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Optimum Design of Vibrating Cantilevers[1]

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Abstract. We determine the optimum tapering of a cantilever carrying an end mass, i.e., the shape which, for a given total mass, yields the highest possible value of the first fundamental frequency of harmonic bending vibrations in the vertical plane.

Three different cases are considered. In the first case, all cross sections are assumed to be geometrically similar. In the second case, the cross sections are assumed to be rectangular and of given width. Finally, we consider a rectangular cross section of given height. This third case is shown to be degenerate in the absence of end mass.

1. Introduction

A straight cantilever beam made of an elastic material can perform small, harmonic, transverse vibrations. The lowest natural frequency ω_0 for such vibrations depends on the length, shape, and material properties of the beam. We shall assume that the material of the beam is homogeneous and isotropic and obeys Hooke's law.

The problem with which we shall deal in this paper is to find the tapering that yields the highest possible value of ω_0 for a given length and volume of the beam.

The corresponding problem for a simply supported beam has been the subject of an earlier paper by one of the authors (Ref. 1). Although our analysis follows closely that of Ref. 1, the extension

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is not trivial, due to the different types of singularity encountered at the free end of a cantilever. It is interesting to note that, while the increment in natural frequency achieved with optimum tapering of simply supported beams (with geometrically similar cross sections) is only approximately 6.6% compared with the uniform beam (Ref. 1), the corresponding figure is as high as 678% for the cantilever, as will be shown here. An increment of 425% is obtained by the best tapering of cantilevers having rectangular cross sections of given width.

We shall also discuss and solve the problem of optimum tapering of cantilevers in the presence of a mass at their tip. It will be seen that even a small mass has a substantial effect on the optimum shape of the beam. For a very large mass at the free end, the mass of the beam itself becomes negligible, and the solution can be given in a closed form (massless cantilever beam of maximum stiffness).

We shall assume throughout this paper that there is a relation of the following form between the bending rigidity EI and the mass per unit length of the beam:

$$I = cA^p, \tag{1}$$

where A is the area of the cross section and c is a constant. We are especially interested in three cases, viz., $p = 1, 2, 3$. The cases $p = 2$ and $p = 3$, corresponding to beams with geometrically similar cross sections and rectangular cross sections of given uniform width, respectively, have so much in common that they can be treated in much the same way. For $p = 1$, however, we have a degenerate case that needs separate treatment. This case corresponds to a beam of given uniform height and tapered width. It will be shown that, in this case, no optimum shape exists in the absence of mass at the tip. This has also been pointed out in another work (Ref. 5), which considered the case corresponding to $p = 1$ for beams with homogeneous boundary conditions.

2. Transverse Vibrations of a Tapered Cantilever Beam

Let us consider small harmonic transverse vibrations of a tapered cantilever beam carrying a mass Q at its tip. If shear deformations and rotary inertia are neglected, the differential equation of motion and the boundary conditions can be written in the following dimensionless form:

$$(\alpha^p y'')'' - \lambda \alpha y = 0, \tag{2}$$

$$y(1) = y'(1) = \alpha^p y''(0) = 0, \qquad (\alpha^p y'')'_{x=0} = \lambda q y(0). \tag{3}$$

Here, y is the amplitude of the lateral deflection in the plane of bending, and a dash indicates differentiation with respect to the dimensionless coordinate $x = \xi/l$. The dimensionless area function is denoted $\alpha = Al/V$, where V is the total volume of the beam. In the last boundary condition (3) we have

$$\lambda = \omega^2(\gamma l^{3+p}/EcV^{p-1}), \qquad q = Q/\gamma V, \tag{4}$$

where c is the constant determined by the relation (1).

From the definition of α, it follows that

$$\int_0^1 \alpha\, dx = 1. \tag{5}$$

The eigenvalues λ of the problem (2)–(3) are uniquely determined by the function $\alpha(x)$. In the sequel, λ will refer to the first fundamental eigenvalue and y to its associated eigenfunction. If the problem under study has an optimum, our aim is to find the nonnegative function $\alpha(x)$, satisfying the condition (5), that renders λ a maximum. We shall employ a variational method for solving our optimization problem.

