A Differential Game of Approach with Two Pursuers and One Evader[1]

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Communicated by N. V. Banichuk

Abstract. A differential game of approach with one evader and two pursuers with a nonconvex payoff function is considered. The duration of the game is fixed. The payoff functional is the distance between the object being pursued and the pursuer closest to it when the game terminates. An explicit form of the game value is found for all possible game positions. The paper is closely related to Refs. 1-12.

Key Words. Value function, programmed maximin function, fundamental equation, singular surfaces.

1. Introduction

Differential games of approach-evasion with many players have attracted the growing attention of specialists (Refs. 6-12). In the present paper, we consider a concrete differential game of approach with a nonconvex payoff function. Such a kind of payoff function occurs frequently in many applied problems; however, the solution process can meet well-known mathematical difficulties.

In the differential game under consideration, the value function coincides with the programmed maximin function, which is a continuous, piecewise-smooth function, made up of four smooth functions. In the game, there is a singular surface on which the value function is nondifferentiable. This fact does not allow the use of the Bellman-Isaacs fundamental equation. In proving the fact that the piecewise-smooth function of programmed maximin is the value of the game under consideration, we use a

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generalization of the Bellman-Isaacs fundamental equation (see Refs. 4 and 5).

2. Problem Formulation

The motion of the pursuers $P(y^{(i)})$ is described by the equations

$$\dot{y}_1^{(i)} = u_1^{(i)}, \qquad \dot{y}_2^{(i)} = u_2^{(i)}, \tag{1a}$$

$$((u_1^{(i)})^2 + (u_2^{(i)})^2)^{1/2} \leq \mu, \qquad \mu > 0, \quad i = 1, 2. \tag{1b}$$

The evader $E(z)$ moves in accordance with the equations

$$\dot{z}_1 = v_1, \qquad \dot{z}_2 = v_2, \tag{2a}$$

$$(v_1^2 + v_2^2)^{1/2} \leq \nu, \qquad \nu > \mu. \tag{2b}$$

Here, $u^{(i)}$, v are the control vectors. The time ϑ at which the game ends is fixed. The game payoff σ is the distance between the object being pursued and the pursuer closest to it at the instant ϑ; i.e.,

$$\sigma = \min_{i=1,2} [(z_1(\vartheta) - y_1^{(i)}(\vartheta))^2 + (z_2(\vartheta) - y_2^{(i)}(\vartheta))^2]^{1/2}. \tag{3}$$

Strategies and motions of players are defined in accordance with Ref. 2.

Suppose that, at the initial instant t_0, the pursuers's coordinates do not coincide. In a plane, we set up a fixed rectangular system of coordinates with axes q_1, q_2. We direct the abscissa axis q_1 from the initial position of the first pursuer $P_1^0(y_0^{(1)})$ to the initial position of the second pursuer $P_2^0(y_0^{(2)})$. We direct the ordinate axis q_2 through the midpoint of the segment $[P_1^0 P_2^0]$, perpendicular to it, so as to obtain a right-oriented system of coordinates (see Fig. 1). The domain of attainability $G^{(i)}(t, y^{(i)}, \vartheta)$ of the pursuers P_i, $i = 1, 2$, from the position $\{t, y^{(i)}(t)\}$ at the instant ϑ is a circle of radius

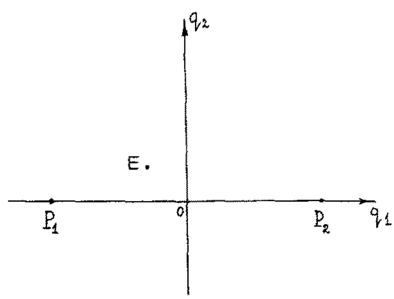

Fig. 1

$r(t)=\mu(\vartheta-t)$, with center at the point $\{y^{(i)}(t)\}$. The domain of attainability $G(t, z, \vartheta)$ of the evader E from the position $\{t, z(t)\}$ is a circle of radius $R(t)=\nu(\vartheta-t)$, with center at the point $\{z(t)\}$. Due to the choice of the coordinate system, the pursuers's positions $P_i\{y_1^{(i)}(t_0), y_2^{(i)}(t_0)\}$, $i=1, 2$, at the initial instant t_0 are such that

$$y_1^{(1)}(t_0)=-y_1^{(2)}(t_0), \qquad y_2^{(1)}(t_0)=y_2^{(2)}(t_0).$$

Suppose that, at the instant t_0, the evader E is at the position $\{z_1(t_0), z_2(t_0\}$ and the attainability domain $G(t_0, z(t_0), \vartheta)$ of the evader intersects the axis q_2 at the points $A_1(0, d_1)$ and $A_2(0, d_2)$ (see Fig. 2), with

$$d_{1,2}=z_2(t_0)\pm(R^2(t_0)-z_1^2(t_0))^{1/2}. \tag{4}$$

We see that the distance between the pursuers P_i and the points A_1, A_2 satisfy the following relations:

$$\operatorname{sign}(|P_iA_1|-|P_iA_2|)=\operatorname{sign}(z_2(t_0)-y_2^{(i)}(t_0)).$$

