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# Perfect Validity, Entailment and Paraconsistency

**Abstract.** This paper treats entailment as a subrelation of classical consequence and deducibility. Working with a Gentzen set-sequent system, we define an entailment as a substitution instance of a valid sequent all of whose premisses and conclusions are necessary for its classical validity. We also define a sequent Proof as one in which there are no applications of cut or dilution. The main result is that the entailments are exactly the Provable sequents. There are several important corollaries. Every unsatisfiable set is Provably inconsistent. Every logical consequence of a satisfiable set is Provable therefrom. Thus our system is adequate for ordinary mathematical practice. Moreover, transitivity of Proof fails upon accumulation of Proofs only when the newly combined premisses are inconsistent anyway, or the conclusion is a logical truth. In either case Proofs that show this can be effectively determined from the Proofs given. Thus transitivity fails where it least matters — arguably, where it ought to fail! We show also that entailments hold by virtue of logical form insufficient either to render the premisses inconsistent or to render the conclusion logically true. The Lewis paradoxes are not Provable. Our system is distinct from Anderson and Belnap's system of first degree entailments, and Johansson's minimal logic. Although the Curry set paradox is still Provable within naive set theory, our system offers the prospect of a more sensitive paraconsistent reconstruction of mathematics. It may also find applications within the logic of knowledge and belief.

## § 0. Introduction

My purpose in this paper is to create a new and systematic theory of entailment satisfying certain explicit conditions of adequacy; and to indicate its applications in the paraconsistent reconstruction of mathematics. The system is provably distinct from any known to me. It is based on an extremely natural and simple semantical relation of entailment, which is captured by an equally natural and simple Gentzen sequent system.

In my paper [1] I defined a *Proof* in a system of natural deduction as an ordinary proof in normal form with no applications of the absurdity rule ("*ex falso quodlibet*"). Main results were

(1) The Lewis paradoxes

$A, -A: B$

$A: B, -B$

are not Provable.

(2) Every unsatisfiable set of sentences is Provably inconsistent.

(3) All logical consequences of any satisfiable set of sentences are Provable therefrom.

(4) Transitivity of Proof fails only where the new combined premisses form an inconsistent set – in which case a Proof of their inconsistency is effectively determinable from the given Proofs.

These results are clearly significant for the paraconsistent reconstruction of mathematics. Let us define Theories by reference to Deductive closure, that is, closure with respect to our new class of Proofs. In our Proof system, contradictions do not Imply arbitrary sentences. There are distinct inconsistent Theories. This raises the possibility that different set Theories, for example, if inconsistent, might be so in "different ways". Some inconsistencies, as it were, might be less harmful than others.

I do not wish at this stage to raise unduly the hopes of naive set theorists. For I shall also show below that Curry's instance of naive abstraction enables us to Prove arbitrary sentences of the language of set theory. I have discussed elsewhere ([2], [3]) how a proper response to the early set theoretical paradoxes is to adopt a *free logic* of sets. The Curry paradox is Provable simply because the first order Proof system is still based on the misguided assumption that every term of the language denotes. Thus the possibility I would still hold out for those interested in paraconsistent mathematics is that, in a *free Logic* of sets there might be interestingly distinct inconsistent Theories. Different large cardinal assumptions in *ZF*, for example, might be inconsistent in different sorts of ways.

Results (2) and (3) above show that our Proof system is adequate to all the demands of our mathematical practice. What are these demands? We are deprived of finitary consistency proofs for interesting theories like arithmetic and set theory. Pending proofs of contradictions from our axioms (such as Peano's postulates, or the Zermelo-Fraenkel axioms) we carry on proving theorems from these axioms. Whenever we do discover inconsistencies in our axioms we do not rejoice in the deluge of easy consequences licensed by the first Lewis paradox, or absurdity rule. Instead we turn our attention back to our starting points, to seek the source of the contradiction. Thus it appears that the demands we make of our logic are two-fold.

(i) The logic should deliver all contradictions, wherever they may be

(ii) The logic should deliver all consequences of our mathematical axioms, should these be collectively consistent.

In our Proof system, as results (2) and (3) above show, these demands can be met. In a nutshell, if *ZF* is inconsistent, then it is Provably so; if it is consistent, then all its theorems are Theorems – that is, they are Deducible from the axioms.

Finally, result (4) shows that Deductive progress is cumulative in the usual way. So by (1) we have excised the Lewis Paradoxes. By (2), (3) and (4) we have done so with minimum mutilation to the deductive fabric of mathematics — indeed, arguably disturbing no part of mathematics as actually practised.

I ended [1] by suggesting that a slightly stronger notion of Proof would preserve results (1) — (4), and also yield

(5) Any Proof of a conclusion other than absurdity from a non-empty set of premisses is a substitution instance of a Proof the premisses of which form a consistent set; and any Proof of a logical truth from a non-empty set of premisses is a substitution instance of a Proof of a contingent conclusion

The gist of (5) is that Proofs contain no Lewis-like features. In a Proof one does not trade illicitly on the inconsistency of the premisses in order to obtain a potentially irrelevant conclusion. Nor does one trade on the logical truth of the conclusion in order to obtain it "from" any premisses. Rather, Proofs establish conclusions from premisses by means only of so much logical detail as is insufficient to reveal either the inconsistency of the premisses or the logical truth of the conclusion (hence the talk about substitution instances in (5)). For example, the Proof

$$\frac{A \ \& \ -A}{A}$$

is a substitution instance of the Proof

$$\frac{A \ \& \ B}{A}$$

the premiss "set" of which is consistent; and the Proof

$$\frac{A}{A \vee -A}$$

is a substitution instance of the Proof

$$\frac{A}{A \vee B}$$

the conclusion of which is contingent. (These are of course trivial examples. The general conjecture (5), however, is not trivial.)

My purpose in this paper, already briefly stated at the beginning, is to re-work all these ideas and results in a Gentzen sequent setting. Proofs of results are thereby simplified, and also slightly improved by symmetric treatment of premiss sets and conclusion sets of sequents. Moreover I am able also to prove conjecture (5) in a suitably Gentzenised form.

