R. E. JENNINGS
P. K. SCHOTCH

The Preservation of Coherence

Abstract. It is argued that the preservation of truth by an inference relation is of little interest when premiss sets are contradictory. The notion of a level of coherence is introduced and the utility of modal logics in the semantic representation of sets of higher coherence levels is noted. It is shown that this representative role cannot be transferred to first order logic via frame theory since the modal formulae expressing coherence level restrictions are not first order definable. Finally, an inference relation, called *yielding*, is introduced which is intermediate between the coherence preserving *forcing* relation introduced elsewhere by the authors and the coherence destroying inference relation of classical logic.

The study of inference has suffered too long from the complacent recitation of a mischievous dogma. It is contained in the dictum to the effect that to say a sentence a follows from a set Σ of sentences is precisely to say that if every sentence in Σ is true, then a must be true as well. Half of this dictum is at best half true; the other half is at best quite false and at worst a nuisance. The less objectionable half of the dictum says only that inferability preserves truth. With this sentiment we must not quarrel overmuch, but offer only this comment: The intellectual enterprise is still often given as the search for truth. This sounds good and with a catch in the voice sounds even better. But much of the energy of the physical sciences is spent warding off contradiction and looking for a suitable language. In such circumstances truth is at best a remote concern, while inference is nevertheless a fact of working life. Inferability also preserves well-confirmedness and also tentatively—suggestedness. There is a deeper mischief worked here. If we had a coherent vehicle compatible with every possible experience and enabling us to say everything that we wanted to say about these experiences, would we care whether the account that we gave in that language were, in addition, true? If we had an unfungible host of such vehicles and such accounts, would we insist that somehow either one of them was true or the truth was yet to be discovered? It seems rather that the notion of truth as a unique commodity is a kind of theological conception to be outgrown. Whatever character the charm of such an account would confer upon its constitutive sentences, we want inferability to preserve.

The other half of this doctrine, that teaches that a is inferable from Σ if the truth of all the sentences of Σ implies the truth of a has run a deeper and, we feel, more malignant course through much recent writing on the theory of inference. It has confounded the theory of inference with the theory of implication, seeming to infect the former with conundra belonging more properly to the realm of the latter. Nowhere more clearly is this to be seen than in the case where Σ is contradictory. On the doctrine, since the supposition that all the sentences of Σ are true is contradictory, therefore anything is inferable from Σ even though Σ contains no single contradictory sentence. The distinction between a set which includes $\{\alpha, \neg\alpha\}$ and one which includes $\{\alpha\wedge\neg\alpha\}$ is one which the doctrine tends to obliterate.

Not everyone has been content to live with the absurd myth of inferentially fulminant sets. But even the brave pioneers who have sought to teach us to pass among these sets with a song in our hearts have worked their reforms in the shadow of the doctrine. Most have wanted to reform our theory of implication so that not even contradictory formulae explode or to alter the notion of truth so as to make contradictory sets satisfiable.[1] Our course has been to recuse the doctrine, to accept contradictory premise sets as an unpleasant fact of inferential life and to ask what is to be done with them. To the researcher, forced to work with contradictory data, it is unlikely to be of interest to be told only what would be the case if all of the data were correct, if the researcher is ultimately after a theory in which that case cannot arise. It is likely to seem to her mere philosophical *badinage* at a moment when she wants advice.

The constraint that our general approach sets upon inference is that, in point of coherence, inferential closure of a premise set should not make matters worse than they are. This we take to be the chief failure of classical inference since it permits the inference of $\alpha\wedge\neg\alpha$ from the set $\{\alpha, \neg\alpha\}$. To make precise what was to be avoided, we introduced in [6] a notion of a *coherence function* which assigns a level of coherence to a set of sentences. In that preliminary and highly specific study, we exploited this notion as a means of relating contradiction-tolerating inference relations, weak modal logics and generalized modal frame theory. In [7], a substantially more general account of level theory makes possible a Gentzen-style presentation of such an inference relation, a presentation of greater concision and elegance than that of [6]. Here we explore more fully the modal and frame theoretic connexions, testing in particular the capacity of modal logic and generalised first order relational frame theory for expressing the central conceptions of level theory.

As in [6] we mold the desired inference relation in the comfortable matrix of modal logic.

Level theory

The central idea of level theory is that of a coherence function from the set $2^{(F-\{\perp\})}$ of subsets of a language F of propositional logic to the

---

[1]For a partial list of these approaches see D. Lewis [5].

set of ordinals up to ω, and satisfying:

0. $\forall a,\ l(a) \leqslant \omega$
1. $l(\emptyset) = 0$
2. $\forall a \in 2^{At},\ \ l(a) = 1$
3. $\forall a \subseteq \{\alpha | \vdash \alpha\},\ \ l(a) = 0$
4. If $a \subseteq b$ then $l(a) \leqslant l(b)$
5. If $l(a) = n$ and $l(b) = m$ then $l(a \cup b) \leqslant (n+m)$

One such function assigns 0 to sets containing only tautologies, 1 to all other classically consistent sets and n to sets partitionable into n but no fewer than n sets of level 1. This is by no means the only function satisfying 0-5. Other examples are discussed in [1]. This function will assign 1 to $\{p\}$, 2 to $\{p, \neg p\}$, 3 to $\{p \wedge q, \neg p \wedge q, p \wedge \neg q\}$ etc. It will assign ω to the set $\{a_i\}_{i \in Nat}$ where for each k, $a_k = \wedge(\neg p_j) \wedge p_k \quad 0 \leqslant j \leqslant k-1$.

Level preserving and level reducing operations

Certain logical operations have the property that level is preserved through the closure under that operation. Thus, for example $C_\vee(\Sigma)$ is the closure of Σ under the operation of taking finite disjunctions of elements of Σ, $l(C_\vee(\Sigma)) = l(\Sigma)$. Other logical operations preserve particular levels but not others. For example, $C_\wedge(\Sigma)$ (taking finite conjunctions) has level 1(0) if $l(\Sigma) = 1(0)$ But $l(C_\wedge(\Sigma))$ is undefined if $l(\Sigma) \geqslant 2$. On the other hand, the operation XX_3^2 of taking disjunctions of pairwise conjunctions of elements of triple subsets of Σ (i.e., $(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3)$ for $\{a_1, a_2, a_3\} \subseteq \Sigma$) preserves any level $\leqslant 2$, but $l(C_{XX_3^2}(\Sigma))$ is undefined if $l(\Sigma) \geqslant 3$. In general, the operation XX_{n+1}^2 preserves any level $\leqslant n$. Of particular interest is that level is preserved by closure under classical consequences of unit subsets and the empty subset. Thus closure under logical implications preserves level as does augmentation by tautologies.