3. Variational Problem

An expression for λ is obtained by multiplying both sides of Eq. (2) by y and integrating between the limits 0 and 1. If the first term is integrated by parts and the boundary conditions (3) are taken into account, we obtain the well-known formula for λ

$$\lambda = \left[\int_0^1 \alpha^p (y'')^2\, dx\right] \Big/ \left[\int_0^1 \alpha y^2\, dx + qy^2(0)\right]. \tag{6}$$

Application of the variational procedure outlined in Ref. 1 leads to the following expression for determining α:

$$p\alpha^{p-1}(y'')^2 - \lambda y^2 = \lambda a^2, \tag{7}$$

where a^2 is a number independent of x. Multiplying both sides by α and integrating, we obtain

$$a^2 = (p-1)\int_0^1 \alpha y^2\, dx + pqy^2(0), \tag{8}$$

which indicates that a^2 is positive for all $p \geqslant 1$.

Solving Eq. (7) for α, we get

$$\alpha = [\lambda(a^2 + y^2)/p(y'')^2]^{1/(p-1)}, \qquad p > 1, \tag{9}$$

which, upon substitution into Eq. (2), yields the following nonlinear differential equation for y:

$$\{[(a^2 + y^2)/p(y'')^2]^{p/(p-1)} y''\}'' - [(a^2 + y^2)/p(y'')^2]^{1/(p-1)} y = 0, \tag{10}$$

which, together with the boundary conditions (3), constitutes a (nonlinear) eigenvalue problem for the parameter a^2 (the eigenvalue).

Substitution of α from (9) into (5) and (8) leads to the following equations for determining a^2 and λ, $p > 1$,

$$\begin{aligned} \int_0^1 [\lambda(a^2 + y^2)/p(y'')^2]^{1/(p-1)}\, dx &= 1, \\ a^2 = (p-1) \int_0^1 [\lambda(a^2 + y^2)/p(y'')^2]^{1/(p-1)} y^2\, dx &+ pqy^2(0) \end{aligned} \tag{11}$$

In the present context, we shall consider three different cases, corresponding to $p = 2, 3, 1$. The last case, corresponding to $p = 1$, is degenerate and will therefore be treated separately.

4. Behavior of the Solution Near the Free End

Before we attempt to solve the differential equation (10), it is expedient to analyze the behavior of the solution near the free end. For this purpose, we assume that the solution $y(x)$ near $x = 0$ can be expanded in a power series of x with a characteristic term bx^k. On substituting $y = bx^k + \cdots$ into Eq. (10) and equating the coefficients of leading terms to zero, we get the following equations for determining, respectively, the smallest noninteger value of k and the smallest integer value of k:

$$\begin{aligned} (-pk_1 + 2p - k_1 + 2)(-pk_1 + p - k_1 + 3) &= 0, \\ (pk_2 + 2p - k_2 + 2)(pk_2 + p - k_2 + 3) - p(p-1)^2 k_2(k_2 - 1) &= 0. \end{aligned} \tag{12}$$

Here, k_1 is the smallest noninteger and k_2 is the smallest integer, $p > 1$.

The respective values for $p = 2$ and $p = 3$ are

$$k_1 = \tfrac{5}{3} \quad \text{and} \quad k_1 = \tfrac{3}{2},$$
$$k_2 = -2 \quad \text{and} \quad k_2 = -1.$$

Taking

$$y = a_1 x + a_2 x^2 + \cdots + b_1 x^{k_1} + \cdots, \tag{13}$$

we satisfy the boundary conditions (3) at $x = 0$.

