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Convex Programs with an Additional Reverse Convex Constraint

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Abstract. A method is presented for solving a class of global optimization problems of the form (P): minimize $f(x)$, subject to $x \in D$, $g(x) \geq 0$, where D is a closed convex subset of R^n and f, g are convex finite functions R^n. Under suitable stability hypotheses, it is shown that a feasible point $\bar{x}$ is optimal if and only if $0 = \max\{g(x) : x \in D, f(x) \leq f(\bar{x})\}$. On the basis of this optimality criterion, the problem is reduced to a sequence of subproblems Q_k, $k = 1, 2, \ldots$, each of which consists in maximizing the convex function $g(x)$ over some polyhedron S_k. The method is similar to the outer approximation method for maximizing a convex function over a compact convex set.

Key Words. Reverse convex constraints, convex maximization, concave minimization, outer approximation methods.

1. Introduction

In this paper we shall be concerned with the following nonconvex optimization problem

$$\text{(P)} \quad \text{minimize } f(x),$$

$$\text{s.t.} \quad h_i(x) \leq 0, \qquad i = 1, 2, \ldots, m,$$

$$g(x) \geq 0,$$

where f, g, $h_i : R^n \to R$ are convex finite functions on R^n. Setting

$$h(x) = \max_{i=1,\ldots,m} h_i(x),$$

$$D = \{x : h(x) \leq 0\}, \qquad G = \{x : g(x) < 0\},$$

we note that the constraint set of this problem is a cavern of the form $D \setminus G$, where D is a closed convex set, G an open convex set.

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A simple example of this type of problems is furnished by the problem of minimizing the distance $f(x) = d(x, M)$ from a convex set M to a point $x \in R^n \setminus G$, where G is an open convex set containing M.

A large class of optimization problems, including convex minimization and convex maximization (or concave minimization) problems, can easily be cast in the form P. For instance, any problem

$$\min\{f(x) - g(x): x \in D\}, \tag{1}$$

where f, g are convex finite functions and D is a closed convex set, can be written as

$$\min\{f(x) - t: x \in D, g(x) - t \geq 0\}, \tag{2}$$

which is obviously a problem of the above type.

The main difficulty with problem P is connected with the presence of the reverse convex constraint $g(x) \geq 0$, which destroys the convexity and possibly even the connectivity of the feasible set. Optimization problems involving such reverse convex constraints were studied earlier by Rosen (Ref. 1), Avriel and Williams (Ref. 2), Mayer (Ref. 3), Ueing (Ref. 4), and more recently by Bansal and Jacobsen (Ref. 5), Hillestad and Jacobsen (Refs. 6 and 7), Tuy (Ref. 8), and Thuong (Ref. 9). Avriel and Williams (Ref. 2) showed that reverse convex constraints may occur in certain engineering design problems. Zaleesky (Ref. 10) argues that reverse convex constraints are likely to arise in many typical economic management applicatons. In an abstract setting, Singer (Ref. 11) related this type of nonconvex constraints to certain problems in approximation theory, when the set of approximation functions is the complement of a convex set.

It should be noted that, although the literature on nonconvex optimization has rapidly increased in recent years, most of the published papers either deal with the theoretical aspects of the problem or are concerned only with finding Kuhn–Tucker points or local solutions rather than global optima. A few papers (Refs. 6–9) have been devoted to the global minimization of a concave (in particular, linear) function under linear and reverse convex constraints, a problem closely related to, but not quite the same as P. Obviously, writing P in the form

$$\min\{t: f(x) \leq t,\ h(x) \leq 0,\ g(x) \geq 0\},$$

we shall convert it into a problem with a linear objective function. But to our knowledge, global optimization problems like P, where convex (nonlinear) and reverse convex constraints are copresent, have been little studied in the literature to date.

The present paper is an outgrowth of an earlier work (Ref. 12), where only the case $D = R^n$ was treated.

Basically, the approach that we shall propose in the sequel consists in viewing the reverse constraint $g(x) \geq 0$ as an additional constraint adjoined to the ordinary convex program

$$\min\{f(x): h(x) \leq 0\}.$$

Using then a duality principle, we interchange the roles of the objective function and the additional constraint, thus reducing the original problem to a convex maximization (i.e., concave minimization) problem, to which available outer approximation algorithms can be applied.

The paper is organized as follows. After the introduction, we state in Section 2 the basic assumptions and some immediate consequences. In Section 3, we give some preliminary results based on the local approach. In Section 4, we establish the duality principle which constitutes the cornerstone of our method (and could possibly be useful in other contexts). In Section 5, this duality principle is applied to provide a convenient optimality criterion and a theoretical solution scheme for stable problems. Section 6 deals with the stabilization of unstable problems. Finally, in Section 7, we present the main algorithm, along with the detailed convergence proof.

2. Basic Assumptions

Throughout this paper (except in Section 4), we shall make the following assumptions:

(i) the functions f, g, h are convex finite throughout R^n; $D \backslash G \neq \varnothing$; and G is bounded;

(ii) a point w is available such that

$$w \in D, \qquad g(w) < 0, \tag{3}$$

$$f(w) < \alpha \overset{\text{def}}{=} \min\{f(x): x \in D \backslash G\}. \tag{4}$$

The first assumption needs no explanation. As for the second assumption, it is quite natural. Indeed, by dropping the constraint $g(x) \geq 0$, we obtain an ordinary convex program

$$\min\{f(x): x \in D\},$$

which can be solved by many available methods. If an optimal solution to this program satisfies the constraint $g(x) \geq 0$, it will obviously solve (P). If this program has no finite optimal solution, P has no finite optimal solution either (since G is bounded). Otherwise, we shall obtain a point w satisfying the required conditions (3)–(4).

It is expedient to indicate here some immediate consequences of the above assumptions.

Lemma 2.1. The functions f, g, h are continuous and subdifferentiable at every point. We have $\{x \in D: f(x) \leq \alpha\} \subset \bar{G}$ (closure of G); and, for any real numbers c, d, the level sets

$$\{x \in D: f(x) \leq c\}, \qquad \{x \in D: g(x) \leq d\}$$

are bounded.