The connexion with modal logic arises from the eminently exploitable fact that modalizing, even under a recognizably strong necessity operator, has the effect of producing a set of level 1 irrespective of the level of the original set. That is $l(C_\Box(\Sigma)) = 1$ even for a $\Box$-operator satisfying [RN] [RR] and [K], whatever $l(\Sigma)$, indeed even if $l(\Sigma)$ is undefined. Only in the presence of [D]$(\vdash \Box p \rightarrow \neg\Box\neg p)$ is $l(\{\Box p, \Box\neg p\}) = 2$. Only in the presence of [Con] $(\vdash \neg\Box\bot)$ is the level of $\{\Box\bot\}$ undefined. This feature of the '$\Box$' operation makes it possible to study sets of level greater than 1 within the framework of classical semantics. While a set Γ may perhaps have level 1, the set $\{a | \Box a \in \Gamma\}$ can have whatever level we please or no level at all.

The derivation of modal semantics

Semantically, a sentence is represented as a set, and a set of sentences as an intersection. A set of sentences having level 0 is represented by the universal set, a set Σ of sentences such that $l(\Sigma) \geqslant 2$ by $\emptyset$. A set of sentences having level 1 can be represented as a non-empty intersection. Thus if $\Box(\Sigma) = \{\alpha | \Box\alpha \in \Sigma\}$ has level $\leqslant 1$, it too can be represented as a non-empty intersection. In particular if $||\alpha||$ is the set representing α, then $||\Box(\Sigma)|| = \bigcap\{||\alpha|| \mid \Box\alpha \in \Sigma\}$.

Define $H(\Sigma) = \big\{\{x\} | x \in ||\Box(\Sigma)||\big\}$, let the sets representing sentences α be sets of maximally consistent sets such that $\alpha \in \Gamma$, and we have the makings of a notion of model structure. Now $H(\Sigma) = \big\{\{\Delta\} | \Box(\Sigma) \subseteq \Delta\big\}$. Each Δ such that $\{\Delta\} \in H(\Sigma)$, is a realization of $\Box(\Sigma)$ and bears an obvious kinship to the frame theoretic notion of an alternative, or an accessible.

So much for the case in which $\Box(\Sigma)$ has level 1. If $l\big(\Box(\Sigma)\big) = 2$, then clearly $\Box(\Sigma)$ is not realizable in a single maximally consistent set. But, by the choice of l and Lindenbaum's lemma, $\Box(\Sigma)$ is realizable in a pair of maximally consistent sets in the sense that there is a function $\pi: \Sigma \to 2$ such that both $\pi^{-1}[0]$ and $\pi^{-1}[1]$ are consistent. We simply redefine $H(\Sigma)$ as $\big\{\{\Delta_0, \Delta_1\}\big\} | \exists f: \Sigma \to 2 \ \& f^{-1}[i] \subseteq \Delta_i (0 \leqslant i \leqslant 1)$. When $\Box(\Sigma)$ has level 2, the natural semantic representation seems to be one in which the accessibilia are pairs. In the general case, where $l\big(\Box(\Sigma)\big) \leqslant n$, we must use the definition

$$H(\Sigma) = \big\{\{\Delta_0, \ldots, \Delta_{n-1}\} \big| \exists f: \Sigma \to n \ \& f^{-1}[i] \subseteq \Delta_i (0 \leqslant i \leqslant n)\big\}$$

a definition whose frame theoretic counterpart seems to be a notion of $n+1$-ary alternativeness.

The connexion can be made precise. A modal logic which admits sets of modalized sentences with level n bears the same relationship to $(n+1)$-ary frames that the standard modal logics tolerating sets of modalized sentences of level $\leqslant 1$ bear to binary frames. We summarize the fundamental results in what follows:

The usual notion of a frame is a notion of a binary frame defined by:

$\mathscr{J} = \langle U, R\rangle$ is a (binary frame) iff
U is a non-empty set and $R \subseteq U^2$ is a binary relation

A model $\mathscr{M}$ on $\mathscr{J}$ is a pair $\langle \mathscr{J}, V\rangle$ where $V: At \to 2^U$ is a valuation. The truth condition for modal formulae is given by:

$$\models_u^{\mathscr{M}} \Box\alpha \Leftrightarrow \forall x, uRx \Rightarrow \models_x^{\mathscr{M}} \alpha$$

The modal logic determined by the universal class $\mathscr{C}_2$ of binary relational frames is the logic $\boldsymbol{K}$ axiomatised by:

[P]	$\vdash_{PC} \alpha \Rightarrow \vdash_K \alpha$
[MP]	$\vdash_K \alpha \to \beta \ \& \vdash_K \alpha \Rightarrow \vdash_K \beta$
[K]	$\vdash_K \Box p \wedge \Box q \to \Box(p \wedge q)$
[RR]	$\vdash_K \alpha \to \beta \Rightarrow \vdash_K \Box\alpha \to \Box\beta$

$$[\text{RN}] \quad \vdash_K \alpha \Rightarrow \vdash_K \Box\alpha$$
$$[\text{US}] \quad \vdash_K \alpha \Rightarrow \vdash_K \alpha[\beta/p]$$

Alternatively $\boldsymbol{K}$ can be axiomatised by adding to an adequate base for $\boldsymbol{PC}$, the single rule:

$$[\text{RT}] \quad \frac{\Gamma \vdash \alpha}{\Box[\Gamma] \vdash \Box\alpha} \qquad (\Box[\Gamma] = \{\Box\alpha \mid \alpha \in \Gamma\})$$

the notion of a n-ary frame is obtained by the obvious generalisation of binary frame.

$\mathscr{I} = \langle U, R\rangle$ is a n-ary frame iff
U is a non-empty set and $R \subseteq U^n$ is an n-ary relation

A model $\mathscr{M}$ is as before, with the truth condition for $\Box\alpha$ given as:

$$\models_u^{\mathscr{M}} \Box\alpha \Leftrightarrow \forall x_1, \ldots, x_{n-1}, uRx_1, \ldots, x_{n-1} \quad \Rightarrow \quad \models_{x_1}^{\mathscr{M}} \alpha \text{ or } \ldots \text{ or } \models_{x_{n-1}}^{\mathscr{M}} \alpha$$

The logic determined by the class $\mathscr{C}_{n+1}$ of $(n+1)$-ary frames is the logic $\boldsymbol{K}_n$ which is most economically axiomatised by replacing [RT] with the rule:

$$[\text{RT}]_n \quad \frac{\Gamma \vdash_n \alpha}{\Box[\Gamma] \vdash \Box\alpha}$$

Here $\Gamma \vdash_n$ abbreviates: $\forall f\colon \Gamma \to n,\ f^{-1}[i] \vdash \alpha$ for some i $(0 \leqslant i \leqslant n-1)$. Alternatively, the axiom [K] is replaced by the axiom [K_n]

$$[\text{K}_n] \quad \Box p_1 \wedge \ldots \wedge \Box p_{n+1} \to \Box \underbrace{\big((p_1 \wedge p_2) \vee \ldots \vee (p_n \wedge p_{n+1})\big)}_{\text{all 2-member conjunctions from } \{p_1, \ldots, p_{n+1}\}}$$

Set $n = 1$ and $[\text{RT}]_n$ is [RT] and [K_n] is [K].