To do this I employ a *semantical* notion of entailment, or validity of sequents. In [1] I passed over this in silence, for the reason that I could not say anything about it. Indeed, I even maintained that one might be able to do without it. In this paper I am happy to address myself to the semantical problem, even if only to secure (5). It might still turn out that all the desired results can be obtained proof-theoretically. Whether one would wish in that case to kick the ladder away is an issue I shall not discuss here.

Once the semantical notion of entailment has been defined, the problem of soundness and completeness results arises for our Proof system. It is interesting that the usual burden of proof is shifted. In this paper it will be the soundness theorems that call for less trivial proof. This, however, might have been expected. The usual problem for the entailment theorist is to show that his Proof system does not let in too much — that it excludes the Lewis paradoxes and other undesirable results. This problem becomes that of proving a soundness theorem with respect to the new semantics that has been designed to invalidate these undesirable results.

At this point it is worth mentioning that I regard disjunctive syllogism

$$A \vee B, \ -A: \ B$$

as a thoroughly desirable result, in opposition to those in the Anderson-Belnap tradition. They preserve unrestricted transitivity at all costs. One of these costs is the rejection of disjunctive syllogism, a mode of inference indispensable in mathematical reasoning. In contrast, I give up transitivity in a very controlled way, arguably where it least matters — arguably, indeed, where it *ought* to be given up! — and thereby preserve disjunctive syllogism. I shall say more about this below.

Apart from its promise for paraconsistent mathematics, the Proof system given below might also be usefully applied in the logic of knowledge and belief. Existing systems treat only of ideal or rational attitudes, consistent and logically closed. What appears to be needed is a logic allowing sensitive discrimination between different inconsistent belief sets. This is a topic, however, that I shall not pursue in this paper.

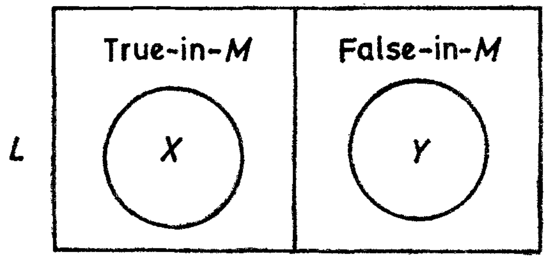
## § 1. Semantics

Let us proceed now to the main semantical idea. I shall generalize the basic idea behind a definition given by Smiley in [4] in connection with entailment, a definition that he gave for a single conclusion system. This generalization requires our being able to speak not just of premisses $A_1$, ..., $A_n$ entailing a (single) conclusion $B$, but of their entailing in general a *set* of conclusions $B_1, \ldots, B_m$. Thus we wish to speak of a sequent of the form

$$A_1, \ldots, A_n: \ B_1, \ldots, B_m$$

being valid, being perfectly valid or being an entailment (to mention the three main notions to be defined in due course). Note that it is *sets* that stand on each side of the colon in a sequent. Order and repetition of premisses on the left, or of conclusions on the right, are irrelevant. Thus also when we come to the Gentzen systems of proof and Proof, it will be *set* sequents with which we shall be dealing, rather than the *sequence* sequents of Gentzen himself. So we shall not be needing rules like Permutation and Contraction in the sequent system — but more of that below.

Classically, a *valid* sequent $X: Y$ is a sequent that cannot be "partitioned" by any interpretation, or model, of the language. That is, there is no way of making all of $X$ true and all of $Y$ false. Diagrammatically, an *invalid* sequent $X: Y$ is thus one for which there exists a model $M$ resulting in a partition of the sentences of the language $L$ thus:




The model $M$ is a counterexample to the argument

All of $X$
*ergo*, At least one of $Y$

or, more simply, a *model of* $X: Y$. Thus a valid sequent is one that has no models.

ø is the empty set. By our definition, $X$: ø is valid just in case $X$ is not satisfiable, and ø: $Y$ is valid just in case $Y$ is not falsifiable. And precisely for this reason the Lewis paradoxes are valid on the classical definition. Neither $\{A, -A\}:\{B\}$ nor $\{A\}:\{B, -B\}$ has a model. (Henceforth I shall omit set braces wherever possible). This is because neither of the respective proper subsequents

$$A, -A: ø \quad ø: B, -B$$

has any models. This motivates the following definition.

A sequent $X: Y$ is *perfectly* valid iff it is valid and has no valid proper subsequents. The Lewis paradoxes, though valid, are not perfectly so. Likewise with the sequent $A, B: A$. The sequent $A \,\&\, B: A$, however, is perfectly valid, as is $A \,\&\, B: B$. The sequent $A \,\&\, -A: B$ is valid, but not perfectly so.

A substitution is a mapping from atoms to formulae. It can be extended to a mapping from formulae to formulae in the obvious way –

$$s(-A) = -s(A)$$

$$s(A \,\&\, B) = s(A) \,\&\, s(B) \text{ etc}$$

and to sets of formulae –

$$s(X) = \{s(A) \mid A \in X\}$$

and thus also to sequents –

$$s(X\colon Y) = s(X)\colon s(Y)$$

We shall represent this diagrammatically as

$$\begin{array}{c} X\colon Y \\ A_1 \ldots A_n \\ s\downarrow \quad\; \downarrow \\ B_1 \ldots B_n \\ s(X\colon Y) \end{array}$$

where $s$ replaces each atom $A_i$ by the (possibly complex) formula $B_i$. Whenever

$$\begin{array}{c} X\colon Y \\ s\downarrow \\ Z\colon W \end{array}$$

we say also that $X\colon Y$ *is a suprasequent of* $Z\colon W$ *via* $s$.

We are now in a position to state our main definition.

A sequent $X\colon Y$ is an *entailment* iff $X\colon Y$ has a perfectly valid suprasequent.