On the other hand, the solution near $x = 0$ must be taken in the following form in order to satisfy the boundary conditions in the absence of a mass at the free end:

$$y = a_1 x + a_2 x^2 + \cdots + b_1 x^{k_2} + \cdots. \tag{14}$$

It should be noted here that, in order to get a correct estimate of the behavior of function y, and hence of $\alpha(x)$, in the transition region from no-mass to very small mass at the free end, it was necessary to analyze the behavior of the solution in more detail at $x = 0$. This involved determination of the next term in the series expansion for y near $x = 0$ by assuming

$$y = b_1 x^{k_2} + b_2 x^{k_2 + \bar{k}} + \cdots$$

and calculating $\bar{k}$ in a manner similar to that described above. The respective values of $\bar{k}$ for $p = 2$ and 3 are: $\bar{k} = 5.4$ and 3.6.

Substitution of (13), or (14) as the case may be, into (9) gives an idea of the variation of α near $x = 0$. For example, substitution of (13) into (9) with $p = 2$ shows that the area function α is proportional to $x^{2/3}$ near $x = 0$ and that the linear dimension—or diameter—of the cross section is thus proportional to $x^{1/3}$. On the other hand, substitution of (14) into (9) shows that, when the beam carries no mass at its tip, the area function α is proportional to x^4 for small values of x, and the linear dimension of the cross section is thus proportional to x^2.

5. Solution by Successive Iterations

As a solution of the differential equation (10) cannot in general be obtained in a closed form, the solution was found numerically by successive iterations; the method is based upon a formal integration of the differential equation, with the introduction of one of the boundary conditions at each integration. Care was taken to separate the differential operator of the highest order on the left-hand side at each step. This was found necessary in order to obtain convergence by successive iterations. We demonstrate the numerical procedure briefly with respect to two specific cases.

5.1. $p = 2$ and $q = 0$. By formal integration of (10) with $p = 2$, we find, after satisfying the boundary conditions at $x = 0$, that

$$y''(x) = \left[(a^2 + y^2)^2 \Big/ 2 \int_0^x \int_0^x [(a^2 + y^2)/(y'')^2] y \, dx^2\right]^{1/3}, \qquad (15)$$

from which we could construct a procedure for successive iterations. However, in order to take care of the singularity in y near $x = 0$, we proceed in a slightly different way. The expansion formula (14), for y near $x = 0$, indicates how to define a finite function $f(x)$ in the closed interval $0 \leqslant x \leqslant 1$ by the following equation:

$$f(x) = x^2 y(x), \qquad (16)$$

such that

$$y'' = x^{-2} f''(x) - 4x^{-3} f'(x) + 6x^{-4} f(x).$$

Furthermore, we define

$$z(x) = x^2 f'' - 4x f' + 6f, \qquad (17)$$

such that

$$y'' = x^{-4} z(x). \qquad (18)$$

The functions $f(x)$ and $z(x)$ introduced above, are regular in the interval $0 \leqslant x \leqslant 1$, and we have $f(1) = f'(1) = 0$, since y and y' vanish at $x = 1$.

By a formal integration of (17) and by taking into account the boundary conditions imposed on $f(x)$ at $x = 1$, we have

$$f(x) = x^2 \int_1^x \int_1^x [z(x)/x^4] \, dx^2, \qquad 0 < x \leqslant 1. \qquad (19)$$

From (15) and (17), we get

$$z(x) = x^4 y'' = x^4 \left[(a^2 + y^2)^2 \Big/ 2 \int_0^x \int_0^x [(a^2 + y^2) y/(y'')^2] \, dx^2\right]^{1/3}, \qquad (20)$$

which can thus be written as

$$z(x) = \left[(a^2 x^4 + f^2)^2 x^4 \Big/ 2 \int_0^x \int_0^x [(a^2 x^4 + f^2) x^2 f/z^2] \, dx^2\right]^{1/3}. \qquad (21)$$

Here, it may be noted that, in numerical integration procedures, it is often preferable to operate on single integrals instead of double integrals. By the rule of transformations, the expression for $f(x)$ and the double integral in the expression for $z(x)$ can be written as

$$f(x) = x^2 \int_1^x [z(\zeta)/\zeta^4](x - \zeta)\, d\zeta,$$

$$(1/x^4) \int_0^x \int_0^x \phi(x)\, x^2\, dx^2 = (1/x^4) \int_0^x \phi(\eta)\, \eta^2(x - \eta)\, d\eta.$$