Frames and levels

At first blush the relationship between frame arity and level seems straightforward enough. We can state the most superficial connexions quite simply. Define $\Box(u)^{\mathscr{M}} = \{\alpha \mid \models_u^{\mathscr{M}} \Box\alpha\}$ and $\Box(u)^{\mathscr{I}} = \bigcap_{\mathscr{M} \text{ on } \mathscr{I}} \{\Box(u)^{\mathscr{M}}\}$. Let us say that $l(\mathscr{I}) \leqslant n \Leftrightarrow l\big(\Box(u)^{\mathscr{I}}\big) \leqslant n$ wherever $l\big(\Box(u)^{\mathscr{I}}\big)$ is defined. Then the fundamental relationship is simply that if $\mathscr{I} \in C_{n+1}$ then $l(\mathscr{I}) \leqslant n$ where it is defined. Our interest in the frame theory-level theory connexion arises from the fact that the corresponding biconditional does not hold. Although the $(n+1)$-arity of the frame relation is sufficient to impose a maximum level it is not necessary. Again, some of the other frame theoretic ways of imposing $l(\mathscr{I}) \leqslant n$ are less interesting than others. First, binarity imposes $l(\mathscr{F}) \leqslant n$ for any $n \geqslant 1$ since it imposes the stronger restriction $l(\mathscr{F}) \leqslant 1$. Secondly, any frame restriction sufficient to validate

$$[\mathrm{T}] \quad \Box p \to p$$

will impose the condition that $l(\mathscr{F}) \leqslant 1$, since a consequence of this principle is that $u \in || \Box(u)^{\mathscr{M}} ||^{\mathscr{M}}$.

In imposing level restrictions two paths seem to be open to us. One is to mess with the frame. The other is to mess with the modal logic. The two are not in general equivalent. Consider the second way first. If our logic has the principle [Con], $\neg \Box \bot$, the class of frames for the logic (the class of frames with respect to which the logic is sound) will have the property that $l\left(\Box(u)^{\mathscr{F}}\right)$ is defined for every $u \in U$. In fact for every model $l\left(\Box(u)^{\mathscr{M}}\right)$ will be defined for every $u \in U$. If our logic has the principle [D], $\Box p \to \neg \Box \neg p$ then the class of frames for the logic will have the property that $l\left(\Box(u)^{\mathscr{F}}\right) \leqslant 1$. If the logic has [K] the class of frames for the logic will have the property that $l\left(\Box(u)^{\mathscr{F}}\right) \leqslant 1$ where it exists. We can see that in the presence of [K], the level theoretic effect of [D] is identical to that of [Con]. This is not surprising since in the presence of [K], [Con] and [D] are deductively equivalent. Finally, if the logic has $[\mathrm{K}_2]$ $\Box p \wedge \Box q \wedge \Box r \to \Box\big((p \wedge q) \vee (p \wedge r) \vee (q \wedge r)\big)$, and $\mathscr{F}$ is in the class of frames for the logic, then $l(\mathscr{F}) \leqslant 2$ and in general, if $\mathscr{F}$ is a frame for a logic having as a theorem $[\mathrm{K}_n]$ $\Box p_1 \wedge \ldots \wedge \Box p_{n+1} \to \Box\big(\vee (p_i \vee p_j)\big)$ $(1 \leqslant j \neq i \leqslant n+1)$ then $l(\mathscr{F}) \leqslant n$. These examples illustrate the way in which by defining a class of frames a modal formula may be seen to impose a level restriction on a frame $\mathscr{F}$, by imposing a level restriction on $\Box(u)^{\mathscr{M}}$ for each $u \in U$ and each model $\mathscr{M}$ on $\mathscr{F}$. How could these levels otherwise be imposed?

One might hope that modal correspondence theory would offer some guidance here. That hope would, however be brief and forlorn. Modal correspondence theory has as one of its central notions, the notion of a modal class. This in its simplest version is the notion of a class $\mathscr{C}$ of frames such that for some modal formula α, $\mathscr{C} \vDash \alpha$ and for any frame $\mathscr{F} \in \mathscr{C}$, $\mathscr{F}$ *non*$\vDash \alpha$; a modal class is a class of frames which is exactly the class of frames for the formula α. Correspondence theory is concerned in part with the question as to which modal classes are also elementary classes and which elementary classes are modal classes. We might ask similar questions concerning levels. Let $\mathscr{C}_{ln}$ be the class of frames $\mathscr{F}$ such that $l(\mathscr{F}) \leqslant n$; call such a class a *coherence class*. Is that class a modal class? We can see immediately that such questions have a significance in the context of level theory different from their correspondence theoretic meaning. What is meant in definability theory as it is usually studied, by the question, "Is the class of frames for α an elementary class?" Is this: "Is there some sentence α' in the first order theory of a single binary relation, such that the class of frames for α is the class of models (in the first order sense) for α'?" We might generalise such a question to the n-ary case; the generalisation is non-trivial since a modal sentence corresponding to a first order theory of a binary relation may correspond to no formula in the first order theory of an n-ary relation for $n \geqslant 2$. See [3] and [4]. But such questions of correspondence arise only against an assumption of arity. What defines a modal formula in the language of binary relations may be inexpressible in the language of ternary relations. So if we liberalise the notion of frame, as it seems we ought, to include ternary, quaternary, and, for each n, n-ary frames, correspondence theory acquires a new (and one feels compelled to say a less spotty) complexion, but certain of its questions lose their force. Certainly the class of frames defined by a particular arity of the frame relation is not a modal class since the logic which (assuming n-arity) corresponds to the universal class of n-ary frames for some n, also corresponds to a proper subclass of the class of all $(n+m)$-ary frames, for any $m \geqslant 1$, and is sound with respect to all frames of arity less than n. Thus, for example, the formula [K] which corresponds to $(x = x)$ in the class of binary frames, corresponds to $uRxy \Rightarrow (x = y) \vee (uRxx) \vee (uRyy)$ in the class of ternary frames. A similar claim can be made for $[K_2]$ and in general for $[K_n]$. In addition each logic $\boldsymbol{K_n}$ is sound with respect to the class of n-ary frames. We will show later than this is true, perhaps surprisingly, even for $\boldsymbol{K}$. In the generalisation to n-ary frames, without an assumption of fixed arity, it is unlikely that any modal formula defines an elementary class. For similar reasons it is doubtful whether any coherence class of relational frames is identical to any elementary class. The question remains whether any coherence class of frames are modally definable or elementarily definable in the context of a fixed arity assumption. Let us look at some of the formulae already mentioned.

As we have noted, the arity of the frame relation imposes a level restriction; in particular if $\mathscr{I}$ is $(n+1)$-ary, then $l(\mathscr{I}) \leqslant n$. But the following points must be stressed:

(a) the universal class of $(n+1)$-ary frames is not an elementary class except trivially within a restriction to the first order theory of a single $(n+1)$-ary relation, in which case it is defined by $x = x$.

(b) this class is not a modal class, since the modal logic $\boldsymbol{K_n}$ whose class of $(n+1)$-ary frames it is, is also sound with respect to a subclass of the class of $(n+2)$-ary frames, and is sound with respect to the class of n-ary frames.

(c) this class is not a coherence class since the $\boldsymbol{K_n}$ $(n+2)$-ary frames also have level $\leqslant n$.

