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TECHNICAL NOTE

On the Relative Leadership Property of Stackelberg Strategies[1]

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Abstract. The relative leadership property of Stackelberg strategies has been investigated via a scalar nonzero-sum, two-person differential game problem. It is shown that, depending on the parameters of the game, there exist three different types of solutions for his class of games.

1. Introduction

In a recent paper (Ref. 1), Simaan and Cruz have obtained the open-loop Stackelberg solution for a class of deterministic nonzero-sum two-person games under the leadership of one of the players. One of the properties of the Stackelberg strategies, as discussed in Ref. 1, is that, if one of the players acts as the leader in the game, then *both* players might benefit from this leadership in the sense that both of them might end up with better payoffs than the ones obtained from the Nash strategies.[3] This property of the Stackelberg solution brings up the question of how to decide on which player should lead the game and which player should follow if this choice is open to the players. As mentioned in Ref. 1, one player may not have enough information

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[3] It is important to note that we are restricting ourselves to noncooperative solution concepts. Otherwise, Pareto-optimal solution should be considered in making the comparison.

to act as leader. In situations where each player has the option of playing leader or follower, there is the further question of whether it is always profitable for either player to act as the leader rather than be the follower.

In this note, we address ourselves to these questions via a scalar nonzero-sum differential game problem which is related to Example 5 of Ref. 1. We show that there are situations in which a player would prefer to be the follower rather than be the leader and that this leads, in general, to three different types of solutions for this class of games. The concept introduced and the conditions derived for the scalar example can readily be extended to encompass a more general class of nonzero-sum game problems.

2. A Linear Quadratic Differential Game

Consider the following generalized form of Example 5 of Ref. 1. The dynamics are described by the scalar linear differential equation

$$\dot{x} = u_1 - u_2, \qquad x(0) = x_0, \tag{1}$$

where u_1 and u_2 are controlled by Player 1 and Player 2, respectively, and are measurable functions of $t \in (0, 1]$ and x_0. The costs to Players 1 and 2 are given by the quadratic payoff functions J_1 and J_2, respectively, where

$$J_1(u_1, u_2) = \tfrac{1}{2}c_1 x_f^2 + (1/2c_p)\int_0^1 u_1^2\, dt, \tag{2-1}$$

$$J_2(u_1, u_2) = \tfrac{1}{2}c_2 x_f^2 + (1/2c_e)\int_0^1 u_2^2\, dt, \tag{2-2}$$

$c_p > 0$, $c_e > 0$, $c_1 \neq 0$, $c_2 \neq 0$, and x_f denotes the terminal state [that is, $x(1)$]. Note that this formulation becomes identical to Example 5 of Ref. 1 when $c_1 = 1$, $c_2 = -1$, and $c_p c_e = 1$.

Now, denote the Stackelberg control of the ith player when the jth player is the leader by u_{isj} and the corresponding Stackelberg payoff $J_i(u_{1sj}, u_{2sj})$ by J_i^j. Then, it follows from Eqs. (48)–(53) of Ref. 1 that the open-loop Stackelberg solution with Player 2 as the leader is given by

$$u_{1s2} = -\{c_1 c_p(1 + c_1 c_p)/[(1 + c_1 c_p)^2 + c_2 c_e]\}\, x_0, \tag{3-1}$$

$$u_{2s2} = \{c_e c_2/[(1 + c_1 c_p)^2 + c_2 c_e]\}\, x_0, \tag{3-2}$$

under the conditions

$$(1 + c_1 c_p) > 0, \qquad (1 + c_1 c_p)^2 + c_2 c_e > 0. \tag{4}$$

and the corresponding Stackelberg payoffs are

$$J_1^2 = \tfrac{1}{2}\{c_1(1 + c_1c_p)^3/[(1 + c_1c_p)^2 + c_2c_e]^2\} x_0^2, \tag{5-1}$$

$$J_2^2 = \tfrac{1}{2}\{c_2/[(1 + c_1c_p)^2 + c_2c_e]\} x_0^2. \tag{5-2}$$

Using a symmetry property of the original differential game, the open-loop Stackelberg solution with Player 1 as the leader can readily be obtained from (3)–(5), that is,

$$u_{1s1} = -\{c_1c_p/[(1 + c_2c_e)^2 + c_1c_p]\} x_0 , \tag{6-1}$$

$$u_{2s1} = \{c_2c_e(1 + c_2c_e)/[(1 + c_2c_e)^2 + c_1c_p]\} x_0 , \tag{6-2}$$

under the conditions

$$(1 + c_2c_e) > 0, \qquad (1 + c_ec_2)^2 + c_1c_p > 0, \tag{7}$$

and the corresponding Stackelberg payoffs are

$$J_1^1 = \tfrac{1}{2}\{c_1/[(1 + c_2c_e)^2 + c_1c_p]\} x_0^2, \tag{8-1}$$

$$J_2^1 = \tfrac{1}{2}\{c_2(1 + c_2c_e)^3/[(1 + c_2c_e)^2 + c_1c_p]^2\} x_0^2. \tag{8-2}$$