Proof. The first assertion follows from the general theory of convex functions (see, e.g., Ref. 13). To prove the second assertion, observe that, for any $x \in D$, if $f(x) < \alpha$, then necessarily $x \in G$ by virtue of the definition of α. Therefore,

$$\{x \in D: f(x) < \alpha\} \subset G,$$

and hence,

$$\{x \in D: f(x) \leq \alpha\} \subset \bar{G}.$$

The boundedness of $\bar{G}$ implies the boundedness of the set $\{x \in D: f(x) \leq \alpha\}$, and hence of all the sets $\{x \in D: f(x) \leq c\}$ (see, e.g., Ref. 13, Corollary 8.7.1). By the same argument, the boundedness of the set $\{x \in D: g(x) \leq 0\} \subset \bar{G}$ implies the boundedness of all the sets $\{x \in D: g(x) \leq d\}$. □

Denote by ∂G the boundary of G,

$$\partial G = \{x: g(x) = 0\}.$$

For every $x \notin G$, let $\pi(x)$ be the point where the line segment $[w; x]$ meets ∂G. Since g is convex, $g(w) < 0$, while $g(x) \geq 0$, it is clear that

$$\pi(x) = tx + (1-t)w,$$

with $t \in (0, 1]$ being uniquely determined from the equation

$$g(tx + (1-t)w) = 0.$$

Lemma 2.2. For every $x \in D$ such that $g(x) > 0$, we have

$$f(\pi(x)) < f(x).$$

Proof. Since $g(x) > 0$, we must have

$$\pi(x) = tx + (1-t)w, \qquad \text{with } t < 1,$$

and it follows from (3) and the convexity of f that

$$f(\pi(x)) \leq tf(x) + (1-t)f(w) < tf(x) + (1-t)f(x) = f(x). \qquad □$$

Corollary 2.1. Every optimal solution to P lies on $D \cap \partial G$.

Thus, under the stated assumptions, problem P is equivalent to finding the minimum of f over $D \cap \partial G$.

3. Local Approach: Stationary Points

In view of the nonconvexity of the feasible set, finding the global minimum of $f(x)$ over this set is generally a difficult problem. Therefore, in many cases it is useful to find a reasonably good (although not optimal) feasible solution, which could subsequently serve as a starting point in the global search procedure.

In this section, we introduce a concept of stationarity (weaker than that of optimality) and discuss a method for finding stationary points.

As usual, let $\partial g(\bar{x})$ denote the subgradient of g at $\bar{x}$ [as pointed out earlier, $\partial g(\bar{x})$ is nonempty at every $\bar{x}$]. Then, for any $p \in \partial g(\bar{x})$, we have

$$\langle p, x-\bar{x}\rangle \leq g(x) - g(\bar{x}) = g(x), \qquad \text{if } g(\bar{x}) = 0.$$

Therefore, provided $\bar{x} \in D \cap \partial G$, we have

$$K_p(\bar{x}) = \{x \in D, \langle p, x-\bar{x}\rangle \geq 0\} \subset D \backslash G,$$

and so, if $\bar{x}$ is optimal, it must achieve the minimum of $f(x)$ over the convex set $K_p(\bar{x})$. This observation leads us to the following proposition.

Proposition 3.1. A necessary condition for a point $\bar{x} \in D \cap \partial G$ to be optimal is that

$$\partial g(\bar{x}) \subset \text{cone } \partial f(\bar{x}) + N_D(\bar{x}), \tag{5}$$

where cone A denotes the cone vertexed at 0 generated by A and

$$N_D(\bar{x}) = \{p: \langle p, x-\bar{x}\rangle \leq 0, \forall x \in D\}$$

is the normal cone to D at $\bar{x}$.

Proof. If $\bar{x} \in D \cap \partial G$ is optimal, then, as seen above,

$$f(x) \geq f(\bar{x}), \qquad \text{for all } x \in K_p(\bar{x}) \text{ and all } p \in \partial g(\bar{x}).$$

Therefore, for every $p \in \partial g(\bar{x})$, there exist real numbers $\lambda \geqslant 0$, $\mu \geqslant 0$, not both zero, such that

$$0 \in \lambda \partial f(\bar{x}) - \mu p + N_D(\bar{x}).$$

One cannot have $\mu = 0$, for $0 \in \lambda \partial f(\bar{x}) + N_D(\bar{x})$ would imply that $\bar{x}$ achieves the minimum of f over the whole set D, contrary to the assumptions (3)-(4). Consequently, one can assume $\mu = 1$. Then,

$$p \in \lambda \partial f(\bar{x}) + N_D(\bar{x}),$$

proving (5). □

Remark 3.1. Applying Proposition 3.1 to problem (2), equivalent to (1), we find the necessary condition for optimality

$$\partial g(\bar{x}) \subset \partial f(\bar{x}) + N_D(\bar{x}),$$

that has been earlier obtained by Polyakova (Ref. 14; see also Ref. 15).

Definition 3.1. A point $\bar{x} \in D \cap \partial G$ is called stationary it if satisfies the condition (5).

Whenever a point $\bar{x} \in D \cap \partial G$ is not stationary, there exists for some $p \in \partial g(\bar{x})$ at least one point z such that

$$z \in D, \qquad \langle p, z - \bar{x} \rangle \geq 0, \qquad f(z) < f(\bar{x}).$$

Then, $x' = \pi(z)$ is a feasible point satisfying

$$x' \in D \cap \partial G, \qquad f(x') < f(z) < f(\bar{x}).$$

This suggests the following procedure for finding a stationary point (assuming that a point $x^1 \in D \cap \partial G$ is available).

Algorithm 3.1. Start from $x^1 \in D \cap \partial G$. Take $p^1 \in \partial g(x^1)$.

Iteration $k = 1, 2, \ldots$. Solve the convex program

$$\min\{f(x): x \in D, \langle p^k, x - x^k \rangle \geq 0\}.$$

If x^k is optimal to this program, stop. Otherwise, let z^k be an optimal solution to this program. Compute $x^{k+1} = \pi(z^k)$; take $p^{k+1} \in \partial g(x^{k+1})$. Go to iteration $k+1$.

Remark 3.2. If a point $x^1 \in D \cap \partial G$ is not readily available, it can be obtained by solving the convex maximization problem $\max\{g(x): x \in D\}$ (see Refs. 16-22); as soon as a point $z^1 \in D$ has been found with $g(z^1) \geq 0$, set $x^1 = \pi(z^1)$.

Proposition 3.2. Suppose that the function g is Gâteaux differentiable [so that $\partial g(x)$ is a singleton] at each point $x \in D \cap \partial G$. Then, the above procedure, whenever infinite, generates a sequence $\{x^k\}$, every cluster point of which is a stationary point.