It can be shown that the optimal programmed strategy for E to evade the pursuers P_i at the instant ϑ from an initial position $\{t_0, z(t_0)\}$ located in the quadrangle $P_1^0A_1P_2^0A_2$ (see Fig. 2) will be the extremal control $v(t)$, $t_0\leq t<\vartheta$, directed toward the point A_1 if

$$z_2(t_0)-y_2^{(i)}(t_0)>0$$

and directed toward the point A_2 if

$$z_2(t_0)-y_2^{(i)}(t_0)<0.$$

The points for which

$$z_2(t)=y_2^{(i)}(t), \quad |z_1(t)|\leq|y_1^{(i)}(t)|, \qquad t_0\leq t<\vartheta, \tag{5}$$

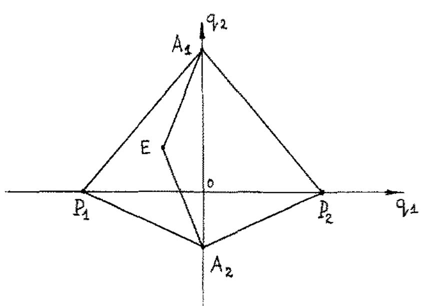

Fig. 2

form a singular set S. The two extremal aiming points A_1 and A_2 correspond to points of the set S. In this case, the optimal programmed evasion strategy of E consists of the two extremal controls $v_1(t)$ and $v_2(t)$, $t_0 \leq t < \vartheta$, directed toward the points A_1 and A_2, respectively. Player E can choose any one of them.

3. Programmed Maximin Function

First, let us determine the programmed maximin function γ_* for the differential game (1)-(3). We introduce the notation

$$\psi(R, z_1) = (\max(0, R^2(t) - z_1^2))^{1/2},$$

$$Q = \{(t, x): t_0 \leq t < \vartheta, \psi(R, Z_1)|z_1|^{-1} \geq (\psi(R, z_1) + |z_2 - y_2^{(1)}|)|y_1^{(1)}|^{-1}\}.$$

If $(t, x) \in Q$, then the programmed maximin function has the form

$$\gamma_* = \max_{v(t)} \min_{u(t)} \sigma = \max(\gamma_1, \gamma_2), \tag{6a}$$

$$\gamma_j(t, x) = ((y_2^{(1)} - d_j)^2 + (y_1^{(1)})^2)^{1/2} - \mu(\vartheta - t), \qquad j = 1, 2. \tag{6b}$$

If $(t, x) \notin Q$ (in particular, this case holds if the attainability domain of E does not intersect the q_2 axis), the game (1)-(3) degenerates into a game with one evader and one pursuer, considered in Ref. 2. In this case, we find that the programmed maximin function has the form

$$\gamma_* = \min(\gamma_3, \gamma_4), \tag{7a}$$

$$\gamma_{i+2}(t, x) = ((z_1 - y_1^{(i)})^2 + (z_2 - y_2^{(i)})^2)^{1/2} + (\nu - \mu)(\vartheta - t), \qquad i = 1, 2. \tag{7b}$$

Finally, if $|P_1 P_2| = 0$, then

$$\gamma_* = ((z_1 - y_1^{(1)})^2 + (z_2 - y_2^{(1)})^2)^{1/2} + (\nu - \mu)(\vartheta - t). \tag{8}$$

It can be verified that, for positions $(t, x) \in \partial Q$, the functions (6) and (7) are equal together with all of their derivatives. Thus, the expressions (6) and (7) define a continuous function γ_*, continuously differentiable for $t_0 \leq t < \vartheta$, everywhere except on the set S.

Below, we shall prove that the programmed maximin function γ_* is identical with the value of the differential game (1)-(3). When proving this fact, we take advantage of the following circumstance. Consider a differential game in which (1) is replaced by the equations

$$\dot{y}_1^{(1)} = u_1, \qquad \dot{y}_2^{(1)} = u_2, \qquad \dot{y}_1^{(2)} = -u_1, \qquad \dot{y}_2^{(2)} = u_2; \tag{9}$$

i.e., in (1), we set

$$u_1^{(1)} = -u_1^{(2)} = u_1, \qquad u_2^{(1)} = u_2^{(2)} = u_2.$$

If we compute the programmed maximin function γ_{**} for the differential game (9), (2), (3), we obtain