This definition of entailment deals only with classical validity and economy of statement in terms of set-inclusion and substitution. It involves no unusual or counterintuitive re-interpretation of the senses of logical operators.

The sequent $A \,\&\, -A{:}A$ is an entailment by virtue of the perfectly valid suprasequent $A \,\&\, B\colon A$. Some perfectly valid sequents have perfectly valid proper suprasequents, e.g.

$$\begin{array}{cc} B,\, -B\colon & -(A \,\&\, B)\colon\; -A \vee C,\; D \vee -B \\ \downarrow & \downarrow \\ -A,\, --A\colon & -(A \,\&\, B)\colon\; -A \vee -B \end{array}$$

Every perfectly valid sequent is a proper suprasequent of some perfectly valid sequent, as can be seen by simply mapping atoms to their double negations. A sequent can be an entailment by virtue of two perfectly valid suprasequents neither of which is a suprasequent of the other, e.g.

$$\begin{array}{ccccc} A \,\&\, C\colon A \vee D & & & & E \,\&\, B\colon F \vee B \\ A \;\; C \;\; D & \searrow & & \swarrow & E \;\; B \;\; F \\ \downarrow \;\; \downarrow \;\; \downarrow & & & & \downarrow \;\; \downarrow \;\; \downarrow \\ A \;\; B \;\; B & & & & A \;\; B \;\; A \\ & & A \,\&\, B\colon A \vee B & & \end{array}$$

The different suprasequents correspond to distinct "lines of argument" that serve to establish the entailment – in this case, extracting right or left conjuncts of the premiss $A \,\&\, B$. Of course, much more radical differences can be expected in more complicated cases.

Our task in the next section will be to define notions of *Proof* and of *perfect proof* in appropriate sequent calculi and to obtain the adequacy result.

The Provable sequents are precisely the entailments.

Our Proof theorywill then tell us some important facts about the entailment relation. The adequacy result is also of course the affirmative solution of conjecture (5) in a Gentzen setting.

## § 2. Syntax

I shall confine myself throughout the body of this paper to the connectives –, &. All the proof theoretical results hold for –, &, ∨, ∃ and ∀ primitive. Towards the end of the paper I shall have more to say about quantification and identity. For the time being, however, the ideas are best illustrated in the simplest possible system.

In the classical sequent system each logical operator has two rules. One tells us how to introduce a dominant occurrence of the operator in a formula on the left of the colon in a sequent derived by means of that rule; the other tells how to do so on the right. For our chosen operators these rules are

$$\frac{X,\ A\colon Y}{X\colon Y,\ -A} \qquad \frac{X\colon Y,\ A}{X,\ -A\colon Y}$$

$$\frac{X\colon Y,\ A \qquad Z\colon W,\ B}{X,\ Z\colon Y,\ W,\ A \,\&\, B} \qquad \frac{X,\ A\colon Y}{X,\ A \,\&\, B\colon Y} \qquad \frac{X,\ B\colon Y}{X,\ A \,\&\, B\colon Y}$$

The displayed formula $A$ in the negation rules is called the *component* for its application. Likewise for $A$, $B$ in the conjunction rules. We have a notational convention whereby the component is assumed not to be a member of the set separated off from it by a comma. Thus in the first negation rule above, we assume $A \notin X$.

In addition to these rules for logical operators there are the following *structural* rules.

Dilution $\dfrac{X:\ Y}{X:\ Y,\ A} \qquad \dfrac{X:\ Y}{X,\ A:\ Y}$

Cut $\dfrac{X:\ Y,\ A \quad A,\ Z:\ W}{X,\ Z:\ Y,\ W}$

Using any of these rules one builds proofs from initial sequents of the form $A{:}A$ in the usual way. For example

$$\dfrac{\dfrac{\dfrac{A:\ A}{:\ A,\ -A}}{:\ A,\ -A\vee -B} \quad \dfrac{\dfrac{B:\ B}{:\ B,\ -B}}{:\ B,\ -A\vee -B}}{\dfrac{:\ A\ \&\ B,\ -A\vee -B}{-(A\ \&\ B):\ -A\vee -B}} \qquad \dfrac{\dfrac{A:\ A \quad B:\ B}{A\vee B:\ A,\ B}}{A\vee B,\ -A:\ B}$$

These proofs of course also involve rules for disjunction, which are dual to those we have given for conjunction. For the record, they are

$$\dfrac{X:\ Y,\ A}{X:\ Y,\ A\vee B} \qquad \dfrac{X:\ Y,\ B}{X:\ Y,\ A\vee B} \qquad \dfrac{X,\ A:\ Y \quad Z,\ B:\ W}{X,\ Z,\ A\vee B:\ Y,\ W}$$

In neither of our two proofs above is the Cut rule applied. This is no accident. As is well known, Gentzen proved the following theorem.

CUT ELIMINATION THEOREM. *Any proof of* $X:Y$ *can be converted into a cut free proof of* $X{:}Y$.

But even in a cut free proof, dilutions can be a source of irrelevancy. This can be shown clearly by the following cut free proof of the first Lewis paradox.

$$\dfrac{\dfrac{A:\ A}{A,\ -A:}}{A,\ -A:\ B}\ \text{dilution}$$

Our next theorem tells us what can be done about this. We define a *Proof* (with uppercase 'P') as a cut free, dilution free proof.

DILUTION ELIMINATION THEOREM. *Any cut free proof of* $X:Y$ *can be converted into a Proof of some subsequent of* $X{:}Y$.

(We shall write $\pi \curvearrowright \Sigma$ for "$\pi$ can be converted into the Proof $\Sigma$".)