Proof. Consider any point

$$\bar{x} = \lim_{\nu \to \infty} x^{k_\nu}.$$

Since $\{x^k\} \subset \partial G$ and G is bounded, the sequence $\{p^k\}$ is bounded (see, e.g., Ref. 13, Theorem 2.4.7). Further, since $f(z^k) \le f(z^1)$, it follows from Lemma 2.1 that the sequence $\{z^k\}$ is bounded too. By taking a subsequence, if necessary, we may then assume

$$p^{k_\nu} \to p \in \partial g(\bar{x}), \quad z^{k_\nu} \to \bar{z}, \qquad \text{with } \bar{x} = \pi(\bar{z}).$$

We have

$$f(z^{k_\nu}) \le f(x), \qquad \text{for all } x \in D \text{ satisfying } \langle p^{k_\nu}, x - x^{k_\nu} \rangle \ge 0.$$

Therefore, if $x \in D$, $\langle p, x - \bar{x} \rangle > 0$, then $\langle p^{k_\nu}, x - x^{k_\nu} \rangle > 0$, and hence

$$f(z^{k_\nu}) \le f(x), \qquad \text{for all large enough } \nu.$$

This implies

$$f(\bar{z}) \le f(x), \qquad \text{for all } x \in D \text{ satisfying } \langle p, x - \bar{x} \rangle > 0;$$

i.e., the convex system

$$x \in D, \qquad \langle p, x - \bar{x} \rangle > 0, f(x) < f(\bar{z})$$

is inconsistent. Consequently, there exist multipliers $\lambda \ge 0$, $\mu \ge 0$, not both zero, satisfying

$$0 \in \lambda \partial f(\bar{x}) - \mu p + N_D(\bar{x}).$$

As in the proof of Proposition 3.1, one can see that $\mu > 0$, whence (5) follows. □

Remark 3.3. If $\partial g(x^k)$ has more than one element, it is easily seen that the above Algorithm 3.1 converges only to points $\bar{x}$ satisfying

$$\partial g(\bar{x}) \cap (\text{cone } f(\bar{x}) + N_D(\bar{x})) \ne \emptyset,$$

which is a weaker condition than (5).

4. Duality between Objective and Constraint in Global Optimization

Turning to the global approach, let us begin with establishing a general and simple duality principle, which will be given a central role in the subsequent development.

As is known, the basic idea of duality consists in the following. Given a problem, we wish to associate with it another problem, called its dual, such that the primal and the dual problems describe in fact two aspects of essentially one and the same situation; solving one problem is just equivalent to solving the other. From a conceptual point of view, this would provide us a better insight into the real situation under study. On the other hand, from a computational point of view, the dual problem may be easier to solve than the original one, or at least may suggest more efficient methods for solving the latter.

Now, there are in every optimization problem two fundamental concepts: the instruments (with costs) and the goal (giving utility), i.e., the constraints on the one hand and the objective on the other. These two concepts deal with two aspects of the reality, which can be viewed as dual to each other in the following manner.

Consider the two problems,

$$(\mathrm{P}_\beta) \quad \inf\{f(x) \colon x \in D, g(x) \geq \beta\},$$

$$(\mathrm{Q}_\alpha) \quad \sup\{g(x) \colon x \in D, f(x) \leq \alpha\},$$

where D is an arbitrary set in R^n, $f: R^n \to R$, $g: R^n \to R$ two arbitrary functions (so we temporarily leave the assumptions about D, f, g made in Section 2), α, β two real numbers.

Definition 4.1. We say that problem P_β is stable, if

$$\lim_{\beta' \to \beta+0} \inf \mathrm{P}_{\beta'} = \inf \mathrm{P}_\beta < +\infty, \tag{6}$$

where $\inf P_\beta$ denotes the value of the infimum in problem P_β. Similarly, problem Q_α is stable, if

$$\lim_{\alpha' \to \alpha-0} \sup \mathrm{Q}_{\alpha'} = \sup \mathrm{Q}_\alpha > -\infty. \tag{7}$$

Proposition 4.1. (a) If Q_α is stable, then

$$\alpha \leq \inf \mathrm{P}_\beta \text{ implies } \beta \geq \sup Q_\alpha. \tag{8}$$

(b) If P_β is stable, then

$$\beta \geq \sup Q_\alpha \text{ implies } \alpha \leq \inf \mathrm{P}_\beta. \tag{9}$$

Proof. (a) Assume that Q_α is stable and $\alpha \leq \inf \mathrm{P}_\beta$. Then, for all $\alpha' < \alpha$, the set $\{x \in D \colon g(x) \geq \beta, f(x) \leq \alpha'\}$ is empty. Hence,

$$\sup\{g(x) \colon x \in D, f(x) \leq \alpha'\} \leq \beta.$$

Thus, $\sup \mathrm{Q}_{\alpha'} \leq \beta$, for all $\alpha' < \alpha$; and, using (7), we conclude that $\sup \mathrm{Q}_\alpha \leq \beta$.

(b) Similarly, if P_β is stable and $\beta \geq \sup Q_\alpha$, then, for all $\beta' > \beta$, the set $\{x \in D: f(x) \leq \alpha,\ g(x) \geq \beta'\}$ is empty. Hence,

$$\inf\{f(x): x \in D, g(x) \geqslant \beta'\} \geqslant \alpha;$$

i.e., $\inf P_{\beta'} \geq \alpha$; and, using (6), $\inf P_\beta \geq \alpha$. □

Corollary 4.1. If both P_β and Q_α are stable, then

$$\alpha \leq \inf P_\beta \Leftrightarrow \beta \geq \sup Q_\alpha.$$

For the purpose of applications, it is important to know under which conditions a given problem is stable.

Lemma 4.1. If $\inf P_\beta < +\infty$, if f is upper semicontinuous (u.s.c.), and β is not a local maximum of g over D, then P_β is stable. Likewise, if $\sup Q_\alpha > -\infty$, if g is lower semicontinuous (l.s.c.), and α is not a local minimum of f over D, then Q_α is stable.