$$\gamma_{**} = \gamma_*. \tag{10}$$

Let ρ_* be the value of the differential game (1)-(3), and let ρ_{**} be the value of the differential game (9), (2), (3). In the game (9), (2), (3), there are additional constraints on the first pursuer. There,

$$\rho_* \leq \rho_{**}. \tag{11}$$

It is well known that the value of a differential game and the programmed maximin function are related by the inequality

$$\rho_* \geq \gamma_*, \qquad \rho_{**} \geq \gamma_{**}. \tag{12}$$

If the equation

$$\rho_{**} = \gamma_{**} \tag{13}$$

holds, then from (10)-(13) it follows that

$$\rho_* = \gamma_*. \tag{14}$$

Let us prove (13). The equations

$$y_1^{(1)}(t) = -y_1^{(2)}(t), \qquad y_2^{(1)}(t) = y_2^{(2)}(t)$$

always hold for the system (9). The function γ_{**} is smooth everywhere in the space of positions, except on the singular set S, where its partial derivatives have discontinuities. In the domain where the function γ_{**} is differentiable, it satisfies the fundamental equation. For example, let the player E at the initial instant be located inside the triangle $P_1A_1P_2$; then, $\gamma_{**} = \gamma_1$ [see (6)]. We introduce the notation

$$r = ((\nu(\vartheta - t))^2 - z_1^2)^{1/2}, \qquad d_1 = z_2 + r,$$
$$R_1 = ((d_1 - y_2)^2 + y_1^2)^{1/2}, \qquad \gamma_1 = R_1 - \mu(\vartheta - t).$$

We find the partial derivatives of these functions,

$$\partial r/\partial t = -\nu^2(\vartheta - t)/r, \qquad \partial r/\partial z_1 = z_1/r,$$
$$\partial d_1/\partial t = -\nu^2(\nu - t)/r, \qquad \partial d_1/\partial z_1 = -z_1/r, \qquad \partial d_1/\partial z_2 = 1,$$
$$\partial R_1/\partial y_1 = y_1/R_1, \qquad \partial R_1/\partial y_2 = -(d_1 - y_2)/R_1,$$
$$\partial R_1/\partial d_1 = (d_1 - y_2)/R_1, \qquad \partial R_1/\partial z_1 = -(d_1 - y_2)z_1/(R_1 r),$$
$$\partial R_1/\partial z_2 = (d_1 - y_2)/R_1, \qquad \partial \gamma_1/\partial t = \mu - \nu^2(\vartheta - t)(d_1 - y_2)/(R_1/r),$$
$$\partial \gamma_1/\partial y_1 = y_1/R_1, \qquad \partial \gamma_1/\partial y_2 = -(d_1 - y_2)/R_1,$$
$$\partial \gamma_1/\partial z_1 = -(d_1 - y_2)z_1/(R_1 r), \qquad \partial \gamma_1/\partial z_2 = (d_1 - y_2)/R_1.$$

The fundamental equation has the form

$$\partial\gamma_1/\partial t + \min_u((\partial\gamma_1/\partial y_1)u_1 + (\partial\gamma_1/\partial y_2)u_2) + \max_v((\partial\gamma_1/\partial z_1)v_1 + (\partial\gamma_1/\partial z_2)v_2) = 0.$$

Since $z_2 \geqslant y_2$, we have $d_1 - y_2 \geqslant 0$. Hence, it follows that

$$\min_u((\partial\gamma_1/\partial y_1)u_1 + (\partial\gamma_1/\partial y_2)u_2) = -\mu,$$

$$\max_v((\partial\gamma_1/\partial z_1)v_1 + (\partial\gamma_1/\partial z_2)v_2) = \nu^2(\vartheta - t)(d_1 - y_2)/(R_1 r).$$

Thus, the fundamental equation is satisfied. Similarly, we can verify that the fundamental equation is satisfied for positions of E located inside the triangle $P_1A_2P_2$.

We consider the other case for the game (9), (2), (3). Let the positions of players P_i and E be such that $(t, x) \notin Q$; i.e., the evader E is outside the quadrangle $P_1A_1P_2A_2$. Then, the game (9), (2), (3) degenerates into a one-to-one game for which the function γ_* [see (7)] is the value of the game (see Ref. 2).

4. Case of a Singular Surface

Let us consider further the positions of the game belonging to the singular set S defined above. In Refs. 4 and 5, necessary and sufficient conditions are formulated for the nonsmooth maximin function to be the value function of the differential game. For the programmed maximin function γ_{**} to be the value of game, it is sufficient to prove the inequality

$$\max_v \min_u \max(d\gamma_1/dt, d\gamma_2/dt) \leqslant 0, \tag{15}$$

i.e., that is a u-stability function (see Refs. 2, 4, 5).