These three facts taken together suggest that if a correspondence is to be found which does not depend upon an assumption of fixed arity, it will be between modal classes and coherence classes. Let us ask more directly, does the logic $\boldsymbol{K}_n$ define a coherence class of frames? Consider the simplest such case, $\boldsymbol{K}$. Under the assumption of binarity, $\boldsymbol{K}$ corresponds to the universal class of binary frames. Under the assumption of ternarity. $\boldsymbol{K}$ corresponds to the first order formula

$$[\mathrm{Di}_3^1] \quad uRxy \Rightarrow uRxx \vee uRyy \vee x = y$$

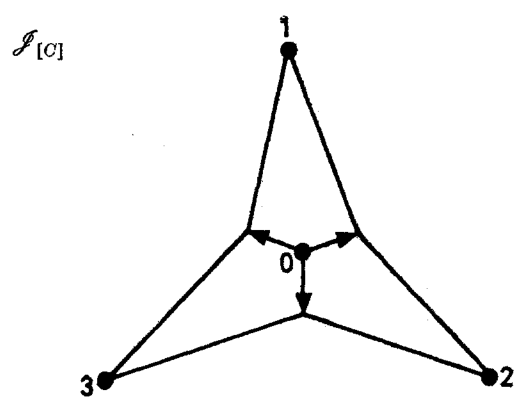
Under an assumption of quaternity, $\boldsymbol{K}$ corresponds to the first order formula

$$[\mathrm{Di}_4^1] \quad uRxyz \Rightarrow uRxxx \text{ or } uRyyy \text{ or } uRzzz \text{ or } x = y = z.$$

and so on. For each $\boldsymbol{K}$ frame $\mathscr{I}$, whatever its arity, $l(\mathscr{I}) \leqslant 1$. Thus $\mathscr{C}_{\boldsymbol{K}} \subseteq \mathscr{C}_{l_1}$. Does the converse hold? Consider the substitution instance

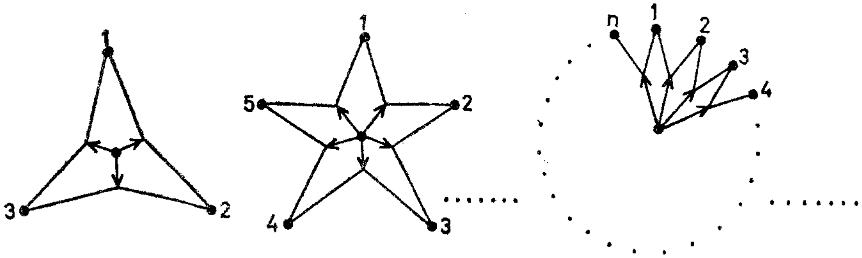
$$[\mathrm{C}] \quad \Box p \wedge \Box \neg p \to \Box \bot$$

of [K]. Since [C] is a theorem of $\boldsymbol{K}$, [C] is valid in every $\boldsymbol{K}$-frame including n-ary frames defined by $[\mathrm{Di}_n^1]$. [C] has the look of a formula which is explicitly about the level of the set of necessities. It 'says' (assuming [US]) that if the set should contain two formulae which would raise the level of the set above 1, then the set has no level. That [C] is a deductively weaker formula than [K] is proved by reference to the ternary frame $\mathscr{I}_{[C]} = \langle U, R\rangle$



where $U = \{0, 1, 2, 3\}$ and $R = \{\langle 0, 1, 2\rangle, \langle 0, 2, 3\rangle, \langle 0, 3, 1\rangle\}$.
[C] is valid or $\mathscr{I}_{[C]}$, but [K] fails at 0 if $V(p) = \{1, 3\}$ and $V(q) = \{2, 3\}$. Unfortunately, this does not answer our question as to whether $\mathscr{C}_{l_1} \subseteq \mathscr{C}_{\boldsymbol{K}}$, one way or the other since $l(\mathscr{I}_{[C]}) \nleqslant 1$. To see that this is so, consider a modal $\mathscr{M}$ on $\mathscr{I}_{[C]}$ in which $V(p_1) = \{1\}$ and $V(p_2) = \{2\}$. Then $\models_0^{\mathscr{M}} \Box(p_1 \vee p_2) \wedge \Box \neg p_1 \wedge \Box \neg p_2$, but non $\models_0^{\mathscr{M}} \Box \bot$. But $l(p_1 \vee p_2, \neg p_1, \neg p_2) = 2$. So by axiom 4 $l(\Box(0)^{\mathscr{M}}) \geqslant 2$ and by our account of frame level, $l(\mathscr{I}_{[C]}) \geqslant 2$ as well.

A proof that even under an assumption of ternarity, [C] is not first order definable is easily adduced which refers to the compactness of first order logic and the following sequence of ternary [C] frames:

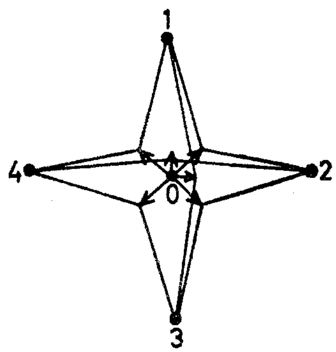


While these remarks about [C] do not answer the question whether [K] defines the class $\mathscr{C}_{l_1}$, they are a first step toward an answer. Consider the model $\mathscr{N}$ on $\mathscr{F}^1_{[C]}$ in which $V(p_1) = \{2, 3\}$ and $V(p_2) = \{1, 3\}$, then $\models^{\mathscr{N}}_0 \Box p_1 \wedge \Box p_2 \wedge \Box(p_1 \wedge p_2)$ but $\models^{\mathscr{N}}_0 \Box\bot$. That is, the formula

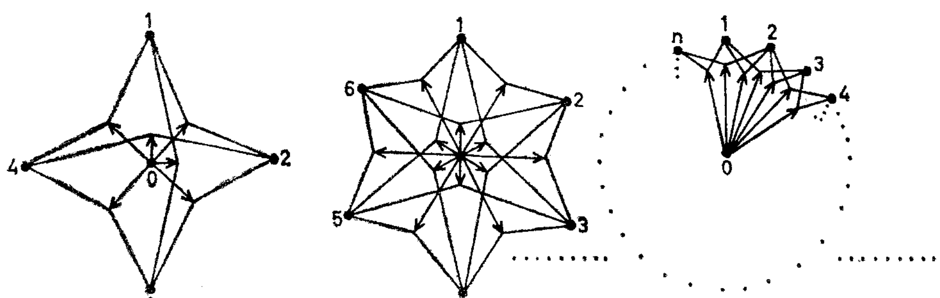
$$[C^1_3] \qquad \Box p_1 \wedge \Box p_2 \wedge \Box\neg(p_1 \wedge p_2) \rightarrow \Box\bot$$

fails on $\mathscr{F}^1_{[C]}$.