PROOF. By induction on the length of proof. The basis is obvious, since $A:\ A$ is already a Proof. In the inductive step we proceed by cases, according to the rule last applied. If any Proof given by the inductive hypothesis establishes a final sequent lacking the relevant component on the relevant side, take it as the required Proof; otherwise, apply the relevant rule to obtain the required Proof. For example, if

$$\overset{\pi}{X: Y, A} \;\frown\; \overset{\Sigma}{Z: W} \qquad (Z \subseteq X, W \subseteq Y)$$

then

$$\overset{\pi}{\dfrac{X: Y, A}{X, -A: Y}} \frown \overset{\Sigma}{Z: W};$$

but if

$$\overset{\pi}{X: Y, A} \;\frown\; \overset{\Sigma}{Z: W, A} \qquad (Z \subseteq X, W \subseteq Y)$$

then

$$\overset{\pi}{\dfrac{X: Y, A}{X, -A: Y}} \frown \overset{\Sigma}{\dfrac{Z: W, A}{Z, -A: W}}$$

Other cases are similar and are left to the reader. Of course, if

$$\overset{\pi}{X: Y} \;\frown\; \overset{\Sigma}{Z: W}$$

then

$$\overset{\pi}{\dfrac{X: Y}{X: Y, A}} \frown \overset{\Sigma}{Z: W} \quad \text{and} \quad \overset{\pi}{\dfrac{X: Y}{X, A: Y}} \frown \overset{\Sigma}{Z: W}$$

This completes the proof of the theorem.

The dilution elimination theorem is the sequent version of the extraction theorem for systems of natural deduction in [1]. Note how in its proof the inductive step cannot be carried out for $\supset$. The sequent rules for $\supset$ are

$$\frac{X, A: Y, B}{X: Y, A \supset B} \qquad \frac{X: Y, A \quad Z, B: W}{X, Z, A \supset B: Y, W}$$

An example of a proof without cut that cannot be converted into a Proof of any subsequent of its final sequent is

$$\cfrac{\cfrac{\cfrac{A: A}{A, -A:}}{A, -A: B}}{-A: A \supset B}$$

We therefore restrict ourselves to a system with $-$, & and $\vee$ as primitive connectives. We can now obtain as corollaries of the cut - and dilution - elimination theorems sequent versions of our results (1) — (4) above.

COROLLARY 1. *The Lewis paradoxes* $A, -A : B$ *and* $A: B, -B$ *are not Provable.*

PROOF. The only possible forms of cut free proof of the first paradox are

$$\frac{\dfrac{A: A}{A: A, B}}{A, -A: B} \qquad \frac{\dfrac{A: A}{A, -A:}}{A, -A: B}$$

both of which involve dilution and are therefore not Proofs. Similarly for the second paradox.

COROLLARY 2. *If* $X$ *is not satisfiable then for some subset* $Z$ *of* $X$ *there is a Proof of* $Z: \varnothing$; *and if* $Y$ *is not falsifiable, then for some subset* $W$ *of* $Y$ *there is a Proof of* $\varnothing: W$.

PROOF. Suppose $X$ is not satisfiable. By classical completeness there is a proof of $X': \varnothing$ for some subset $X'$ of $X$. By eliminating cuts and then dilutions we obtain a Proof of $Z: \varnothing$ for some subset $Z$ of $X'$, hence of $X$, as required. The second half is proved similarly.

COROLLARY 3. *If* $X$ *is satisfiable and logically implies* $A$, *then for some subset* $Z$ *of* $X$ *there is a Proof of* $Z: A$.

PROOF. Suppose $X$ is satisfiable and logically implies $A$. By classical completeness, cut- and dilution-elimination there is a Proof of $Z: \varnothing$ or of $Z: A$, for some subset $Z$ of $X$. By satisfiability of $X$ and classical soundness, only the latter can be the case.

Corollary 3 has been formulated with an emphasis on single conclusions, but this is unnecessary. A general and symmetric statement is

> If $X$ is satisfiable and $Y$ is falsifiable and $X: Y$ is valid, then for some non-empty subsets $Z$, $W$ of $X$, $Y$ respectively there is a Proof of $Z: W$.

Note that a proof (hence also a Proof) of a subsequent $Z: W$ of $X: Y$ tells us something stronger than a proof of $X: Y$ itself. It is much harder for a sequent to be valid (provable), the fewer sentences it has — 'fewer' in the sense of proper inclusion. Hence the subsetting mentioned in Corollaries 2 and 3 (and 4 below) does not detract at all from the result — in fact, it enhances it. If one can 'winnow down' a sequent without loss of validity or of provability, one is improving matters. One might even come to learn that the premisses were inconsistent or the conclusions not falsifiable, or that fewer of the premisses served to ensure that the truth lay among fewer of the conclusions. This epistemic gain should therefore be borne in mind in assessing the next result.

COROLLARY 4. *For* $1 \leqslant i \leqslant n$, *let* $\underset{X_i:Y_i,A_i}{\pi_i}$ *be a Proof, and let* $\underset{X_0,A_1,\ldots,A_n:Y_0}{\Sigma}$ *be a Proof. Then there is a Proof* $\underset{X:Y}{\pi}$ *for some subsets* $X$, $Y$ *of the respective unions of the* $X_i$ *and of the* $Y_i$.

PROOF. By $n$-fold cut obtain a proof of the sequent $\bigcup_{i=0}^{n} X_i : \bigcup_{i=0}^{n} Y_i$. By cut- and dilution-elimination turn this into a Proof as required.

Thus if we have Proofs of Axioms: Lemma 1 and of ... and of Axioms: Lemma $n$ and of Axioms, Lemmata $1-n$: Theorem, then we can determine from them a Proof either of Axioms: Theorem or of Axioms: ø or of ø: Theorem. In the last two cases – where transitivity of Proof "fails" – we learn either that our axioms are inconsistent (in which case we are hardly likely to mourn the loss of Theorem) or that Theorem was a logical truth anyway (in which case we are pleased to have a Proof of it outright). This is why I insist that transitivity of Proof fails where it least matters – indeed, where it *ought* to fail.

Note that every substitution instance of a Proof is a Proof.

We now introduce the notion of *perfect* proof, designed to capture perfect validity. But note that not all perfect proofs will be Proofs, nor will all Proofs be perfect proofs, as will become evident in due course. First we need some notation and terminology.