By a change of variables, the expressions for $f(x)$ and $z(x)$ can be further reduced to the following expressions containing only single integrals between limits 0 and 1:

$$f(x) = x^2(1 - x^2) \int_0^1 z(\eta - \eta x + x)[\eta/(\eta - \eta x + x)^4]\, d\eta, \tag{22}$$

$$z(x) = \left[(a^2x^4 + f^2)^2 \Big/ 2 \int_0^1 \phi(x\psi)(\psi^2 - \psi^3)\, d\psi\right]^{1/3}, \tag{23}$$

where $\phi(x)$ is defined by

$$\phi(x) = (a^2x^4 + f^2)f/z^2. \tag{24}$$

The scheme for successive iterations can now be written as follows, where only finite functions are subject to numerical treatment[4]:

(i) $f_n(x) = x^2(1 - x^2) \int_0^1 z_n(\eta - \eta x + x)[\eta/(\eta - \eta x + x)^4]\, d\eta,$

(ii) $a_n^4 = [\int_0^1 (f_n/z_n)^2\, dx]/[\int_0^1 (x^4/z_n)^2\, dx],$

(iii) $\phi_{n+1}(x) = (a_n^2x^4 + f_n^2)f_n/z_n^2,$

(iv) $z_{n+1}(x) = [(a_n^2x^4 + f_n^2)^2/2 \int_0^1 \phi_{n+1}(x\psi)(\psi^2 - \psi^3)\, d\psi]^{1/3}.$

The sequence of successive iterations is started with an arbitrary regular function $z_0(x)$ (not necessarily satisfying the boundary conditions). The new functions z_1 and f_1 will satisfy all the boundary conditions. The sequence of iterates z_n, f_n converged very rapidly to functions z and f, from which the solution $y(x)$ and its derivatives were determined using equations (16) and (17). The optimum eigenvalue λ (corresponding to ω_0) was then found from the following equation, which is obtained by solving the system of equations (11) in a^2 and λ:

$$\lambda = 2\Big/\left\{\int_0^1 (fx^2/z)^2\, dx + \left[\int_0^1 (f^2/z)^2\, dx \int_0^1 (x^4/z)^2\, dx\right]^{1/2}\right\}. \tag{25}$$

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[4] The expression for a^4 is obtained by solving the system of equations (11).

Using the relation (9), we finally computed the area function $\alpha(x)$ and, hence, the linear dimension $\sqrt{\alpha}$ of the optimum cantilever in the interval $0 \leqslant x \leqslant 1$.

5.2. $p = 3$ and $q =$ finite. The formal integration procedure leads to the following equation for $y(x)$:

$$y(x) = -\int_x^1 - \int_x^1 \frac{[(a^2 + y^2)/3]^{3/4}}{\left[\int_0^x \int_0^x A(x) y \, dx^2 + \int_0^x qy(0) \, dx \int_0^1 A(x) \, dx\right]^{1/2}} dx^2, \quad (26)$$

where

$$A(x) = [(a^2 + y^2)/3]^{1/2} (1/y''). \quad (27)$$

The expansion formula (13) indicates that we can define a finite function $g(x)$ in the closed interval $0 \leqslant x \leqslant 1$ as follows:

$$g(x) = \sqrt{x}\, y''(x). \quad (28)$$

Furthermore, we define

$$A(x) = A^*(x) \sqrt{x}, \quad (29)$$

where

$$A^*(x) = [(a^2 + y^2)/3]^{1/2} [1/g(x)]. \quad (30)$$

The functions $g(x)$ and $A^*(x)$ introduced now are regular in the interval $0 \leqslant x \leqslant 1$.