However, consider the frame



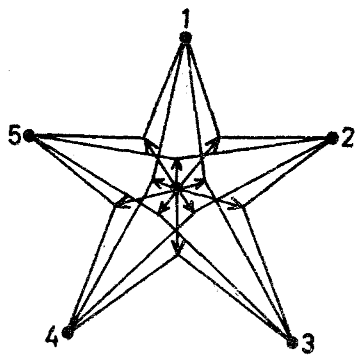
Here both [C] (which in anticipation we call $[C^1_2]$) and $[C^1_3]$ are valid. Again, similar remarks can be made about $[C^1_3]$) and $\mathscr{F}_{[C^2_3]}$. First, we can show that $[C^1_3]$ is not triadically first order definable by reference to the sequence of frames



Secondly, the frame $\mathscr{I}_{[C_3^1]}$ has a level greater than 1. Let $\mathscr{M}$ be a model on $\mathscr{I}_{[C_3^1]}$ such that $V(p_1) = \{2, 3, 4\}$, $V(p_2) = \{1, 3, 4\}$ and $V(p_3) = \{1, 2, 4\}$. Then $\models_0^{\mathscr{M}} \Box p_1 \wedge \Box p_2 \wedge \Box p_3 \wedge \Box(p_1 \wedge p_2 \wedge p\, 3)$, and *non* $\models_0^{\mathscr{M}} \bot$. But then $l(\Box(n)^{\mathscr{M}}) \geqslant 2$; $(\mathscr{I}_{[C_3^1]}) \geqslant 2$; and the formula

$$[C_4^1] \qquad \Box p_1 \wedge \Box p_2 \wedge \Box p_3 \wedge \Box \neg(p_1 \wedge p_2 \wedge p_3) \rightarrow \Box \bot$$

fails on $\mathscr{I}_{[C_3^1]}$. To validate this formula, we construct



$\mathscr{I}_{[C_4^1]}$ validates $[C_2^1]$, $[C_3^1]$ and $[C_4^1]$, does not validate $[C_5^1]$, and is the first member of a sequence of frames by reference to which $[C_4^1]$ may be proved triadically first order undefinable.

Clearly a general theorem is in the offing. First, we can give the general form of $[C_n^1]$:

$$[C_n^1] \qquad \Box p_1 \wedge \ldots \wedge \Box p_{n-1} \wedge \Box \neg(p_1 \wedge \ldots \wedge p_{n-1}) \rightarrow \Box \bot$$

The m th frame $\mathscr{I}^1_{[C_n^1]}$ is the pair $\langle U^m_{[C_n^1]}, R^m_{[C_n^1]}\rangle$ where $U^m_{[C_n^1]} = \{0, 1, \ldots, k = n+2m-1\}$, and $R^m_{[C_n^1]}$, treated as a function $U \rightarrow 2(U^2)$ is defined as follows:

$$R^m_{[C_n^1]}(i) = \emptyset \text{ if } 1 \leqslant i \leqslant k$$
$$= \{\langle h, j\rangle | 1 \leqslant h \neq j \leqslant k \;\&\; ({}_{h-j}^{\;k}) < n\} \text{ if } i = 0$$

Of particular interest to us is the sequence of frames $\mathscr{I}^1_{[C_n^1]}$. The m-th frame, $\mathscr{I}^1_{[C_m^1]}$ of this sequence is the pair $\langle U^1_{[C_m^1]}, R^1_{[C_m^1]}\rangle$ where

$U^1_{[C_m^1]} = \{0, 1, \ldots, m+1\}$, and
$R^1_{[C_m^1]}$ regarded as a function $U \rightarrow 2(U^2)$ is defined:

$$R^1_{[C^1_m]}(i) = \emptyset \text{ if } 1 \leqslant i \leqslant m+1$$
$= \binom{m+1}{2}$ if $i = 0$ (where $\binom{m+1}{2}$ is the set of pairs of distinct objects drawn from $\{1, \ldots, m+1\}$.

Each frame, $\mathscr{F}^1_{[C^1_m]}$ validates $[C^1_n]$ for every n $(2 \leqslant n \leqslant m)$. But for each m, $[C^1_{m+1}]$ fails on $\mathscr{F}^1_{[C^1_m]}$. Thus no frame in the sequence has level 1 and no finite subset of the sequence defines the class $\mathscr{C}_{l_1}$. What of the sequence as a whole?

More can be said, however. Let C^1 be the logic generated by replacing [K] with the sequence $\{C^1_i | i \in Nat\}$ in the axiomatization of $\boldsymbol{K}$. It is easy to prove the following:

THEOREM. *If $\mathscr{F}$ is a frame for C^1, then $l(\mathscr{F}) = 1$.*

We prove this via two intermediate results.

LEMMA. *Let $\mathscr{M}$ be a model on $\mathscr{F}$ and n a point in $\mathscr{F}$. Then for every finite subset Δ of $\Box(n)^{\mathscr{M}}$, $l(\Delta) \leqslant 1$.*

PROOF. Let Δ be a finite subset of $\Box(n)^{\mathscr{M}}$ such that $l(\Delta) = m$ for $m > 1$. Let f be a function: $\Delta \to m$. Then $\Delta' = f^{-1}[0] \cup f^{-1}[1]$ is a finite subset of $\Box(n)^{\mathscr{M}}$ and $l(\Delta') = 2$.

Let Θ be a minimal subset of Δ' having level 2. Let $\Theta = \{a_1, \ldots, a_k\}$. Then $\vdash a_1 \to (\neg a_2 \vee \ldots \vee \neg a_k)$. Therefore $(\neg a_2 \vee \ldots \vee \neg a_k) \in \Box(n)^{\mathscr{M}}$. Therefore by $[C^1_{k+1}]$, $\bot \in \Box(n)^{\mathscr{M}}$. Then $l\big(\Box(n)^{\mathscr{M}}\big)$ is undefined, contrary to hypothesis.

As a second result, we prove a compactness property for level theory.

FINITENESS THEOREM FOR LEVEL THEORY. $\forall \Gamma \in 2^F$, $l(\Gamma) \leqslant n$ iff for every finite subset Δ of Γ, $l(\Delta) \leqslant n$.

PROOF. $\Rightarrow$ trivial by axiom 4.
$\Leftarrow$ by induction on $l(\Gamma)$; for $l(\Gamma) = 0$, the proof is trivial and for $l(\Gamma) = 1$, the result follows from compactness of $\vdash$. Suppose that the result holds $l(\Gamma) = n$. Let $l(\Gamma) = n+1$ and let $f: \Gamma \to n+1$ be such that $\overset{n}{\underset{j=0}{\forall}} j$, $f^{-1}[j]$ *non* $\vdash \bot$. Define $\Pi_f(\Gamma) = \{f^{-1}[j] | 0 \leqslant j \leqslant n\}$. Clearly, $\forall a \in \big(\Pi f_n(\Gamma)\big)$, $l(\bigcup a) = n$.