Proof. We need only prove the first assertion, since the second can be established by a similar method. Suppose that $\inf P_\beta < +\infty$, f is u.s.c., while β is not a local maximum of g over D, and consider a sequence $\{x^k\} \subset D$ such that

$$g(x^k) \geq \beta, \qquad f(x^k) \leq c_k, \qquad c_k \searrow \inf P_\beta.$$

If, for some k, $g(x^k) > \beta$, then, for all β' sufficiently close to β, $g(x^k) \geq \beta'$; hence,

$$\inf P_{\beta'} \leq f(x^k) \leq c_k;$$

hence,

$$\lim_{\beta' \to \beta+0} \inf P_{\beta'} \leq c_k.$$

Therefore, if the inequality $g(x^k) > \beta$ holds for infinitely many k, then

$$\lim_{\beta' \to \beta+0} \inf P_{\beta'} \leq \inf P_\beta;$$

hence, (6) follows. On the other hand, if $g(x^k) = \beta$ for all but finitely many k, then, since β is not a local maximum of g over D, there exist for each k with $g(x^k) = \beta$ a sequence $x^{k,\nu} \to x^k$ such that $x^{k,\nu} \in D$, $g(x^{k,\nu}) > \beta$. Then, for all β' sufficiently close to β, we have $g(x^{k,\nu}) \geq \beta'$; hence,

$$\inf P_{\beta'} \leq f(x^{k,\nu});$$

hence,

$$\lim_{\beta' \to \beta+0} \inf P_{\beta'} \leq f(x^{k,\nu}).$$

By making $\nu \to \infty$ and using the u.s.c. of f, we obtain

$$\lim_{\beta' \to \beta+0} \inf P_{\beta'} \leq f(x^k) \leq c_k,$$

which yields (6) as $k \to \infty$. □

Remark 4.1. Suppose that both P_β and Q_α are stable. Then,

$$\alpha = \min \mathrm{P}_\beta \Leftrightarrow \beta = \max Q_\alpha.$$

Indeed, if $\alpha = \min \mathrm{P}_\beta$, then $\beta \geq \sup \mathrm{Q}_\alpha$. But one cannot have $\beta > \sup \mathrm{Q}_\alpha$, because there exists $\bar{x} \in D$, such that $g(\bar{x}) \geq \beta$, $f(\bar{x}) = \alpha$. Therefore, $\beta = \sup \mathrm{Q}_\alpha$. Furthermore, one cannot have $g(\bar{x}) > \beta$, hence $g(\bar{x}) = \beta$, i.e., $\beta = \max \mathrm{Q}_\alpha$. In a similar way, $\beta = \max \mathrm{Q}_\alpha$ implies $\alpha = \min \mathrm{P}_\beta$.

Thus, the minimal cost necessary to obtain an utility level β is equal to α, if and only if the maximal utility level that can be obtained with a cost α is just equal to β.

5. Solution Method for Stable Problems: Reduction to Convex Maximization

Let us now return to the original problem P, subject to all the assumptions specified in Section 2.

From the results of the previous section, we can readily derive the following optimality criterion.

Proposition 5.1. In order that a feasible solution $\bar{x}$ to P be optimal, it is necessary and, if P is stable, also sufficient that

$$0 = \max\{g(x): x \in D, f(x) \leq f(\bar{x})\}. \tag{10}$$

Proof. Apply Proposition 4.1, with

$$\alpha = \min\{f(x): x \in D \backslash G\} = \min \mathrm{P}, \qquad \beta = 0.$$

Observe that here f, g are continuous throughout R^n. The stability of Q_α then follows from Lemma 4.1 and assumptions (3)-(4), which implies that α is not a local minimum of f over D. Therefore, if $\bar{x}$ is optimal [i.e., $f(\bar{x}) = \alpha$], then, by Proposition 4.1,

$$0 = \max\{g(x): x \in D, f(x) \leq f(\bar{x})\},$$

where one must have the equality, because $g(\bar{x}) = 0$. Conversely, if P is stable and (10) holds, then by the same proposition and Remark 4.1,

$$f(\bar{x}) = \min \mathrm{P} = \alpha.$$ □

Thus, given any feasible solution x^1 to P, to check whether x^1 is optimal, we can solve the subproblem

$$(\mathrm{Q}(x^1)) \quad \max\{g(x): x \in D, f(x) \leq f(x^1)\}. \tag{11}$$

This is a convex maximization problem (or concave minimization problem), which consists in finding the global maximum of the convex function $g(x)$ over the compact convex set $\{x \in D: f(x) \leq f(x^1)\}$. For such problems, there are at the present time several available algorithms (Refs. 16-20; see also Refs. 21-22). Some of these algorithms are practical for problems of small size or having some special structure (Refs. 18, 20, 23).

If the optimal value in (11) is zero, we are done. Otherwise, we obtain an optimal solution z^1 of $Q(x^1)$ with $g(z^1) > 0$. Then, $x^2 = \pi(z^1)$ yields a new feasible solution such that, according to Lemma 2.2,

$$f(x^2) < f(z^1) \leq f(x^1).$$

We are thus led to the following procedure.

Algorithm 5.1. Start from any point $x^1 \in D \cap \partial G$ (for example, x^1 is a stationary point found by Algorithm 3.1).

Iteration $k = 1, 2, \ldots.$ Solve the subproblem

$$(\mathrm{Q}(x^k)) \quad \max\{g(x): x \in D, f(x) \leq f(x^k)\},$$

and obtain an optimal solution z^k to this subproblem. If $g(z^k) = 0$, stop. Otherwise, set $x^{k+1} = \pi(z^k)$, and go to iteration $k+1$.

Proposition 5.2. If the above algorithm is infinite, it generates a sequence $\{x^k\} \subset D \cap \partial G$, every cluster point of which yields a feasible solution to P satisfying condition (10), and hence an optimal solution to P if this problem is stable.

Proof. Let

$$\bar{x} = \lim_{\nu \to \infty} x^{k_\nu}.$$

Since the sequences $\{x^k\}$ and $\{z^k\}$ are bounded, we may, by taking a subsequence if necessary, assume that $x^{k_\nu+1} \to \tilde{x}$, $z^{k_\nu} \to z$. Clearly, $f(x^k)$ is a monotone decreasing sequence. Therefore, for every $x \in D$ satisfying $f(x) \leq f(\bar{x})$, we have

$$f(x) \leq f(x^k), \qquad k = 1, 2, \ldots.$$

This implies, by the definition of z^{k_ν}, $g(x) \leq g(z^{k^\nu})$; hence, by making $\nu \to \infty$: $g(x) \leq g(z)$. Thus, z is an optimal solution of the subproblem $\mathrm{Q}(\bar{x})$. Suppose

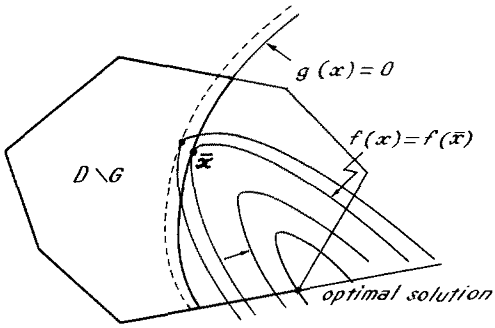

Fig. 1. Unstable problem (f convex nonlinear).

that $g(z)>0$. Since $x^{k_{\nu+1}}=\pi(z^{k_\nu})$, it is easily seen that $\tilde{x}=\pi(z)$, and hence, by Lemma 2.2, $f(\tilde{x})<f(z)$. But

$$f(x^{k_{\nu+1}})\leqslant f(x^{k_\nu+1})<f(z^{k_\nu})\leq f(x^{k_\nu}).$$

This yields, by letting $\nu\to\infty$,

$$f(\bar{x})\leqslant f(\tilde{x})\leq f(z)\leq f(\bar{x}),$$

conflicting with the just established inequality $f(\tilde{x})<f(z)$. Therefore, $g(z)=0$, and hence, (10) is satisfied. By virtue of Proposition 5.1, if P is stable, this ensures the optimality of $\bar{x}$. □

Note that subproblems of the type $\mathrm{Q}(x^k)$ were also used in Refs. 5 and 7.