The iteration process, involving integration of finite functions only, is carried out in the following sequence:

(i) $y_n'(x) = -\int_x^1 [g_n(x)]^{-1/2} \, dx$,

(ii) $y_n(x) = -\int_x^1 y_n'(x) \, dx$,

(iii) $a_n^2 = 2[\int_0^1 A_n^*(x) \sqrt{x}\, y_n^2(x) \, dx]/[\int_0^1 A_n^*(x) \sqrt{x} \, dx] + 3qy_n^2(0)$,

where

$$A_n^*(x) = [(a_n^2 + y_n^2(x))/3]^{1/2} [1/g_n(x)].$$

(iv) $M_{n+1}(x) = \{\int_0^x \int_0^x A_n^*(x) \sqrt{x}\, y_n(x) \, dx^2 + \int_0^x gy_n(0) \, dx \int_0^1 [A_n^*(x)]^{1/2} \, dx\}$,

(v) $g_{n+1}(x) = y_{n+1}''(x) \cdot \sqrt{x} = \sqrt{x}[(a_n^2 + y_n^2)/3]^{3/4}/[M_{n+1}(x)]^{1/2}$.

The expression for a^2 and λ are obtained by solving the system of equations (11). As the expression for a^2 is an implicit one, a_n^2 is determined in an *inner loop* in the iteration scheme. Similar situations often arise in optimization problems of vibrating plates (Ref. 2).

As in the previous case, the iterations were started with an arbitrary regular function $g_0(x)$. The iterates g_n, A_n^*, y_n', y_n converged rapidly to the functions g, A^*, y' and y, from which y'' was obtained using equation (28). The optimum eigenvalue λ was then found from the following expression, which is obtained by solving the system of equations (11) with the designation (30):

$$\lambda = 1/\left[\int_0^1 A^*(x)\sqrt{x}\,dx\right]^2. \tag{31}$$

Using the relations (9) and (30), we did finally compute the area function $\alpha(x)$ of the optimum cantilever for various values of non-dimensional mass q from the expression

$$\alpha(x) = \sqrt{\lambda}\,A^*(x)\sqrt{x}. \tag{32}$$

Numerical integration was performed by subdividing the interval $0 \leqslant x \leqslant 1$ into a number of equal parts and applying a polynomial formula. The mesh length d was varied, and the result was extrapolated to $d = 0$ by means of Newton's formula. Starting with an arbitrary initial function ($z_0 \equiv 1$ for $p = 2$ and $g_0 \equiv 1$ for $p = 3$), the accurate estimate of solution $y(x)$ was obtained within a few iterations.

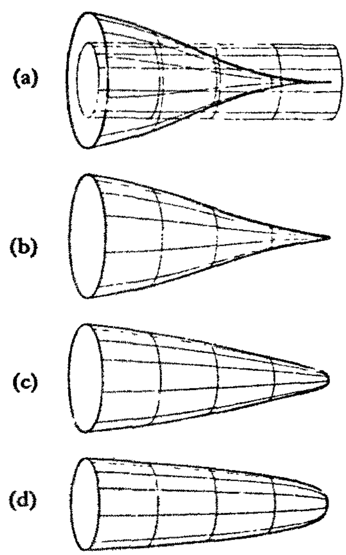

Fig. 1

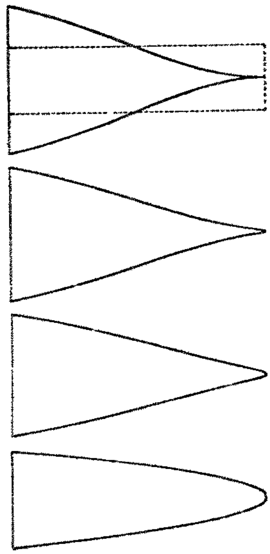

Fig. 2

Figures 1 and 2 show the variation of linear dimension of cross section as a function of nondimensional x for various values of nondimensional mass q for $p = 2$ and 3, respectively.

Dotted lines in the figures for $q = 0$ indicate a cantilever beam of uniform cross section and the same length and volume as the optimum beam. The corresponding percentage increase in the lowest natural frequency ω_0 in comparison with that of the cantilever beam with uniform cross section and the same volume, material, and length as the optimum beam is indicated in Table 1.