For each a, choose $\Theta_a \subseteq \bigcup a$ such that Θ_a is finite and $l(\Theta) = n$. For each Θ_a, define $f_{\Theta_a}: \Theta_a \to n+1$: $\overset{n}{\underset{j=0}{\forall}} j$, $f^{-1}\Theta_a[j] \subseteq f^{-1}[j]$. Define $b_j = \bigcup \big\{f^{-1}_{\Theta_a}[j] | a \in \big(\Pi f_n(\Gamma)\big)\big\}$. Clearly each b_j is finite and of level 1, and $l(\bigcup_{j=0}^{n} \{b_j\}) = n+1$. Moreover $\bigcup_{j=0}^{n} \{b_j\} \subseteq \Gamma$ and $\bigcup_{j=0}^{n} \{b_j\}$ is finite.

The desired theorem follows immediately. The logic C^1 imposes a frame level 1. That is $\mathscr{C}_{C^1} \subseteq \mathscr{C}_{l_1}$. We are now in a position to prove that $\boldsymbol{K}$ does not define $\mathscr{C}_{l_1}$.

Define a frame $\mathscr{J}_\omega = \langle U_\omega, R_\omega \rangle$ with

$U_\omega = \{u\} \cup Nat$
$R_\omega(u) = \binom{Nat}{2}$ and $R_\omega(i) = \emptyset$ for $i \in Nat$
$\mathscr{J}_\omega$ is a frame for C^1, but $\mathscr{J}_\omega$ does not validate [K].
Therefore $\mathscr{J}_\omega \in \mathscr{C}_{l_1}$. Therefore $\mathscr{C}_{l_1} \subseteq \mathscr{C}_K$.

We have of course established at the same time that *non* $\vdash_{C^1}$ [K]. Thus C^1 is a proper sublogic of K and the class of K frames is a proper subclasss of the class of C^1 frames.

We can now prove that C^1 *defines* the coherence class $\mathscr{C}_{l_1}$, by demonstrating

LEMMA. *If* $l(\mathscr{J}) = 1$ *then* $\mathscr{J} \in \mathscr{C}_{C^1}$.

PROOF. Suppose that $\mathscr{J} \notin \mathscr{C}_{C^1}$. Then for some model $\mathscr{M}$ on $\mathscr{J}$ and some point $u \in \mathscr{J}$, there is some $[C^1_m]$ such that $\vDash^{\mathscr{M}}_u [C^1_m]$
That is, $\exists a_1 \ldots a_m$: $\vDash^{\mathscr{M}}_u \Box(a_1 \vee \ldots \vee a_m) \wedge \Box\neg a_1 \wedge \ldots \wedge \Box\neg a_m$ & *non* $\vDash^{\mathscr{M}}_u \Box\bot$
Then $l(\Box(n)^{\mathscr{M}}) \geqslant 2$. Therefore $l(\mathscr{J}) \geqslant 2$ contrary to hypotheses.

Thus we may assert:

THEOREM. $\mathscr{C}_{C^1} = \mathscr{C}_{l_1}$.

The general case

What we have been able to prove about [K] and the class $\mathscr{C}_{l_1}$ of frames can be proved in general, *mutatis mutandis* about $[K_n]$ and the class $\mathscr{C}_{l_n}$ of frames. The class $\mathscr{C}_{l_n}$ is defined, not by $[K_n]$ but by the infinite sequence $\{C^n_i \mid i > n\}$. For each n, the first formula in this sequence is:

$$[C^n_{n+1}] \quad \bigwedge_{i=0}^{n} \{\Box(p_0 \wedge \ldots \wedge \neg p_i \wedge \ldots \wedge p_n)\} \to \Box\bot$$

To understand the forms of later formulae in the sequence, we must first become acquainted with the notion of a *trace*. The topic of traces is a mathematically rich one, of which we nevertheless give here only a sketch.

A trace over a set A is a finite collection of finite subsets of A which has a rather special property, namely that for every partition A up to a given size, some member of the collection survives the partition intact. If T is a collection of subsets of A having this property for partitions up to m-fold partitions, then we say that T is a m-trace over A. We call the set of such traces over A, $\mathscr{T}_m(A)$. More precisely, we may define an m-trace over A as follows:

DEFINITION. Let $T \subseteq 2^A$ be finite, and of finite sets. Then $T \in \mathscr{T}_m(A)$ iff $\forall f: A \to m$, $\exists a \in A$, $\overset{m-1}{\underset{i=0}{\exists}} i: a \subseteq f^{-1}[i]$.

Now our particular interest is in traces over sets of formulae, and this gives rise to the notion of a *formulated trace* or *trace formula*. Let Δ be a set of formulae and let $T = \{a_1 \dots a_k\}$ be a m-trace over Δ. Then $\bigvee_{i=l}^{k}\left(\wedge(a_i)\right)$ is the formulation of T. The importance of formulated traces lies in the following:

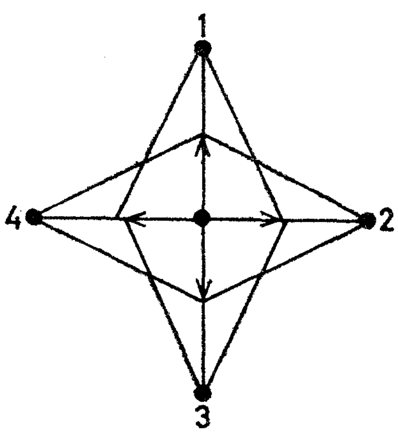
PROPOSITION. *Let* $\Delta = \{\alpha_1 \dots \alpha_m\}$ *and* β *the formulation of some trace,* T in $\mathscr{T}_m(\Delta)$. *Then* $\vdash_{\mathbf{K}_m} \Box\alpha_1 \wedge \dots \wedge \Box\alpha_m \rightarrow \Box\beta$.

The proof of this proposition is a non-trivial exercise in propositional logic. That the formula is valid in $(m+1)$-ary frames follows straightforwardly from the definitions of m-trace and trace formulation and the truth condition for $\Box$-formulae on $(m+1)$-ary frames. With the aid of this notion we may outline the proof that $\mathbf{K}_n$ does not define $\mathscr{C}_{l_n}$. We illustrate the case for $n = 2$.

$$[\mathrm{C}_3^2] \quad \Box(\neg p_0 \wedge p_1 \wedge p_2) \wedge \Box(p_0 \wedge \neg p_1 \wedge p_2) \wedge \Box(p_0 \wedge p_1 \wedge \neg p_2) \rightarrow \Box\bot$$

is universally valid on ternary frames, but also valid in the quaternary frame

$\mathscr{I}^1_{[C_3^2]}$



in which $[\mathrm{K}_2]$ fails. The formula

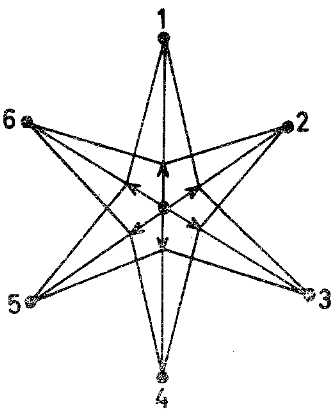
$$[\mathrm{C}_4^2] \quad \Box p_0 \wedge \Box p_1 \wedge \Box p_2 \wedge \Box\neg\left((p_0 \wedge p_1) \vee (p_0 \wedge p_2) \vee (p_1 \wedge p_2)\right) \rightarrow \Box\bot$$

fails on $\mathscr{I}^1_{[C_3^2]}$. Therefore $(\mathscr{I}^1_{[C_3^2]}) > 2$. Both are valid in $\mathscr{I}^1_{[C_4^2]} = \langle U^1_{[C_4^2]}, R^1_{[C_4^2]}\rangle$, where $U^1_{[C_4^2]} = \{0, 1, 2, 3, 4, 5\}$ and

$$R^1_{[C_4^2]}(i) = \emptyset \quad \text{if} \quad 1 \leqslant i \leqslant 5$$
$$= \binom{\{1, \dots, 5\}}{3} \quad \text{if} \quad i = 0$$

The sequence of frames $\mathscr{I}^1_{[C_n^2]}$ (and for $\mathscr{I}^1_{[C_n^m]}$ is the quaternary $(m+2)$-ary version of the sequence $\mathscr{I}^1_{[C_n^1]}$ described above.