6. Stabilization of Unstable Problems

It is likely that many problems encountered in practice are stable, hence can be solved by Algorithm 5.1. However, examples of unstable problems can easily be constructed. Figures 1 and 2 illustrate typical situations where instability may occur.

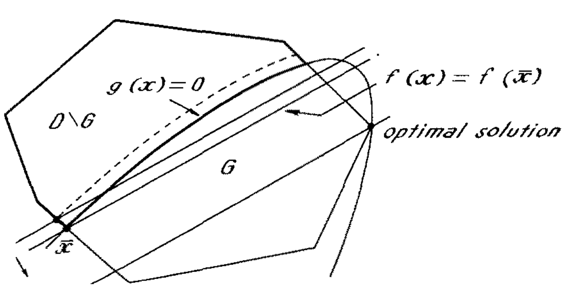

Fig. 2. Unstable problem (f linear).

If problem P is not stable, Algorithm 5.1 furnishes only a point $\bar{x} \in D \cap \partial G$ satisfying condition (10), i.e.,

$$0 = \max\{g(x): x \in D, f(x) \leq f(\bar{x})\}.$$

Such a point may not be an optimal solution (see Figs. 1 and 2). Nevertheless, this condition being necessary, the optimal solutions to P must be sought among the optimal solutions to the program

$$(\mathrm{Q}(\bar{x})) \quad \max\{g(x): x \in D, f(x) \leq f(\bar{x})\}.$$

Hence, if $g(\bar{x}) = 0$ and $\bar{x}$ is the unique optimal solution to this program, then it will solve P. Otherwise, the set of all optimal solutions to $\mathrm{Q}(\bar{x})$ is a union of faces of the compact convex set

$$D(\bar{x}) = \{x \in D: f(x) \leq f(\bar{x})\};$$

see, e.g., Ref. 13, Corollary 32.1.1. Thus, having found by Algorithm 5.1 a point $\bar{x}$ satisfying (10), we still have to solve a number of residual problems, each of which consists in minimizing the convex function $f(x)$ over some face of $D(\bar{x})$. Very often, these faces are easy to determine; for instance, if $g(x)$ is strictly convex, they can only be extreme points of $D(\bar{x})$. But, generally speaking, a direct study of the set of faces of $D(\bar{x})$ where $g(x)$ achieves its maximum may be hard.

Therefore, it is important to know a simple method for overcoming instability, at least in the most important cases. We first prove the following proposition.

Proposition 6.1. If $\bar{G} \subset \operatorname{int} D$ (in particular, if $D = R^n$), problem P is stable.

If D is a polyhedral convex set and $g(x)$ is a strictly convex function which does not vanish at any vertex of D, then P is stable.

Proof. If $\bar{G} \subset \operatorname{int} D$, then every extreme point x of D satisfies $g(x) > 0$. Since a local maximum of g in D must always be attained at an extreme point of D, zero cannot be a local maximum of g in D. Therefore, by Lemma 4.1, problem P is stable. In the case where D is a polyhedral convex set and $g(x)$ a strictly convex function, the only points of D where g may achieve a local maximum over D are the vertexes of D. Therefore, if $g(x) \neq 0$ at every vertex x of D, zero cannot be a local maximum of g over D. Consequently, again by Lemma 4.1, P is stable. □

Proposition 6.2. If D is a polyhedral convex set, there is $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$, the perturbed problem

$$(P(\epsilon)) \quad \min\{f(x): x \in D, g(x) + \epsilon(|x|^2 + 1) \geq 0\}$$

is stable. If x_ϵ is an optimal solution to $P(\epsilon)$ and $x_\epsilon \to \bar{x}$ for $\varepsilon \to 0$, then $\bar{x}$ is an optimal solution to P.

Proof. Let V_0 denote the set of vertexes x of D where $g(x)=0$, and let V_1 denote the set of all remaining vertexes of D. Let

$$\delta = \min\{|g(x)|: x \in V_1\} > 0,$$

and let $\epsilon_0 > 0$ be so small that

$$\epsilon_0(|x|^2+1) < \delta, \qquad \text{for all } x \in V_1.$$

Then, for every $\epsilon \in (0, \epsilon_0)$, we have

$$g(x)+\epsilon(|x|^2+1) \geq \epsilon > 0, \qquad \text{for all } x \in V_0,$$

$$|g(x)+\epsilon(|x|^2+1)| \geq \delta - \epsilon_0(|x|^2+1) > 0, \qquad \text{for all } x \in V_1.$$

Since the function $g(x)+\epsilon(|x|^2+1)$ is strictly convex and does not vanish at any vertex of D, it follows from the previous proposition that P is stable.

Let x_ϵ be any optimal solution to $P(\epsilon)$, so that

$$f(x_\epsilon) \leq f(x), \qquad \text{for all } x \in D \text{ satisfying } g(x)+\epsilon(|x|^2+1) \geq 0;$$

and let $x_\epsilon \to \bar{x}$ for $\epsilon \to 0$. Then, for all $x \in D$ satisfying $g(x) \geq 0$, we have

$$g(x)+\epsilon(|x|^2+1) > 0, \text{ hence } f(x_\epsilon) \leq f(x), \text{ so that } f(\bar{x}) \leq f(x).$$

Since $\bar{x} \in D$, $g(\bar{x}) \geq 0$, $\bar{x}$ is an optimal solution to P. □

On the basis of this proposition, in order to solve a given problem P, where D is polyhedral convex, it suffices to solve problem P(ϵ), with $\epsilon > 0$ arbitrarily small, and then make $\epsilon = 0$ in the result.

7. Improved Algorithm

Algorithm 5.1 reduces problem P to a sequence of subproblems Q(x^k), $k = 1, 2, \ldots$, each of which is in fact a relaxed form of the problem

$$\text{(Q)} \quad \max\{g(x): x \in D, f(x) \leq \alpha\}.$$

Therefore, the procedure can be viewed as a special outer approximation method applied to problem Q (see Ref. 20). However, since the constraint sets of these subproblems Q(x^k), i.e., the sets

$$D(x^k) = \{x \in D, f(x) \leq f(x^k)\}$$

are nonpolyhedral, these subproblems cannot in general be solved exactly by finitely many operations, using currently available methods for convex maximization.