Knowing that $\lambda_c = 12.35$ for a cantilever beam of uniform circular cross section or with uniform rectangular cross section (Ref. 3), we conclude that the most appropriate (optimum) tapering—keeping the

Table 1. Ratio of $\sqrt{(\lambda/\lambda_c)}$ for various values of q ($p = 2, 3$).

	$q = 0$	$q = 0.0003$	$q = 0.03$	$q = 100$
$p = 2$	6.78 (Fig. 1a)	5.48 (Fig. 1b)	3.36 (Fig. 1c)	1.27 (Fig. 1d)
$p = 3$	4.25 (Fig. 2a)	3.71 (Fig. 2b)	2.30 (Fig. 2c)	1.33 (Fig. 2d)

total volume constant—will increase the lowest natural frequency by as much as 678% in the former case ($p = 2$), and 425% in the latter ($p = 3$).

It may be noted that optimum shapes corresponding to $q = 100$ closely approximate the classical solution obtained by considering the corresponding static problem of a massless cantilever with a heavy mass at its tip.

6. Cantilever of Given Constant Height and Varying Width ($p = 1$)

It is evident from Eq. (9) that, for $p = 1$, we have a degenerate case. Furthermore, α drops out of Eq. (7) for $p = 1$, suggesting that the problem under study does not have an optimum in the sense used in this text. For a cantilever beam carrying no mass at its tip, this fact —which is not so evident at the first sight—can be explained physically in the following way.

For a cantilever of given height and varying width, which is symmetrical with respect to the plane of bending, we can always divide the cantilever into two symmetrical halves along the plane of bending, each half vibrating with the frequency of the original cantilever. Now, by symmetrically superimposing one-half on the other near the clamped end, thereby retaining the volume of the original cantilever, the frequency of vibrations of the resulting cantilever will be increased. Following this process of division of the cantilever into two symmetrical halves indefinitely with subsequent superimposition of one-half on the other symmetrically near the clamped end, the frequency of the resulting cantilever can similarly be increased indefinitely while at the same time retaining its total volume. However, the same argument does not hold in the case of a cantilever carrying a mass at its tip. In fact, as will be shown later in this section, such a cantilever does have an optimum shape in the present sense. We will presently try to give a mathematical interpretation of the former case. We will show briefly that, for a certain family of curves defining the area function $\alpha(x)$, the first fundamental frequency of the cantilever can be increased indefinitely with an increase of the characteristic parameter of the family of curves, the total volume being retained.

Let us assume that $\alpha = kx^n$ is a family of curves defining the area function, where n can assume any value between $-\infty$ and ∞. The coefficient k is determined by the condition that the total volume of the cantilever is constant (5).

From (6), we obtain for $q = 0$ and $\alpha = kx^n$

$$\lambda = \left[\int_0^1 x^n (y'')^2\, dx\right] \Big/ \left[\int_0^1 x^n y^2\, dx\right].$$

The upper limit of the first eigenvalue λ_1 can be obtained by Rayleigh's method, taking $y = (1 - x)^2$, which satisfies the kinematic boundary conditions at $x = 1$. After performing the simple integrations, we get

$$\lambda_1 \leqslant 2n^2 + 10n + 12.$$

Obviously, the upper limit of the first fundamental eigenvalue of the cantilever tends to approach ∞ for $n \to \infty$. However, it is also necessary to determine the lower bound of the first eigenvalue. To do so, we adopt a procedure similar to the Dunkerley method involving Green's function.