When $X$ and $Y$ are non-empty sets of formulae, $X$ & $Y$ is the set of all conjunctions with left conjuncts in $X$ and right conjuncts in $Y$. When $X$ is a non-empty set of formulae, and $A$ is a formula, then $X$ & $A$ is the set $\{B \,\&\, A / B \in X\}$. Likewise for $A$ & $X$. $\sim X$ is $\{\sim A / A \in X\}$.

Now *perfect* proofs are built up from initial sequents $A$: $A$ (with $A$ atomic), without cut or dilution, by means of the following 'Frobenian' rules having *sets* of formulae in general as components:

$$\frac{X:\ Z,\ Y}{X,\ -Z:\ Y} \qquad \frac{X,\ Z:\ Y}{X:\ Y,\ -Z}$$

$$\frac{X, Z: Y}{X, Z \,\&\, A: Y} \qquad \frac{X: Y, Z \quad U: V, W}{X, U: Y, V, Z \,\&\, W}$$

$$\frac{X,\ Z:\ Y}{X,\ A \,\&\, Z: Y}$$

where $A$ is an atom that does not occur in the top sequent (i.e. $A$ is a fresh atom)

where the upper sequents have no atoms in common (i.e. are vocabulary disjoint)

PERFECTION THEOREM *Any Proof* $\underset{X:Y}{\pi}$ *can be converted into a perfect proof* $\underset{Z:W}{\Sigma}$ *for some suprasequent* $Z$: $W$ *of* $X$: $Y$.

PROOF. We obtain $\Sigma$ by working from the top down in $\pi$. At &: we re-label so as to achieve foreignness of atoms, and at : & we re-label so as to achieve vocabulary disjointness. Via the substitution mapping thus induced at each stage we keep track of what subsets within the new sequents thus formed "on the way to" $\Sigma$ correspond to the components of each step in the original Proof $\pi$; and as we work down $\pi$ we mimic its steps in the new perfect proof $\Sigma$ under construction in the appropriately "Frobenian" fashion.

EXAMPLES. Our Proof above of disjunctive syllogism $A \vee B,\ -A : B$ is already a perfect proof. Our Proof of the de Morgan sequent $(A\ \&\ B)$: : $-A \vee -B$, however, is not. Bearing in mind that the rules for $\vee$ are simply dual to those for &, so that the same sorts of considerations apply, we can turn our Proof of the de Morgan sequent into the following perfect proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{A:\ A}{:A,\ \ -A}}{:A,\ \ -A\vee C}\qquad\dfrac{\dfrac{B:\ B}{:B,\ \ -B}}{:B,\ D\vee -B}}{:A\ \&\ B,\ \ -A\vee C,\ \ D\vee -B}}{-(A\ \&\ B):\ \ -A\vee C,\ \ D\vee -B}\qquad\qquad \begin{array}{rcccc} & A & B & C & D\\ s & \downarrow & \downarrow & \downarrow & \downarrow\\ & A & B & -B & -A\end{array}$$

Note how in the third line $C$ and $D$ are foreign atoms for $:\vee$, and how the sequents in the third line are "vocabulary disjoint" for : &. As a final example, consider the Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{A:\ A}{:A,\ \ -A}}{:A\ \ \ -A\vee -B}\qquad\dfrac{\dfrac{B:\ B}{:B,\ \ -B}}{:B,\ \ -A\vee -B}}{:A\ \&\ B, -A\vee -B}}{-(-A\vee -B):A\ \&\ B}$$

with the perfected version

$$\dfrac{\dfrac{\dfrac{\dfrac{A:\ A}{:A,\ \ -A}}{:A,\ \ -A\vee C}\qquad\dfrac{\dfrac{B:\ B}{:B,\ \ -B}}{:B,\ D\vee -B}}{:A\ \&\ B,\ \{-A\vee C,\ \ D\vee -B\}}}{-(-A\vee C),\ \ -(D\vee -B):A\ \&\ B}\qquad\qquad \begin{array}{rcccc} & A & B & C & D\\ s & \downarrow & \downarrow & \downarrow & \downarrow\\ & A & B & -B & -A\end{array}$$

The final step here is an application of the "Frobenian" rule $-$: of perfect proof. The set braces are inserted to make clear what the component is for the application of the rule. Note how under the substitution $s$ this set component "collapses" to the single formula $-A \vee -B$ which, in the original Proof, was the component for the application of the negation rule in question.

## § 3. Soundness and Completeness Results

COMPLETENESS THEOREM. *Every entailment is Provable.*

PROOF. Suppose $X : Y$ is an entailment. Let $Z : W$ be a perfectly valid suprasequent of $X : Y$ via $s$, as required by the definition of entailment. By Corollaries 2 and 3 above there is a Proof $\underset{Z':W'}{\pi}$ for some subsets $Z'$, $W'$ of $Z$, $W$ respectively. By soundness of proof, hence of Proof, $Z' : W'$ is valid. By perfect validity of $Z: W$, $Z = Z'$ and $W = W'$. Substituting via $s$ in $\pi$ we obtain a Proof of $X: Y$.

PERFECT COMPLETENESS THEOREM. *Every perfectly valid sequent has a perfectly provable suprasequent.*

PROOF. Let $X : Y$ be perfectly valid. By classical completeness there is a proof of $X: Y$, which by cut- and dilution-elimination, and then perfection, can be turned into a perfect proof $\underset{Z:W}{\pi}$ for some $Z: W$ such that

$$\begin{array}{cc} & Z:\ W \\ s & \downarrow \\ & X':\ Y' \subseteq X:\ Y \end{array}$$

Since $X: Y$ is perfectly valid, $X = X'$ and $Y = Y'$. Hence the result.

COROLLARY. *Every entailment has a perfectly provable suprasequent.* Thus we can generate the entailments not only by Proof, but also by perfect proof and substitution.