The question arises as to whether the usual outer approximation method can be applied directly to problem Q, thus reducing P to a sequence of subproblems of the form

$$(\mathrm{Q}_k) \quad \max\{g(x): x \in S_k\},$$

where each S_k is a polyhedral convex set and

$$S_1 \supset S_2 \supset \cdots \supset D_\alpha = \{x \in D: f(x) \leq \alpha\}.$$

The following algorithm proceeds along this line [it is interesting to compare this algorithm with an outer approximation algorithm developed in Ref. 20 for maximizing $f(x)$ over $D \backslash G$].

Recall that

$$D = \{x: h(x) \leq 0\},$$

where h is a convex finite function on R^n.

Algorithm 7.1. Start from a feasible point $\bar{x}^1 \in D \cap \partial G$. Set $\alpha_1 = f(\bar{x}^1)$, and select a polytope S_1 containing the compact convex set $\{x \in D: f(x) \leq \alpha_1\}$.

Iteration $k = 1, 2, \ldots$ Solve the subproblem

$$(\mathrm{Q}_k) \quad \max\{g(x): x \in S_k\}$$

by a finite algorithm (see, e.g., Refs 19, 20, 24). Let z^k be an optimal solution to (Q_k). If $g(z^k) = 0$, stop. Otherwise, find the point x^k where the line segment $[w; z^k]$ meets the surface

$$\max\{f(x) - \alpha_k, g(x)\} = 0.$$

(a) If $x^k \in D$ [i.e., $h(x^k) \leq 0$], choose $p^k \in \partial f(x^k)$, and let

$$l_k(x) = \langle p^k, x - x^k \rangle. \tag{12}$$

(b) If $x^k \notin D$ [i.e., $h(x^k) > 0$], choose $p^k \in \partial h(x^k)$, and let

$$l_k(x) = \langle p^k, x - x^k \rangle + h(x^k). \tag{13}$$

Form S_{k+1} by adjoining to S_k the new constraint

$$l_k(x) \leq 0.$$

Set

$$\bar{x}^{k+1} = x^k, \qquad \text{if } x^k \in D,\ g(x^k) = 0,$$
$$\bar{x}^{k+1} = \bar{x}^k, \qquad \text{otherwise.}$$

Set $\alpha_{k+1} = f(\bar{x}^{k+1})$. Then, go to iteration $k+1$.

We shall establish the convergence of this algorithm under the following additional assumption (iii) [aside from the assumptions (i), (ii) already stated in Section 2]:

(iii) $w \in \operatorname{int} D$.

In view of assumption (ii), this is equivalent to requiring that D have a nonempty interior; for then, by slightly moving w, one can simultaneously satisfy (ii) and (iii).

Observe that, for every $k = 1, 2, \ldots$, $\bar{x}^k$ is the best feasible solution that has been computed up to step k, while $\alpha_k = f(\bar{x}^k)$ is the best objective function value up to this step.

Lemma 7.1. For every k, we have

$$\{x \in D\colon f(x) \leq \alpha_k\} \subset S_k.$$

Proof. Let $x \in D$, $f(x) \leq \alpha_k$. Since $\alpha_k \leq \alpha_1$

and

$$\{y \in D\colon f(y) \leq \alpha_1\} \subset S_1,$$

it follows that $x \in S_1$. Furthermore, for $i < k$, if $x^i \in D$, then $p^i \in \partial f(x^i)$, so that

$$l_i(x) = \langle p^i, x - x^i \rangle \leq f(x) - f(x^i) \leq f(x) - \alpha_k \leq 0.$$

If $x^i \notin D$, then $p^i \in \partial h(x^i)$ and

$$l_i(x) = \langle p^i, x - x^i \rangle + h(x^i) \leq h(x) \leq 0.$$

Therefore, $x \in S_k$. □

Proposition 7.1. If $g(z^k) = 0$, then

$$0 = \max\{g(x)\colon x \in D, f(x) \leq \alpha_k\}. \tag{14}$$

If the sequence z^k has a cluster point $\bar{x}$ such that $g(\bar{z}) = 0$, then

$$0 = \max\{g(x)\colon x \in D, f(x) \leq \alpha_*\}, \tag{15}$$

where

$$\alpha_* = \inf\{f(\bar{x}^i)\colon i = 1, 2, \ldots\}.$$

Proof. Since

$$\alpha_k \geq \alpha = \min\{f(x)\colon x \in D, g(x) \geq 0\},$$

we have from Lemma 7.1 that

$$\{x \in D\colon f(x) \leq \alpha\} \subset \{x \in D\colon f(x) \leq \alpha_k\} \subset S_k. \tag{16}$$

By Proposition 5.1,

$$0 = \max\{g(x) \colon x \in D, f(x) \leq \alpha\}; \tag{17}$$

and, by hypothesis,

$$g(z^k) = \max\{g(x) \colon x \in S_k\} = 0.$$

Hence, (14) follows.

To prove the second assertion, observe that, in view of the inequality

$$g(z^k) \geq g(x), \qquad \forall x \in S_k,$$

and the continuity of g,

$$g(\bar{z}) \geq \max\left\{g(x) \colon x \in \bigcap_{k=1}^{\infty} S_k\right\}. \tag{18}$$

On the other hand, from (16) we have

$$\{x \in D \colon f(x) \leq \alpha\} \subset \{x \in D \colon f(x) \leq \alpha_*\} \subset \bigcap_{k=1}^{\infty} S_k. \tag{19}$$

Since $g(\bar{z}) = 0$, the relations (17)-(19) altogether imply (15). □

Lemma 7.2. For any extended real number $\bar{\alpha}$, if

$$0 = \max\{g(x) \colon x \in D, f(x) \leq \bar{\alpha}\}, \tag{20}$$

then $\bar{\alpha} \leq \alpha$, provided problem P is stable.

Proof. For any real number $\alpha' < \bar{\alpha}$, we have from (20)

$$0 \geq \max\{g(x) \colon x \in D, f(x) \leq \alpha'\}.$$

Consequently, by Proposition 4.1 [see (9)], $\alpha' \leq \alpha$. This implies $\bar{\alpha} \leq \alpha$. □

Proposition 7.2. Assume that problem P is stable. If $g(z^k) = 0$, then $\bar{x}^k$ is an optimal solution to P. If the sequence z^k has a cluster point $\bar{z}$ satisfying $g(\bar{z}) = 0$, then any cluster point $\tilde{x}$ of the sequence $\bar{x}^k$ is an optimal solution to P.