Additionally, each formula $[C_n^2]$ ($[C_n^m]$) can be shown to be quadradically ($(m+2)$-adically) first order undefinable by an obvious generalisation of the proof outlined above. The frame $\mathscr{J}^2_{[C_3^2]}$ is this:



Unary frames

We earlier made the claim that each logic $\boldsymbol{K}_n$ was sound with respect to the universal class of n-ary frames, and that this held even for $\boldsymbol{K}$. We conclude our modal logical remarks by showing briefly that this is so. A unary frame is simply a frame whose relation is a set of singletons. Understood as a function, R assigns to each object of the frame a (possibly empty) collection of 0-tuples. Thus we may identify the relation with the set of points to which it assigns a non-zero collection of 0-tuples, in other words the serial points. At these points the truth-conditions for modal formulae are unsatisfiable and therefore we have:

$$\{x\} \in R \Rightarrow \forall \alpha, \ \models_x^{\mathscr{M}} \Box \alpha.$$

At the non-serial points, which in this case are the points whose singletons are excluded from the relation, the truth-conditions are vacuously satisfied and therefore we have:

$$\{x\} \notin R \Rightarrow \forall \alpha, \ non \models_x^{\mathscr{M}} \Box \alpha.$$

Clearly, in these frames, the formula $[K_1]$ is sound. But the aggregation principle whose class of unary frames is the universal class $\mathscr{C}_1$ is the correspondingly stronger aggregation principle:

$[K_0]$ $\quad \Box p \to \Box(p \wedge q)$

On these frames, the formula

$[C_0]$ $\quad \Box p \to \Box \bot$

is of course also valid. Thus all unary frames are frames of coherence level 0. Again, $[K_0]$ has a class of binary frames, those which are modals for

$$[\mathrm{Di}_2^0] \quad uRx \Rightarrow x \neq x.$$

The inferential connexion

In numerous other places, we have argued against the logic ***K*** that the strong aggregative principle [K] makes the modal operator incapable of a doxastic or deontic interpretation. This was precisely because the formulae $[C_n^1]$ are theorems of ***K***, and entail that conflicting beliefs commit us to every belief and conflicting obligations commit us to every obligation. Put another way, we have argued that neither the set of one's beliefs nor the set of one's obligations is closed under classical inference. We have put forward, as plausible alternatives, various inference relations which lack the unrestricted $\wedge$-introduction rule:

$$[\wedge \mathrm{I}] \quad \frac{\Sigma \vdash \alpha, \Sigma \vdash \beta}{\Sigma \vdash \alpha \wedge \beta}.$$

It might be argued that $[\wedge \mathrm{I}]$ is deontically and doxastically unobjectionable provided that $l(\{\alpha, \beta\}) = 1$, that we have, in effect, thrown away with the bath water a rather attractive rubber duck. This study of level theory enables us in some measure to put the matter right.

In [4] we offered a study of the $\boldsymbol{K_n}$ logics and showed that the limit of this decreasing sequence of modal logics is the logic[2] ***S*** axiomatised with [RN] and [RR] as the only modal principles. Since we now have before us the logics $\boldsymbol{C^n}$ axiomatised by adding to [RN] and [RR] exactly those principles to which we object, we have at our disposal a means of saying more exactly what it is we do want. What we do not want in each case is the set of theorems $\boldsymbol{W_n} = (\boldsymbol{C^n} - \boldsymbol{S})$. What we do want in each case is the set $\boldsymbol{K_n^*} = (\boldsymbol{K_n} - \boldsymbol{W_n})$. In the particular case which we consider here, the logic is the logic $\boldsymbol{K^*} = (\boldsymbol{K_1} - \boldsymbol{W_1})$. Now this logic is a queer one in requiring axiomatisation without unrestricted uniform substitution; its frame theory, if it has one, remains a mystery. But the logic is straightforwardly axiomisable by the rule:

$$[\mathrm{RT}] \quad \frac{\Sigma \succ \alpha}{\Box[\Sigma] \vdash \Box \alpha}.$$

Here $\Box[\Sigma] = \{\Box \alpha | \alpha \in \Sigma\}$, and $\succ$ is an inference relation explained as follows: We define a ***singular set*** as a set which is empty, or a unit set or a set of level $\leqslant 1$. We call the set of singular subsets of a set Σ, $\boldsymbol{Sing}(\Sigma)$.

DEFINITION $\succ^*$. $\Sigma \succ^* \alpha \Leftrightarrow \exists \Delta \in Sing(\Sigma): \ \Delta \vdash \alpha$

[2]The "***S***" is for 'Segerberg'. This logic is called '***N***' in [2].

DEFINITION $C_{\succ*}$. $C_{\succ*}(\Sigma) = \{a | \Sigma \succ^* a\}$.

DEFINITION $\Sigma \succ a$ (read Σ derives a) $\Leftrightarrow \exists n: C_{\succ*}(\Sigma) \succ^* a$.

$\succ^*$ satisfies the most important two structural requirements for an inference relation, namely:

[Ref] $a \in \Sigma \Rightarrow \Sigma \succ^* a$ and

[Mon] $\Sigma \succ^* a \ \& \ \Sigma \subseteq \Delta \Rightarrow \Delta \succ^* a$. It follows that $\succ$ satisfies [Ref] and [Mon] as well. $\succ^*$ does not, however, satisfy the usual structural requirement.

[Cut] $\Sigma \cup \{a\} \succ^* \beta \ \& \ \Sigma \succ^* a \Rightarrow \Sigma \succ^* \beta$

to which the following is a counterexample:

Let $\Sigma = \{p \wedge \neg q,\ p \to q\}$ $l(\Sigma) = 2)$. Let $\Theta = \{p \wedge \neg q\}$ and $\Delta = \{p \to q\}$. Then $l\Theta = l(\Delta \cup \{p\}) = 1$ and both $\Delta \cup \{p\} \vdash q$; $\Sigma \cup \{p\} \succ^* q$ and $\Theta \vdash p$; $\Sigma \succ^* p$. But neither $\Theta \vdash q$ nor $\Delta \vdash q$. Therefore Σ *non* $\succ^* q$. It is easily shown, however, that [Cut] holds for $\succ^*$ in the special cases in which $l(\Sigma) = 1$ or is undefined. Moreover, we may also prove:

THEOREM. [Cut] *holds unrestrictedly for* $\succ$.