PERFECT SOUNDNESS THEOREM. *Every perfectly provable sequent is perfectly valid.*

PROOF. Initial sequents are perfectly valid. It is clear that the rules of perfect proof preserve ordinary validity. Thus it remains to show that, if their upper sequents are *perfectly* valid, then so are their lower sequents. To do this we assume the perfect validity of the upper sequents, and show that any proper subsequent of a lower sequent has a model (i.e. is invalid). We proceed by cases, according to the rule in question.

*The rule* :$-$

Consider $\dfrac{X,\ Z\colon\ Y}{X\colon\ Y,\ -Z}$

Suppose the upper sequent $X,\ Z\colon\ Y$ is perfectly valid (i.e any proper subsequent of $X,\ Z\colon\ Y$ has a model). Consider any proper subsequent $T\colon\ U$ of the lower sequent $X\colon\ Y,\ -Z$. So $T\colon\ U$ is the result of dropping at least one formula $A$ say in $X\colon\ Y,\ -Z$.

(i) Suppose $A$ is dropped from $X$. Let the result be $X'$. Then $X', Z\colon\ Y$ is a proper subsequent of $X,\ Z\colon\ Y$ and thus has a model $M$ say. $M$ is a model of $X'\colon\ Y,\ -Z$, hence also of $T\colon\ U$.

(ii) Suppose $A$ is dropped from $Y$. Let the result be $Y'$. Then $X,\ Z\colon\ Y'$ is a proper subsequent of $X,\ Z\colon\ Y$ and thus has a model $M$ say. $M$ is a model of $X\colon\ Y',\ -Z$, hence also of $T\colon\ U$.

(iii) Suppose $A$ is dropped from $-Z$. Then the result is of the form $-Z'$ for some proper subset $Z'$ of $Z$. $X,\ Z'\colon\ Y$ is a proper subsequent of $X, Z\colon\ Y$ and thus has a model $M$ say. $M$ is a model of $X\colon\ Y,\ -Z'$, hence also of $T\colon\ U$.

The reasoning for the rule $-$: is similar.

*The rule* &:

Consider $\dfrac{X, Z\colon Y}{X, Z\ \&\ B\colon Y}$ where $B$ is a foreign atom. (The reasoning for $B\ \&\ Z$ is similar.) Suppose the upper sequent $X, Z\colon Y$ is perfectly valid. Consider any proper subsequent $T\colon\ U$ of the lower sequent $X,\ Z\ \&\ B\colon\ Y$. So $T\colon\ U$ is the result of dropping at least one formula $A$ say in $X, Z\ \&\ B\colon\ Y$.

(i) Suppose $A$ is dropped from $X$. Let the result be $X'$. Then $X', Z\colon\ Y$ is a proper subsequent of $X,\ Z\colon\ Y$ and thus has a model $M$ say. Since $B$ is foreign, extend $M$ to a model $N$ in which $B$ is true. $N$ is a model of $X',\ Z\ \&\ B\colon\ Y$, hence also of $T\colon\ U$.

(ii) If $A$ is dropped from $Y$, the reasoning is similar.

(iii) Suppose $A$ is dropped from $Z\ \&\ B$. Then the result is of the form $Z'\ \&\ B$ for some proper subset $Z'$ of $Z$. Now $X, Z'\colon Y$ has a model $M$ say. Extend $M$ to a model $N$ in which $B$ is true. $N$ is a model of $X,\ Z'\ \&\ B{:}Y$, hence also of $T{:}U$.

*The rule* :&

Consider $\dfrac{X\colon Y, Z \quad U\colon V, W}{X,\ U\colon\ Y,\ V, Z\ \&\ W}$ with the upper sequents vocabulary disjoint and perfectly valid. Consider any proper subsequent $T$: $R$ of the lower sequent, resulting from it by dropping at least one formula $A$ say.

(i) Suppose $A$ is dropped from $X$. Let $X'$ be the result. $X'$: $Y$, $Z$ has a model $M$ say. By truth table for &, $M$ is a model of $X'$: $Y, Z$ & $W$. $U$: $V$ has a model $N$ say. By vocabulary disjointness the union of $M$ and $N$ is a model of $X'$, $U$: $Y$, $V$, $Z$ & $W$, hence also of $T$: $R$.

(ii) If $A$ is dropped from $U$, $Y$ or $V$ the reasoning is similar.

(iii) Suppose $A$ is dropped from $Z$ & $W$. Suppose $A$ is $B$ & $C$, where $B$ is in $Z$ and $C$ is in $W$. $X$: $Y$, $Z \setminus \{B\}$ has a model $M$ say, and $U$:$V$, $W \setminus \{C\}$ has a model $N$ say. By vocabulary disjointness the union of $M$ and $N$ is a model of $X$, $U$: $Y$, $V$, $Z$ & $W \setminus \{B$ & $C\}$, hence of $T$: $R$. This holds no matter how the union model may have to be extended to deal with atoms involved in $B$ and $C$ that might not be assigned values in $M$ and $N$ respectively.

SOUNDNESS THEOREM. *Every Provable sequent is an entailment.*

PROOF. Take any Proof $\underset{X:Y}{\pi}$. Then it can be turned into a perfect proof $\Sigma$ of some suprasequent $Z$: $W$ of $X$:$Y$. By perfect soundness, $Z$: $W$ is perfectly valid. Hence $X$: $Y$ is an entailment.

In a readily graspable sense explained in the proof of the perfection theorem, $\pi$ is a "substitution instance" of $\Sigma$, even though $\pi$ may not be perfect. Moreover if $X$ and $Y$ are non-empty, so are $Z$ and $W$, with $Z$ satisfiable and $W$ falsifiable (by virtue of perfect validity of $Z$: $W$). Hence conjecture (5) above has been answered affirmatively.

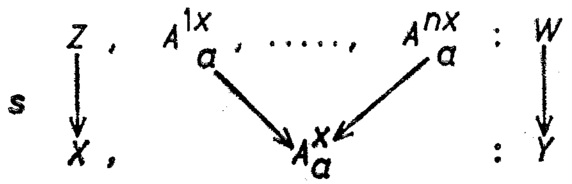
## § 4 Quantifiers and Identity

So far we have been discussing only propositional logic. Now that the reader is familiar with the main ideas, we can indicate how to extend the treatment to deal with the quantifiers. For the time being we shall consider first order logic *without* identity.