Green's function $G(x, \xi)$ is the deflection at x due to a unit load at ξ which, in terms of the normal modes v_n and by making use of the Bessel inequality (Ref. 4), can be written as

$$[G(x, \xi)]_{x=\xi} = \sum_{n=1}^{\infty} [(1/\omega_n^2)\, v_n^2(x)].$$

Multiplying both sides of the above expression by $m(x)$ (mass per unit length at x) and integrating over the length of the cantilever, we get

$$\sum_{n=1}^{\infty} (1/\omega_n^2) = \int_0^1 G(x, x)\, m(x)\, dx,$$

since

$$\int_0^1 m(x)\, v_n^2(x)\, dx = 1.$$

Obviously,

$$1/\omega_1^2 < \int_0^1 G(x, x)\, m(x)\, dx,$$

where ω_1 is the first fundamental frequency of the cantilever. With the assumed family of area functions, Green's function $G(x, x)$ turns out to be[5]

$$G(x, x) = (12/Ek)[2/(n-1)(n-2)(n-3)(1-x)^{n-3} - 2/(n-1)(n-2)(n-3) - 2x/(n-1)(n-2) - x^2/(n-1)], \qquad n \neq 1, 2, 3.$$

[5] For the sake of convenience, the origin has been taken at the clamped end.

Thus, we get

$$\omega_1^2 > (E/12\gamma)[2(n-1)(n+1)(n-2)(n+3)/[(1-2n)(n+2)(n+3)+(3n+4)(n+1)]].$$

As the degree of n in the numerator is greater than that in the denominator, the lower limit of ω_1 tends to approach infinity as $n \to \infty$.

We have therefore been able to show that, for a cantilever of prescribed height and with no mass as its free end, the first fundamental frequency can be increased indefinitely (at least for the family of curves used here for defining the area function) or, in other words, that the problem at hand does not have an optimum solution in the sense used in this paper.

However, as mentioned earlier, the same is not true if the cantilever carries a mass at its tip. We proceed here in a sort of *inverse* way, as will become clear later. We specify the first eigenvalue λ and try to find the optimum width function and the corresponding q giving this λ. Clearly, a backward interpretation leads us to the optimum design in the sense it has been used so far in the text.

7. Cantilever of Given Height and Varying Width ($p = 1$) with a Mass at the Free End

Equation (7) can be rewritten as

$$(y'')^2 = \lambda(a^2 + y^2). \tag{33}$$

Making the substitution

$$y = af(x \sqrt[4]{\lambda}), \tag{34}$$

we get, instead of (33), the following differential equation for $f(x \sqrt[4]{\lambda})$, which is independent of a:

$$(f'')^2 = 1 + f^2. \tag{35}$$

Note the difference in the arguments of functions y and f. For convenience, the origin is taken at the clamped end, so we have

$$f(0) = f'(0) = 0. \tag{36}$$

After satisfying the kinematic boundary conditions (36), we see that the solution of differential equation (35) is given by

$$f = \int_0^{x\sqrt[4]{\lambda}} \sqrt{[\tfrac{1}{2}f\sqrt{(1+f^2)} + \operatorname{arcsinh}(f)]}\, d(x \sqrt[4]{\lambda}). \tag{37}$$

The function y and its derivatives can thus be readily obtained from (34) to the accuracy of constant a, knowing that

$$f' = -\sqrt{[\tfrac{1}{2} f \sqrt{(1+f^2)} + \operatorname{arccosh}(f)]}, \qquad f'' = \sqrt{(1+f^2)}.$$

Having thus found the function y and its derivatives, we return to the original differential equation (2) in order to find the corresponding optimum shape function.

By a formal integration of (2) and by satisfying the boundary conditions (3), we get

$$\alpha(x) = [\sqrt{\lambda}/y''(x\sqrt[4]{\lambda})] \left\{ \int_x^1 \int_x^1 \alpha(x)\, y(x\sqrt[4]{\lambda})\, dx^2 + \int_x^1 q y(\sqrt[4]{\lambda})\, dx \right\}. \tag{38}$$

From (6), we have

$$\lambda = \left[\int_0^1 \alpha a^2 \lambda y''^2\, dx\right] \Big/ \left[\int_0^1 \alpha a^2 y^2\, dx + q a^2 y^2(\sqrt[4]{\lambda})\right],$$

whence

$$q = [1/y^2(\sqrt[4]{\lambda})] \left[\int_0^1 \alpha(x)\, y''^2(x\sqrt[4]{\lambda})\, dx - \int_0^1 \alpha(x)\, y^2(x\sqrt[4]{\lambda})\, dx\right]. \tag{39}$$