Proof. If $g(z^k) = 0$, then, by Proposition 7.1, (20) holds with $\bar{\alpha} = \alpha_k = f(\bar{x}^k)$. If $g(\bar{z}) = 0$ for some cluster point $\bar{z}$ of the sequence z^k, then, by

Proposition 7.1, (20) holds with $\bar{\alpha} = \alpha_* = f(\bar{x})$ for any cluster point $\bar{x}$ of the sequence $\bar{x}^k$. The conclusion then follows from Lemma 7.2. □

One last lemma needed for the proof of our basic convergence theorem is the following cutting plane convergence principle.

Lemma 7.3. Let C be an arbitrary set, and let z^k be a bounded sequence in R^n. Assume that, for every $k = 1, 2, \ldots$, there is an affine function $l_k(\cdot)$ such that

$$\text{(A)} \quad l_k(w) \leq 0, \text{ for some fixed } w;\ l_k(z^k) > 0; \tag{21}$$

$$\text{(B)} \quad l_j(z^k) \leq 0, \text{ for all } j < k; \tag{22}$$

(C) for any subsequence z^{k_ν} such that $z^{k_\nu} \to \bar{z} \notin C$ and $l_{k_\nu}(z) \to l(z)$ for every $z \in R^n$, we have $l(\bar{z}) > 0$.

Then, every cluster point of z^k belongs to C.

Proof. See Ref. 20 or 23. □

We are now in a position to state our basic result.

Theorem 7.1. Assume (i), (ii), (iii) and that problem P is stable. If Algorithm 7.1 terminates at iteration k, then $\bar{x}^k$ is an optimal solution to P. If the algorithm is infinite, then every cluster point $\bar{x}$ of the sequence $\bar{x}^k$ is an optimal solution to P.

Proof. The first part of the theorem follows from Proposition 7.2. To prove the second part, it suffices, by the same proposition, to show that any cluster point $\bar{z}$ of $\{z^k\}$ satisfies $g(\bar{z}) = 0$. According to Lemma 7.3, we need only check that all conditions of this lemma are fulfilled by

$$C = \partial G = \{z: g(z) = 0\}$$

and $\{z^k\}$. Condition (B) is obvious. To verify condition (A), observe that, if l_k is of the form (12), then

$$l_k(w) = \langle p^k, w - x^k \rangle \leq f(w) - f(x^k) < 0$$

[see (4)], $l_k(x^k) = 0$, hence $l_k(z^k) > 0$, because

$$z^k = x^k + t_k(x^k - w), \qquad \text{with } t_k > 0.$$

Similarly, if l_k is of the form (13), then $l_k(w) \leq \langle p^k, w - x^k \rangle + h(x^k) \leq h(w) < 0$ [by (iii)], $l_k(x^k) = h(x^k) > 0$, hence $l_k(z^k) > 0$. Thus, it only remains to verify condition (C). Suppose that $z^{k_\nu} \to \bar{z}$, $l_{k_\nu}(z) \to l(z)$, for every z and $\bar{z} \notin C$ [hence $g(\bar{z}) > 0$, because $g(z^k) > 0$, for every k]. By taking a subsequence if necessary, we may assume that either of the following cases occurs.

(a) $x^{k_\nu} \in D$, for all ν. Then l_{k_ν} has the form (12); and, since $x^k \in S_1$, it follows from the compactness of S_1 that the sequence p^k is bounded (see, e.g., Ref. 13, Theorem 24.7). Again, by taking a subsequence if necessary, we may assume $x^{k_\nu} \to \bar{x}$, $p^{k_\nu} \to p \in \partial f(\bar{x})$, so that $l(z) = \langle p, z - \bar{x} \rangle$. Since $g(\bar{z}) > 0$, we have

$$\bar{z} = \bar{x} + t(\bar{x} - w), \qquad \text{with } t > 0.$$

Noting that

$$l(w) = \langle p, w - \bar{x} \rangle \leq f(w) - f(\bar{x}) < 0, \qquad l(\bar{x}) = 0,$$

we then deduce $l(\bar{z}) > 0$, as required by condition (C).

(b) $x^{k_\nu} \notin D$, for all ν. Then, l_{k_ν} has the form (13); and, as previously, we may assume $x^{k_\nu} \to \bar{x}$, $p^{k_\nu} \to p \in \partial h(\bar{x})$, so that

$$l(z) = \langle p, z - \bar{x} \rangle + h(\bar{x}).$$

Clearly,

$$l(w) = \langle p, w - \bar{x} \rangle + h(\bar{x}) \leq h(w) < 0, \qquad l(\bar{x}) = h(\bar{x}) \geq 0,$$

because $h(x^{k_\nu}) > 0$. Since

$$\bar{z} = \bar{x} + t(\bar{x} - w), \qquad \text{with } t > 0,$$

we conclude, as before, that $l(\bar{z}) > 0$.

Thus, all conditions of Lemma 7.3 are fulfilled. By this lemma we have $\bar{z} \in C$, i.e., $g(\bar{z}) = 0$, completing the proof. □

Remark 7.1. In practice, given a tolerance number $\epsilon > 0$, we terminate when

$$g(z^k) < \epsilon. \tag{23}$$

Since any cluster point $\bar{z}$ of the sequence z^k satisfied $g(\bar{z}) = 0$, (23) must occur after finitely many iterations. Suppose that

$$\gamma = \max\{g(x) \colon x \in D\} > 0, \tag{24}$$

which necessarily holds if the problem is stable, and that $\epsilon < \gamma$. Then,

$$\max\{g(x): x \in D, f(x) \leq \alpha_k\} \leq g(z^k) < \epsilon < \gamma,$$

which implies $\alpha_k < +\infty$, i.e., there is $\bar{x}^k \in D$ such that $g(\bar{x}^k) = 0, f(\bar{x}^k) = \alpha_k$. Furthermore, there is no $x \in D$ such that $f(x) \leq \alpha_k$, $g(x) \geq \epsilon$. Hence,

$$f(\bar{x}^k) = \alpha_k < \min\{f(x): x \in D, g(x) \geq \epsilon\}. \tag{25}$$

Thus, with the stopping rule (23), where $\epsilon < \gamma$, the algorithm is finite and provides an ϵ-optimal solution in the sense (25) (note that this solution may not be ϵ-optimal in the usual sense). This conclusion holds under assumptions (i), (ii), (iii), and (24), no matter whether the problem is stable or not.