An important point to note is that perfecting Proofs is a process that produces *sets* of sentences in general as components for applications of rules. Consider now the rule for introducing the existential quantifier on the left.

$$\frac{X, A^x_a : Y}{X, \exists\, xA : Y} \text{ where } a \text{ does not occur in the lower sequent.}$$

A proof $\underset{X,A^x_a:Y}{\pi}$ might be perfected as $\underset{Z,A^{1x}_a,\ldots,A^{nx}_a:W}{\Sigma}$ where the associated substitution mapping is




Our rules in the propositional case mentioned "foreignness of atoms" and "vocabulary disjointness". In the case of first order logic without identity we now understand this to apply to function and predicate expressions, but not necessarily to names or parameters. In perfecting a Proof we can leave the pattern of name- and parameter-occurrences untouched, save for the proliferation of their occurrences under the set-creation just mentioned. To apply $\exists$: in the perfect proof $\Sigma$ above, we need the notion of a *set conjunction*

$$\& \, (A_a^{1x}, \ldots, A_a^{nx})$$

so that we can form its existential closure

$$\exists x \, \& \, (A^1, \ldots, A^n)$$

The set conjunction is a new kind of "sentence" the truth conditions of which are that every member should be true. Likewise for $\forall$ we shall need a notion of set disjunction.

Under substitution via $s$, the existential sentence just given collapses to $\exists x \, \& \, (A)$, which we simply identify with $\exists x A$. Equipped with these notions, the reader can carry out all the proofs above for first order logic without identity. Most importantly, the perfect soundness theorem requires that we be able to form model unions and extensions. By the respective disjointness and foreignness conditions, this is easy, given that the language does not contain identity. For, consider what is involved in forming the union of two models. If they differ in cardinality, add indiscernibles to the smaller model in order to make the domains have the same cardinality. Then define a 1-1 onto map between the domains by assigning to named individuals their namesakes in the other model, and extending the map arbitrarily on nameless individuals. Vocabulary disjointness then ensures the consistency of the model formed by the union of the relational structures via the 1-1 map just constructed, in the obvious way. In this union, an individual has all the relational properties that it (via the map) has in either of the two models forming the union.

In the language of the first order logic with identity the identity predicate itself will not be distinguished. Identity will be treated axiomatically. Logical truths of the first order logic of identity will be just those sentences that follow from the axioms of identity, namely all instances of reflexivity and substitutivity.

## § 5. Comparison with other systems

The system of entailment and Proof set out above is distinct from the Anderson-Belnap system of first degree entailment. For disjunctive syllogism is Provable, but is not a first degree entailment.

Nor will the restriction of $Y$ to singletons make our system of Provable sequents $X: Y$ coextensive with minimal logic. For disjunctive syllogism is not provable in minimal logic either. Moreover, the sequent $A, -A:$ $: -B$ *is* provable in minimal logic, but is not Provable.

Note that Johansson obtained minimal logic from Gentzen's sequent system for intuitionistic logic simply by dropping dilution. But the rules: & and :∨ had the more restrictive form

$$\frac{X: Y, A \quad X: Y, B}{X: Y, A \,\&\, B} \qquad \frac{X, A: Y \quad X, B: Y}{X, \ A \vee B: \ Y}$$

In our Proof system, the rules

$$\frac{X: Y, A \quad Z: W, B}{X, Z: Y, W, A \,\&\, B} \qquad \frac{X, A: Y \quad Z, B: W}{X, Z, A \vee B: Y, W}$$

ironically allow in a little of the dilution that would otherwise be required to top up the upper sequents to the same $X$ and $Y$ before applying the more restrictive rules.

## § 6. The Curry paradox in naive set theory

As noted by Meyer, Routley and Dunn [7], some people had hoped that a paraconsistent logic might be found in which naive set theory, despite its inconsistency, would not collapse onto the whole language. As they show, the relevance logic $\boldsymbol{R}$ cannot serve this purpose, because using $\boldsymbol{R}$ one can derive arbitrary sentences as theorems of naive set theory by choosing suitable substitution instances, due to Curry, of the naive comprehension axiom scheme. In this section I show that the same is true of the logical system of this paper.

Remembering that we do not have ⊃ primitive, let $\gamma$ be the set abstract $\{x| -(x \in x \;\&\; -q)\}$. Let $p$ be the sentence $\gamma \in \gamma$. The naive comprehension schema has the instance

$$\forall z\big(z \in \gamma \equiv -(z \in z \;\&\; -q)\big)$$

which in $(-, \;\&)$ language is

$$(A) \qquad \forall z\Big(-\big(z \in \gamma \;\&\; --(z \in z \;\&\; -q)\big) \;\&\; -\big(-(z \in z \;\&\; -q) \;\&\; -z \in \gamma\big)\Big).$$

Taking $\gamma$ for $z$ we obtain by universal elimination the instance

$$-\big(p \;\&\; --(p \;\&\; -q)\big) \;\&\; -\big(-(p \;\&\; -q) \;\&\; -p\big)$$

