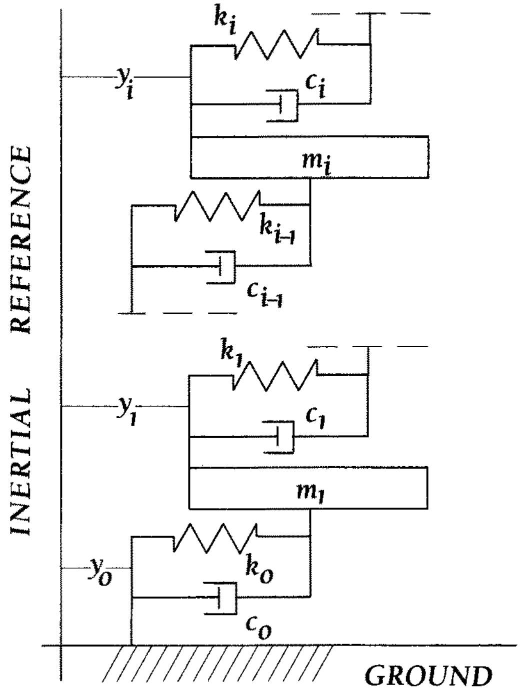

Fig. 15. Lumped parameter model, absolute coordinates.

7. Appendix B

Consider a linear, lumped-parameter model of an N-story building in planar motion; see Fig. 16. Let y_0 be the ground displacement relative to an inertial reference frame and z_i be the displacement of the ith floor relative to the ground. All other quantities are as defined in Appendix A.

Then,

$$m_1(\ddot{y}_0+\ddot{z}_1)=-c_0\dot{z}_1-k_0z_1+c_1(\dot{z}_2-\dot{z}_1)+k_1(z_2-z_1),$$

$$m_i(\ddot{y}_0+\ddot{z}_i)=-c_{i-1}(\dot{z}_i-\dot{z}_{i-1})-k_{i-1}(z_i-z_{i-1})+c_i(\dot{z}_{i+1}-\dot{z}_i)+k_i(z_{i+1}-z_i), \quad (8)$$

$$m_N(\ddot{y}_0+\ddot{z}_N)=-c_{N-1}(\dot{z}_N-\dot{z}_{N-1})-k_{N-1}(z_N-z_{N-1}).$$

Thus, we see that in this description (i.e., in terms of coordinates *relative* to the ground z_i), the ground-motion-induced disturbance $m_i\ddot{y}_0$ affects every floor. This disturbance involves the ground acceleration $\ddot{y}_0$; it cannot be reduced by the design of the base isolation system.

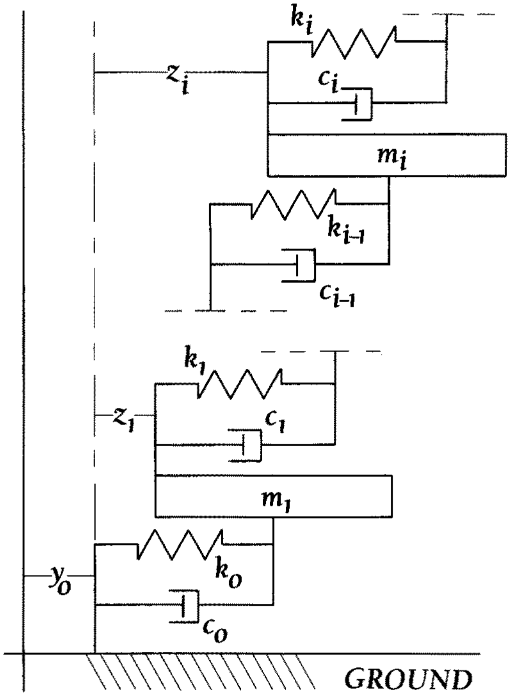

Fig. 16. Lumped parameter model, relative coordinates.

8. Appendix C

Consider a dynamical system

$$\dot{x}(t) = [A + \Delta A(r(t))]x(t) + [B + \Delta B(s(t))]u(t) + Cv(t), \tag{9a}$$

$$x(t_0) = x_0, \tag{9b}$$

where

$$x \in R^n, \quad u \in R^m, \quad v \in R^l, \quad r \in R^p, \quad s \in R^q,$$

and

$$A \in R^{n \times n}, \quad B \in R^{n \times m}, \quad C \in R^{n \times l}$$

are known constant matrices, and

$$\Delta A(\cdot): R^p \to R^{n \times n}, \quad \Delta B(\cdot): R^l \to R^{n \times m}$$

are known continuous functions.