Remark 7.2. The subproblems Q_k can be solved by any available algorithm for maximizing a convex function over a polytope (see Refs. 18, 19, 20, 24). Since, however, Q_k differs from Q_{k-1} by just one additional constraint $l_k(x) \leq 0$, in order to economize the computational effort one should use for solving Q_k an algorithm which could take advantage of the information obtained in solving Q_{k-1}. Such an algorithm is provided, for instance, in Refs. 18, 20. Following this algorithm, the starting polytope S_1 is chosen so that all its vertexes are known or can be computed easily. At iteration $k > 1$, we already have in hand the vertex set of S_{k-1}. Let $l_k(x) \leq 0$ be the new constraint adjoined to S_{k-1} for defining S_k. Then, compute in the following way the vertex set of S_k. Consider all pairs u, w of vertices of S_{k-1} such that $l_k(u) < 0$, $l_k(w) > 0$, and $[u, w]$ is an edge of S_{k-1} (i.e., $n-1$ linearly independent constraints are simultaneously binding at u and w). For each of these pairs u, w, find the point v on the line segment $[u, w]$ that satisfies $l_k(v) = 0$. The vertex set of S_k then consists of all the points v obtained in that way, along with all the vertices u of S_{k-1} that satisfy $l_k(u) \leq 0$. For each new vertex v of S_k, compute $g(v)$, and let v^k be the new vertex with maximal value of g. If $g(v^k) > g(z^{k-1})$, an optimal solution to Q_k is $z^k = v^k$; otherwise, $z^k = z^{k-1}$ (this is based on the property that the maximum of a convex function over a polytope is achieved in at least one vertex).

Remark 7.3. A matter of concern in Algorithm 7.1, as in other outer approximation methods, is that the number of constraints of the subproblems Q_k increases systematically by one at each iteration and may thus become excessively large as the algorithm proceeds. To circumvent this drawback, one can propose an alternative, more flexible, strategy for forming the subproblems.

In fact, it was shown in Ref. 23 that Lemma 7.3 remains valid if condition (22) is replaced by a weaker one. Namely, instead of (22) we can assume that

$$l_j(z^k) = 0, \text{ for all } j < k \text{ such that}$$

$$l_j(z^i) > 0, \text{ for at least } N \text{ indices } i < j, \tag{26}$$

where N is any fixed natural number chosen beforehand.

With this strong form of Lemma 7.3 in mind, let us choose a natural number N. At each iteration j, let ν_j denote the number of points z^i with $i < j$ such that $l_j(z^i) > 0$ (i.e., the number of previously generated points z^i that violate the current constraint). N_0 being a fixed natural number greater than N, we now modify as follows the rule for forming the subproblems Q_k:

> At every iteration $k \geq N_0$, if $\nu_{k-1} \geq N$, then form Q_{k+1} by adjoining the newly generated constraint $l_k(x) \leq 0$ to Q_k; otherwise, form Q_{k+1} by adjoining the newly generated constraint to Q_{k-1}.

It is easily seen that, in this way, any constraint $l_j(x) \leq 0$, with $\nu_j < N, j > N_0$, is used just once (in the subproblem Q_{j+1}) and will be dropped in all subsequent iterations $k > j+1$. Intuitively, only those constraints are retained that are sufficiently efficient, having discarded at least N previously generated points z^i. In view of Lemma 7.3, where (22) is replaced by (26), this constraint dropping device does not adversely affect the convergence of the algorithm. Of course, the choice of N and N_0 is up to the user, who should be aware, however, that while a larger value of N allows having a smaller number of constraints for the subproblem in each iteration, this advantage can be offset by a greater number of required iterations.

8. Conclusions

The adjunction of just one reverse convex constraint to an ordinary convex program transforms it into a very difficult problem. In this paper, we have developed for this problem an algorithm whose complexity is apparently the same as that of the outer approximation methods for maximizing convex functions over compact convex sets, as described in Refs. 18 and 20. Computational experiments have shown that most currently available methods of convex maximization over compact convex sets are practical only for problems of rather small size. Very likely, the same will be true for our Algorithm 7.1, considering the close relationship exhibited above between convex maximization problems and problem P.

There are, however, two points worth noting in connection with the practical implementation of the algorithm. First, the algorithm could be combined in several conceivable ways with other (stochastic, local) approaches to yield practical results in a given case. For example, in a first stage, one could use the algorithm to find only a rough approximation of the global optimum: then, in a second stage, starting from this approximate global optimum, apply the local approach to find a stationary point. Or one could do the converse: first, find an approximate optimum x^0 by some other method, then apply the algorithm while replacing the set D by $D \cap \{x: f(x) \leq f(x^0)\}$. Second, if the problem has a special structure, one can hope to improve considerably the algorithm by a rational exploitation of this structure. For example, a problem of the form

$$\text{minimize } \varphi(u), \qquad \text{s.t. } Au + Bv + c \leq 0,\ \psi(v) \geq 0,$$

where $\varphi: R^{n_1} \to R$, $\psi: R^{n_2} \to R$ are convex functions, $c \in R^m$, $A \in R^{m \times n_1}$, $B \in R^{m \times n_2}$, can be reduced easily to problem P in R^{n_2}, using the decomposition method developed in Ref. 23. Therefore, provided n_2 is relatively small, Algorithm 7.1 can be applied, even if n_1 is fairly large.

Finally, let us observe that the above method applies to the more complicated case where problem P involves not just one single reverse convex constraint $g(x) \geq 0$, but several such constraints:

$$g_j(x) \geq 0, \qquad j = 1, 2, \ldots, s, \tag{27}$$

with g_j being convex functions on R^n. Indeed, the above system is equivalent to the single inequality

$$g(x) \geq 0,$$

where $g(x) = \min\{g_1(x), g_2(x), \ldots, g_s(x)\}$. But

$$g(x) = p(x) - q(x),$$

with

$$p(x) = g_1(x) + g_2(x) + \cdots + g_s(x),$$

$$q(x) = \max\left\{ \sum_{j \neq r} g_j(x) : r = 1, 2, \ldots, s \right\}.$$

Since both $p(x)$ and $q(x)$ are convex functions, by introducing an additional variable t we can now rewrite (27) in the form

$$p(x) - t \geq 0, \qquad q(x) - t \leq 0.$$

Here, the first inequality is reverse convex, while the second is convex (and hence, can be incorporated into the convex constraints $h_i(x) \leq 0$, $i = 1, 2, \ldots, m$). Thus, at the cost of an additional variable, any convex program with several additional reverse convex constraints can be converted into the form P (with just one single reverse convex constraint) and treated by the above method.

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