Denoting the Nash payoffs of Players 1 and 2 by J_1^N and J_2^N, respectively, it is certainly true that

$$J_1^1 < J_1^N, \qquad J_2^2 < J_2^N; \tag{9}$$

that is, the leader will always do better (in the sense of achieving a lower payoff) than his Nash solution. The inequalities in (9) are strict, because of the assumption $c_1 \neq 0$, $c_2 \neq 0$. However, relation (9) does not necessarily imply that the best that each player can do (in a noncooperative sense) is to be the leader in the game. In fact, such a statement will not always be true as will be shown in the sequel.

In order to derive the conditions under which $J_1^1 \leqslant J_1^2$ and/or $J_2^2 \leqslant J_2^1$, we will first have to require relations (4) and (7) to be satisfied. Then, $J_1^1 \leqslant J_1^2$ implies (after some straightforward but extensive manipulations) either

$$\text{(i)} \quad c_2 > 0 \tag{10-1}$$

or

$$\text{(ii)} \quad c_2 < 0, \qquad -2/c_2c_e \leqslant 1 + (2 + c_1c_p)/(1 + c_1c_p)^2. \tag{10-2}$$

That is, if either (10-1) or (10-2) is satisfied, Player 1 can achieve the lowest possible payoff (in a noncooperative sense and assuming that Player 2 acts rationally) by being the leader in the game. Note that, if $J_i^i = J_i^j$, $i \neq j$, then Player i achieves the same payoff by being either the leader or the follower. In such a paradoxical situation, we assume that he acts as a leader.

Similarly, the conditions under which Player 2 would rather prefer to be the leader (i.e., $J_2^2 \leqslant J_2^1$) are either

$$\text{(iii)} \quad c_1 > 0 \tag{11-1}$$

or

$$\text{(iv)} \quad c_1 < 0, \qquad -2/c_1 c_p \leqslant 1 + (2 + c_2 c_e)/(1 + c_2 c_e)^2, \tag{11-2}$$

provided that relations (4) and (7) are also satisfied.

To summarize these results and to indicate their immediate implications in a compact form, denote the set of $c_1 \neq 0$, $c_2 \neq 0$, $c_p > 0$, $c_e > 0$ which satisfy (4) and (7) by Ω. Further, denote the quadruple $\{c_1, c_2, c_p, c_e\}$ by α. Let Γ_1 be the set of $\alpha \in \Omega$ which satisfy either (10-1) or (10-2), and Γ_2 be the set of $\alpha \in \Omega$ which satisfy either (11-1) or (11-2). Then, we have the following conclusions (note that any given α specifies the game completely):

(i) $\Gamma_1 \subset \Omega, \quad \Gamma_2 \subset \Omega$;

(ii) $\Gamma_1 \cap \Gamma_2 \neq \phi$ (this will be proven in the sequel via a numerical example, see Example 2.2);

(iii) $(\Omega - \Gamma_1) \cap (\Omega - \Gamma_2) \neq \phi$ (this will also be proven in the sequel via a numerical example, see Example 2.4);

(iv) Player i wants to be the *leader* iff $\alpha \in \Gamma_i$;

(v) Player i wants to be the *follower* iff $\alpha \in \Omega - \Gamma_i$;

(vi) both players want *Player i* to be the *leader* iff $\alpha \in \Gamma_i \cap (\Omega - \Gamma_j)$, $i \neq j$;

(vii) each player wants himself to be the *leader* iff $\alpha \in \Gamma_1 \cap \Gamma_2$;

(viii) each player wants himself to be the *follower* iff $\alpha \in (\Omega - \Gamma_1) \cap (\Omega - \Gamma_2)$.

Hence, associated with the nonzero-sum differential game considered in this note, we have three different types of solutions, depending on the parameters defining the game.