We put

$$r = ((\nu(\vartheta - t))^2 - z_1^2)^{1/2}, \qquad R = (r^2 + y_1^2)^{1/2}.$$

We find the partial derivatives of the functions

$$\partial\gamma_1/\partial t = \partial\gamma_2/\partial t = \mu - (\nu^2(\vartheta - t))/R,$$

$$\partial\gamma_1/\partial y_1 = \partial\gamma_2/\partial y_1 = y_1/R, \qquad \partial\gamma_1/\partial z_1 = \partial\gamma_2/\partial z_1 = -z_1/R,$$

$$\partial\gamma_1/\partial y_2 = -\partial\gamma_2/\partial y_2 = -r/R, \qquad \partial\gamma_1/\partial z_2 = -\partial\gamma_2/\partial z_2 = r/R,$$

$$d\gamma_j/dt = \partial\gamma_j/\partial t + (\partial\gamma_j/\partial y_1)u_1 + (\partial\gamma_j/\partial y_2)u_2 + (\partial\gamma_j/\partial z_1)v_1 + (\partial\gamma_j/\partial z_2)v_2, \qquad j = 1, 2.$$

The inequality (15) has the form

$$\mu - (\nu^2(\vartheta - t) + \max_v \min_u \varphi(y_1, z_1, u, v))/R \leqslant 0,$$

$$\varphi(y_1, z_1, u, v) = y_1 u_1 - z_1 v_1 + r|u_2 - v_2|.$$

Thus, it is necessary to prove the inequality

$$\max_v \min_u \varphi(y_1, z_1, u, v) \leqslant \nu^2(\vartheta - t) - R\mu. \tag{16}$$

We note that (see Fig. 2)

$$y_1 \leqslant 0, \qquad r \geqslant 0.$$

The inequality

$$\mu(\vartheta - t) < R \tag{17}$$

follows from (2) and (5). It follows from (17) and from the inequality

$$r \leqslant \nu(\vartheta - t)$$

that

$$\nu \geqslant \mu r R^{-1}.$$

Let $v_2 \in [\mu r R^{-1}, \nu]$. Then,

$$\max_v \min_u \varphi(y_1, z_1, u, v) = \max_v (r v_2 - z_1 v_1) - R\mu$$

$$= \nu(r^2 + z_1^2)^{1/2} - R\mu = \nu^2(\vartheta - t) - R\mu.$$

Let $v_2 \in [0, \mu r R^{-1}]$. Then,

$$\max_v \min_u \varphi(y_1\ z_1, u, v) = \max_v (y_1(\mu^2 - v_2^2)^{1/2} - z_1 v_1)$$

$$= ((\nu^2 R^2 - \mu^2 r^2)^{1/2}|z_1| - \mu|y_1|(R^2 - r^2)^{1/2})/R.$$

So, we need to prove the inequality

$$|z_1|(\nu^2 R^2 - \mu^2 r^2)^{1/2} - y_1^2 \mu \leqslant (\nu^2(\vartheta - t) - R\mu)R.$$

The validity of this inequality follows from the relations

$$R \geqslant \mu(\vartheta - t), \qquad R\nu^2(\vartheta - t) \geqslant r^2\mu,$$

$$|z_1|(\nu^2 R^2 - \mu^2 r^2)^{1/2} \leqslant R\nu^2(\vartheta - t) - r^2\mu.$$

In this way, the proof of Inequality (15) is completed. We have proved that the programmed maximin function γ_* is the value of the differential game (1)-(3).

5. Remark

For the case when $\mu \geqslant \nu$, i.e., the pursuers have the advantage in speed [see (2)], it can be proved that the value function has the form [see (6), (7)]

$$\gamma = \max(\gamma_*, 0).$$

This fact can also be proved by using the results of Refs. 4 and 5. All the constructions in the paper can be generalized easily to this case.

6. Conclusions

We have found the value function in the game of approach with two pursuers and one evader, in which the players have simple motions.

In Refs. 11 and 12, the value function of the differential game of approach with two pursuers and one evader was constructed for other concrete problems of approach with a nonconvex payoff function. The value function was found for all possible positions.

In Ref. 11, the differential game of approach involving two inertial pursuers and a noninertial evader was considered. In this problem, the value function does not coincide with the programmed maximin function. The value of the game is determined from the solution of a system of nonlinear algebraic equations.

In Ref. 12, a similar differential game of approach for dynamical players is considered. In this problem, the motion of the players is caused by the action of control forces and friction. A numerical example of pursuit-evasion is described.

Knowing the value of the differential game, we can construct ϵ-optimal player's feedback strategies which provide the saddle point of the game within a prescribed accuracy ϵ. This question is considered in Ref. 3.

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