PROOF. Assume that $\Sigma \cup \{a\} \succ \beta$ and $\Sigma \succ a$. Then $\exists m$: $C^m_{\succ}(\Sigma) \succ^* a$. Then $a \in C^{m+1}_{\succ}(\Sigma)$. Therefore $\Sigma \cup \{a\} \subseteq C^{m+1}_{\succ}(\Sigma)$. Therefore $C^{m+1}_{\succ}(\Sigma) \succ^* \beta$. (Mon) i.e. $\exists$n: $C^{m*}_{\succ}\big(C^{m+1}_{\succ}(\Sigma)\big) \succ^* \beta$ But $C^n_{\succ}{}^*\big(C^{m+1}_{\succ}(\Sigma)\big) = C^l_{\succ}{}^*(\Sigma)$ for some l. Therefore $\exists l$: $C^{m*}_{\succ}(\Sigma) \succ^* \beta$ Therefore $\Sigma \succ \beta$. Thus $\succ$ is a genuine inference relation, in the technical sense of [8]. It is also a natural inference relation inasmuch as its definition is in accord with the ordinary inferential procedure of treating consequences of premise sets as newly gained premises. It preserves the distinction which we have argued elsewhere ought to be preserved between premise sets which contain a contradictory n-tuple of premises and those which contain an explicitly contradictory sentence. Only in the latter case does inferential detonation occur. In the former case, we may retain, with certain restrictions, nontrivial classical procedures. In this respect $\succ$ is like the forcing relation $[\vdash]$ of [6]. It differs from forcing chiefly in admitting a relatively strong version of $\wedge$-introduction.

$$[\wedge I_{\succ}] \quad \frac{\Sigma \succ a,\ \Sigma \succ \beta}{\Sigma \succ a \wedge \beta} \quad (l\{a,\ \beta\}) = 1)$$

as well as the mixed introduction rule:

$$[\mathsf{XXI}_{\succ}] \quad \frac{\Sigma \succ a_1, \ldots, \Sigma \succ a_{l(\Sigma)+1}}{\Sigma \succ \vee (a_i \wedge a_j)\ (\{i, j\} \in \binom{l(\Sigma)+1}{2}}.$$

The other operator rule requiring mention is the rule

$$[\to E] \qquad \frac{\Sigma \vdash \alpha,\ \Sigma \vdash \alpha \to \beta}{\Sigma \vdash \beta}.$$

There is no such rule for forcing which survives coherence levels greater than 1. In the case of $\succ$, the rule survives in a restricted form even when $l(\Sigma) > 1$:

$$[\to \mathrm{E}_{\succ}] \qquad \frac{\Sigma \succ \alpha,\ \Sigma \succ \alpha \to \beta}{\Sigma \succ \beta} \quad \big(l(\{\alpha,\ \beta\}) = 1\big).$$

The rule may actually survive in the unrestricted form, depending upon the character of the language in which inferences are drawn. In a language in which, if $\vdash \alpha \to \neg\beta$ but *non* $\vdash \neg\beta \to \alpha$, there is an α' such that $\vdash \alpha \equiv \alpha' \wedge \neg\beta$, the classical form of the rule is preserved intact.

Forcing, in our sense, is well-named. Certainly if we at once abandon the distinction between incoherent premise sets and premise sets containing contradictions, we reasonably regard as forced upon us anything which follows from some cell of every least partition into consistent sets. But, as the name implies, forcing is an extremely conservative account of inferability. We may well ask whether an account of inference that conservative is quite what we are after. Perhaps we want to note not only the inferences which we are forced to draw, but also those which in conscience we ought to draw, or those which we *can* draw withouth playing upon the conflicts among our sources, but only upon their agreements. This last conveys the relatively liberal posture that *yielding* is intended to reflect. Put another way *forcing* represents an account of inferability according to which conflicting sources are treated as suspect inasmuch as they conflict; *yielding* treats distinct sources of premises as suspect only insofar as they conflict.

There is, however, a positive connexion between yielding and a subsidiary notion of forcing discussed in [7] under the name of A-forcing. This notion results from a restriction on the class of permissible partitions of a set Δ of premises. In particular it contemplates the possibility of corpora of premises which form natural units and which we do not permit our partitions to disperse. For ease of exposition, it was assumed that these sets are disjoint, but we might easily adopt an account of forcing in which decompositions replace partitions, and in which therefore the set A of exempt sets need not be a disjoint collection. Suppose that we have made this alteration. (It will make no difference to what forces what.) Then $\succ^*$ is that limiting case of A-forcing which results when we take A to be the set of consistent sets of the language. There will, in this case, be exactly one acceptable decomposition of Δ into consistent sets, namely the decomposition into the largest consistent subsets of Δ. We have seen that $\succ^*$ is not an inference relation in the sense of Tarski.

In [7] we also consider a generalisation of forcing called Σ-forcing. This is the inference relation which results when a set Σ of contingent sentences replaces the null set in the difinition inconsistency and related definitions. When the adoption of an otherwise useful mathematical model, as say in Quantum field theory, has a theoretically undesirable consequence such as that a particle could have infinite potential energy, that theory, while not contradictory, is less than perfectly coherent in a physical sense. It may nevertheless be a useful theory provided that the consequences drawn from it do not depend upon the physically suspect consequence cited or any other of like kind. In possession of such a theory, our position is clearly analogous to that of a person having to process contradictory information from diverse sources. Σ-forcing seems a natural procedure to adopt, with only one flaw. Σ-forcing like natural forcing represents a conservative strategy, missing any inferences requiring premise sets which do not survive partitions. The Σ-generalisation of yielding will inherit the usefulness of Σ-forcing whilst maintaining the comparatively liberal character of yielding.

References

[1] R. E. Jennings and P. K. Schotch, ***Inference and Necessity***. Monograph in preparation.

[2] R. E. Jennings and P. K. Schotch, *Remarks on (weakly) weak modal logics*, ***Notre Dame Journal of Formal Logic***, forthcoming.

[3] R. E. Jennings, P. K. Schotch and D. K. Johnston, *The n-adic undefinability of the Geach formula*, ***Notre Dame Journal of Formal Logic***, forthcoming.

[4] R. E. Jennings, P. K. Schotch and D. K. Johnston, *Universal first order definability in modal logic*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*** 26 (1980), pp. 327–330.

[5] D. Lewis, *Logic for Equivocators*, ***Nous***, 16 (1982), pp. 431–441.

[6] P. K. Schotch and R. E. Jennings, *Inference and Necessity*, ***Journal of Philosophical Logic*** 9 (1980), pp. 327–340.

[7] P. K. Schotch and R. E. Jennings, *On detonating*, Unpublished.

[8] D. Scott, *Completeness and axiomatization of many-valued logics*. In: ***Proceedings of the Tarski Symposium***, pp. 411–435 AMS.

Simon Fraser University

Dalhonsic University

Received July, 1981; *revised November*, 1981