Type A: Concurrent Solution. If $\alpha \in \Gamma_i \cap (\Omega - \Gamma_j)$, $i \neq j$, it follows from item (vi) that the players mutually benefit from the leadership of the ith player and, hence, they *collectively* decide to play the game under player ith leadership (even though it is a noncooperative game). We call this a *concurrent* solution, since there is no reason for either player to deviate from the corresponding Stackelberg solution which was computed under mutual agreement.

Type B: Nonconcurrent Solution. If $\alpha \in \Gamma_1 \cap \Gamma_2$, either player knows that he will do best (in the noncooperative sense) if he himself is the leader (item vii). Hence, either player will try to announce his strategy first and thus force the other player to pick the Stackelberg strategy under his leadership. In this case, the one who can process his data faster will certainly be the leader and announce his policy first. However, if the *slower* player does not actually know that the other player can process his data faster than he does and or if there is a delay in the information exchange between the two players (which is the case in many economic situations), then he might tend to announce a Stackelberg strategy under his own leadership quite unaware of the announcement of the other player; this certainly results in a non-equilibrium situation.

Type C: Stalemate Solution. If $\alpha \in (\Omega - \Gamma_1) \cap (\Omega - \Gamma_2)$, then neither player wants to be the leader (item viii). Both players will rather prefer to wait for the opponent to announce his policy first, which will result in a *stalemate.* In order to come up with a reasonable solution for this case, one has to introduce some negotiation or bargaining between the players. The question of the existence and nature of the bargaining procedure that would result in a concurrent solution is yet an open problem that requires further investigation.

We next consider numerical examples to illustrate these three different types of solutions.

Example 2.1

$$\alpha = \{c_1 = 1, c_2 = -1, c_p = 1, c_e = 0.5\}.$$

It can readily be checked that $\alpha \in \Gamma_2 \cap (\Omega - \Gamma_1)$ and, hence, this example admits a Type A solution with Player 2 being the leader. This example can also be considered as a velocity-controlled pursuit–evasion game of the nonzero-sum variety in which the pursuer (Player 1) has less weight on his control than the evader (Player 2), that is, $1/c_p < 1/c_e$. Under this condition, the pursuer would prefer to wait and act second, and the evader would rather prefer to act first.

Different Stackelberg payoffs for this example are

$$J_1^1 = 0.400x_0^2, \qquad J_2^1 = -0.040x_0^2,$$
$$J_1^2 = 0.326x_0^2, \qquad J_2^2 = -0.143x_0^2.$$

Example 2.2

$$\alpha = \{c_1 = 1, c_2 = -1, c_p = 0.2, c_e = 0.8\}.$$

For this game $\alpha \in \Gamma_1 \cap \Gamma_2$ and, hence, it admits a Type B solution, i.e., neither Player 1 (the pursuer) nor Player 2 (the evader) want to be the follower. Note that the only difference of this example from the previous one is that now the pursuer has more weight on his control than the evader.

Different Stackelberg payoffs for this example are

$$J_1^1 = 2.08x_0^2, \qquad J_2^1 = -0.069x_0^2,$$
$$J_1^2 = 2.11x_0^2, \qquad J_2^2 = -0.78x_0^2.$$

Example 2.3

$$\alpha = \{c_1 = 1, c_2 = 1, c_p = 1, c_e = 0.5\}.$$

This example also admits a Type B solution, that is, $\alpha \in \Gamma_1 \cap \Gamma_2$. Different Stackelberg payoffs for this game are

$$J_1^1 = 0.154x_0^2, \qquad J_2^1 = 0.160x_0^2,$$
$$J_1^2 = 0.198x_0^2, \qquad J_2^2 = 0.111x_0^2$$

Example 2.4

$$\alpha = \{c_1 = -1, c_2 = -1, c_p = \tfrac{1}{3}, c_e = \tfrac{1}{3}\}$$

For this final example, $\alpha \in (\Omega - \Gamma_1) \cap (\Omega - \Gamma_2)$ and, hence, it admits a Type C solution. Both players want to be the follower and this leads to a *stalemate* solution. One has to introduce some cooperation between the players in order to derive a concurrent solution (if such a solution exists). Different Stackelberg payoffs for this example are

$$J_1^1 = -4.5x_0^2, \qquad J_2^1 = -12x_0^2,$$
$$J_1^2 = -12x_0^2, \qquad J_2^2 = -4.5x_0^2.$$

Reference

1. SIMAAN, M., and CRUZ, J. B., JR., *On the Stackelberg Strategy in Nonzero-Sum Games*, Journal of Optimization Theory and Applications, Vol. 11, No. 5, 1973.