Uncertainties in the system matrix, input matrix, and input, respectively, are modelled by the unknown Lebesgue measurable functions

$$r(\cdot): R \to \mathcal{R}, \quad s(\cdot): R \to \mathcal{S}, \quad v(\cdot): R \to \mathcal{V},$$

where $\mathcal{R}$, $\mathcal{S}$, $\mathcal{V}$ are known compact subsets of the appropriate spaces. Thus, the only information concerning the unknown elements $r(t)$, $s(t)$, $v(t)$ resides in their bounding sets $\mathcal{R}$, $\mathcal{S}$, $\mathcal{V}$.

Concerning system (9a)-(9b), we assume that the uncertainties belong to the range space of the input matrix B or, more precisely, there exist continuous functions $D(\cdot): R^p \to R^{m\times n}$, $E(\cdot): R^l \to R^{m\times m}$ and constant matrix $F \in R^{m\times l}$ such that the so-called matching conditions are met; namely,[5]

$$\Delta A(r) = BD(r), \qquad \forall r \in \mathcal{R}, \tag{10a}$$

$$\Delta B(s) = BE(s), \qquad \forall s \in \mathcal{S}, \tag{10b}$$

$$C = BF, \tag{10c}$$

and

$$\max_{s\in\mathcal{S}} \|E(s)\| < 1. \tag{11}$$

Furthermore, we assume that the matrix pair (A, B) is stabilizable; that is, there exists a constant matrix $K \in R^{m\times n}$ such that $\bar{A} = A + BK$ is stable.[6] Of course, (A, B) is stabilizable if it is controllable; e.g., see Ref. 15.

Before proceeding, we state two desirable properties to be achieved for the uncertain system (9a)-(9b) by appropriate choice of control $u(t)$.

Property P1. *Uniform Boundedness.* Given $x_0 \in R^n$, there is a positive $d(x_0) < \infty$ such that, for all solutions $x(\cdot): [t_0, t_1) \to R^n$, $x(t_0) = x_0$,

$$\|x(t)\| \leq d(x_0), \qquad \forall t \in [t_0, t_1).$$

Property P2. *Uniform Ultimate Boundedness.* Given $x_0 \in R^n$ and $S = \{x \in R^n \mid \|x\| \leq \delta > 0\}$, there is a nonnegative $T(x_0, S) < \infty$ such that, for all solutions $x(\cdot): [t_0, \infty) \to R^n$, $x(t_0) = x_0$,

$$x(t) \in S, \qquad \forall t \geq t_0 + T(x_0, S).$$

Loosely speaking, uniform boundedness implies that every solution emanating from initial state x_0 remains within a bounded neighborhood whose radius may depend on x_0. Uniform ultimate boundedness implies that every solution starting at x_0 will enter and remain within a neighborhood of prescribed radius δ after a finite time which may depend on x_0 and δ. These two properties, sometimes stated in a slightly different but equivalent form, are the main ingredients of *practical stability*; e.g., see Refs. 16-18.

---

[5] While $\|\cdot\|$ may be any norm, it is convenient to deal with the Euclidean norm of vectors and the corresponding matrix norm.

[6] The real parts of the eigenvalues of $\bar{A}$ are strictly negative.

Now, consider a control

$$u = Kx + p_\epsilon(x) \tag{12}$$

such that, given $\epsilon > 0$, there holds[7]

$$p_\epsilon(x) = \begin{cases} (-B^T Px/\|B^T Px\|)\rho(x), & \text{if } \|B^T Px\| \geqslant \epsilon, \\ (-B^T Px/\epsilon)\rho(x), & \text{if } \|B^T Px\| < \epsilon, \end{cases} \tag{13}$$

where $P \in R^{n\times n}$ is the symmetric, positive-definite solution of

$$P\bar{A} + \bar{A}^T P + Q = 0 \tag{14}$$

for given symmetric, positive definite $Q \in R^{n\times n}$, and

$$\rho(x) = [1 - \max_{s\in\mathcal{S}} \|E(s)\|]^{-1}[\max_{r\in\mathcal{R}} \|D(r)x\|$$

$$+ \max_{s\in\mathcal{S}} \|E(s)Kx\| + \max_{v\in\mathcal{V}} \|Fv\|]. \tag{15}$$

Now, provided the stated assumptions are met, *control* (12) *guarantees practical stability for uncertain system* (9*a*)-(9*b*) and, in particular, Properties P1 and P2 for every possible realization of uncertain elements $r(\cdot)$, $s(\cdot)$, and $v(\cdot)$; e.g., see Ref. 16, where one can also find explicit expressions for $d(x_0)$ and $T(x_0, S)$. Here, it should be noted that $\delta = \delta(\epsilon)$ and can be made arbitrarily small by choice of ϵ; namely, decreasing ϵ results in a decrease of the radius of the ultimate boundedness set. Furthermore, if the initial state belongs to the Lyapunov ellipsoid which defines the ultimate boundedness set, the whole solution remains within it.

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[7] Superscript T denotes transpose.

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