The following Proof $\pi$ shows that this entails $p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p: p}{p \mathbin{\&} -q: p} \qquad \dfrac{\dfrac{p \mathbin{\&} -q: p \mathbin{\&} -q}{p \mathbin{\&} -q,\ -(p \mathbin{\&} -q):}}{p \mathbin{\&} -q:\ --(p \mathbin{\&} -q)}}{p \mathbin{\&} -q: p \mathbin{\&} --(p \mathbin{\&} -q)}}{p \mathbin{\&} -q,\ -(p \mathbin{\&} --(p \mathbin{\&} -q)):}}{-(p \mathbin{\&} --(p \mathbin{\&} -q)): -(p \mathbin{\&} -q)} \qquad \dfrac{p: p}{: p,\ -p}}{-(p \mathbin{\&} --(p \mathbin{\&} -q)):\ -(p \mathbin{\&} -q) \mathbin{\&} -p,\ p}}{-(p \mathbin{\&} --(p \mathbin{\&} -q)),\ -(-(p \mathbin{\&} -q) \mathbin{\&} -p):\ p}}{-(p \mathbin{\&} --(p \mathbin{\&} -q)) \mathbin{\&} -(-(p \mathbin{\&} -q) \mathbin{\&} -p),\ -(-(p \mathbin{\&} -q) \mathbin{\&} -p): p}}{-(p \mathbin{\&} --(p \mathbin{\&} -q)) \mathbin{\&} -(-(p \mathbin{\&} -q) \mathbin{\&} -p):\ p}$$

Abbreviate the final sequent to $r$: $p$. We can then continue the Proof to one of $A$: $q$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ r: p\end{array} \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ r: p\end{array} \quad \dfrac{q: q}{: q,\ -q}}{r: p \mathbin{\&} -q,\ q}}{r,\ -(p \mathbin{\&} -q): q}}{r:\ --(p \mathbin{\&} -q),\ q}}{r:\ p \mathbin{\&} --(p \mathbin{\&} -q),\ q}}{r,\ -(p \mathbin{\&} --(p \mathbin{\&} -q)):\ q}}{r: q}}{A: q}$$

(The reader might, out of interest, try to perfect this Proof!)

Thus our Proof system cannot save naive set theory from triviality. Nevertheless, this need not count against the possibility of discovering distinct inconsistent set theories – perhaps even theories extending *ZF*. The problem with naive set theory is that it is so thoroughly naive! More precisely, the first order logic on which it is based is naive. The first and most obvious response to the inconsistency of naive set theory is to adopt a free logic, freed of the assumption that every set abstract denotes. Curry's paradox has been shown above to arise within a non-free logic from a naive axiom schema. We might, however, prefer to treat the latter inferentially, by incorporating into a "Logic of sets" the two sequent rules

$$\dfrac{X:\ Y,\ F^x_t}{X:\ Y,\ t \in \{x|F\}} \qquad \dfrac{X,\ F^x_t:\ Y}{X,\ t \in \{x|F\}:\ Y.}$$

Of course we can then no longer prove that every (classical) proof can be converted into a cut-free version, so the new Proof system (in which cut is prohibited as before, as well as dilution) does not automatically satisfy results (1) – (4) about Proofs above, results that relied on cut-elimination in the underlying proof system. Nevertheless, the suggested "Logic of sets" is not without interest, being such a simple extension to set theory of our earlier proof system. Neither Russell's paradox nor Curry's paradoxical instances appear to admit of Proof in this system. It would be most interesting to investigate just how much of naive set theory could be thus Developed.

## § 7. Varia

Our definition of entailment in this paper is reminiscent of Smiley's definition ([4], p.240) of an entailment relation $\vdash$:

> $A_1, \ldots, A_n \vdash B$ if and only if the implication $(A_1 \,\&\, \ldots \,\&\, A_n) \supset B$ is a substitution instance of a tautology $(A_1' \,\&\, \ldots \,\&\, A_n') \supset B'$, such that neither $\vdash B'$ nor $\vdash -(A_1' \,\&\, \ldots \,\&\, A_n')$.

Tony Dale has pointed out to me that on this definition the premisses $(A \vee B)$, $-(A \vee B)$ do not entail $A \,\&\, B$. Now on my account $-(A \vee B)$ entails each of $-A$, $-B$, so by applying disjunctive syllogism twice one would expect $A \,\&\, B$ to be entailed by the given premisses. And indeed the following Proof shows this to be the case:

$$\dfrac{\dfrac{\dfrac{A:A \quad \dfrac{B:B}{B:A\vee B}}{A\vee B:\ A\vee B,\ A}}{A\vee B,\ -(A\vee B):A} \qquad \dfrac{\dfrac{\dfrac{A:A}{A:A\vee B} \quad B:B}{A\vee B:\ A\vee B,\ B}}{A\vee B,\ -(A\vee B):\ B}}{A\vee B,\ -(A\vee B):\ A\,\&\,B}$$

The perfected version of this Proof is

$$\dfrac{\dfrac{\dfrac{A:A \quad \dfrac{B:B}{B:C\vee B}}{A\vee B:\ C\vee B,\ A}}{A\vee B,\ -(C\vee B):\ A} \qquad \dfrac{\dfrac{\dfrac{E:E}{E:E\vee D} \quad F:F}{E\vee F:\ E\vee D,\ F}}{E\vee F,\ -(E\vee D):\ F}}{A\vee B,\ E\vee F,\ -(C\vee B),\ -(E\vee D):\ A\,\&\,F}$$

Note how the re-lettering with $C$ and $D$ brings out the different lines of argument indicated in the remarks above. Note also how by liberalizing to sets of premisses on the left of a colon, rather than conjoining them to form the antecedent of an implication, we achieve an important degree of freedom in seeing how one argument (i.e. a sequent) can be a substitution instance of another. Substitution can importantly "merge" previously distinct formulae, as has happened in our example, in which the distinct formulae $-(C \vee B)$, $-(A \vee D)$ merge, upon substitution of $A$ for $C$ and $B$ for $D$, into the single formula $-(A \vee B)$.

The relation of entailment in propositional logic is decidable on finite $X$ and $Y$, because there are only finitely many suprasequents (up to isomorphism via re-lettering) to check for perfect validity.

Entailment for first order logic is compact and undecidable because ordinary logical consequence is.

I conjecture that the mutual entailment of two sentences is a sufficient condition for their interreplaceability *salva veritate* in all statements of entailment.

Finally, it is worth noting an agreeable philosophical stability in our choice of a Logic. If all the background theory of sets etc. that has been used in our metalogical treatment is consistent, then by Corollary 3 above we have secured all our meta-results about our Logic using the same as our metaLogic!

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