Substitution of (39) into (38) gives the following implicit expression for $\alpha(x)$:

$$\alpha(x) = [\sqrt{\lambda}/y''(x\sqrt[4]{\lambda})] \left[\int_x^1 \int_x^1 \alpha(x)\, y(x\sqrt[4]{\lambda})\, dx^2 \right.$$
$$\left. + \int_x^1 [1/y(\sqrt[4]{\lambda})] \left\{ \int_0^1 \alpha(x)\, y''^2(x\sqrt[4]{\lambda})\, dx - \int_0^1 \alpha(x)\, y^2(x\sqrt{\lambda})\, dx \right\} dx \right]. \tag{40}$$

Having found $\alpha(x)$ from (40), we can now determine the nondimensional mass factor q from (39).

Note that condition (5) is not, in general, satisfied, but this only changes the numerical values of the ordinates and not the shape of the function itself.

8. Solution by Successive Iterations

The iteration scheme used for determining y and $\alpha(x)$ as a function of x is in itself quite simple and will, therefore, not be described here. However, it is necessary to analyze the behavior of f (and hence y)

and α in order to choose proper functions for their approximation. For this purpose, let us first consider the differential equation (35).

For $x \sqrt[4]{\lambda} \to 0$ (which also corresponds to very small λ or indirectly very large q), f is very small compared with unity, and hence (35) can be approximated by

$$f'' \simeq 1, \tag{41}$$

the solution of which for all values of λ, after satisfying the kinematic boundary conditions, is

$$f = \tfrac{1}{2}x^{*2}, \qquad x^* = x \sqrt[4]{\lambda}.$$

On the other hand, for $x \sqrt[4]{\lambda} \gg 1$ (which corresponds to very large λ or, indirectly, very small q), f is very large compared with unity, and hence (35) can be approximated by

$$f'' = f,$$

the solution of which is

$$f = A \exp(x^*). \tag{42}$$

Note that (42) does not satisfy the kinematic boundary conditions (36) and hence cannot be carried through to the clamped end.

It is found that (41) gives a good approximation up to $x^* = 1$, which corresponds to $\lambda = 1$; and (42) gives a good approximation for $x^* > 4$, which corresponds to $\lambda > 256$. Between these two extremes, f is determined from (37).

Let us next investigate the behavior of α for small values of x. For a small value of x, we obtain from the differential equation (2) and its solution (41) the following differential equation for α for all values of λ:

$$d^2\alpha/dx^2 - \lambda \cdot (x^2/2)\alpha = 0. \tag{43}$$

Assuming that α can be expanded in a series of x near $x = 0$ in the form

$$\alpha = A - Bx + Cx^2 + Dx^3 + Ex^4 + \cdots, \tag{44}$$

and substituting (44) into the differential equation (43), we get, after equating the coefficients of like powers of x,

$$C = D = 0, \qquad E = (\lambda/24)A.$$

Therefore, for small x,

$$\alpha = A - Bx + (\lambda/24)\, Ax^4,$$

which means that, for small values of λ, α is linear with negative slope.

For larger values of x (and, thus, λ), α has to be determined from (40) by a process of successive iterations.

Having assumed λ, we perform a numerical integration by subdividing the interval $0 \leqslant x \sqrt[4]{\lambda} \leqslant \sqrt[4]{\lambda}$ into a number of equal parts and applying a polynomial formula in the interval $0 \leqslant x \sqrt[4]{\lambda} \leqslant 0.2$ and an exponential formula in the rest of the interval. The sequence of successive iterations is started with an arbitrary $f_0 \equiv 1$ in expression (37) and $\alpha_0(x) \equiv 1$, with $\alpha_0(1) = 0$ in expression (38). The sequence of iterates converges very rapidly to f and $\alpha(x)$. Knowing $\alpha(x)$, we determine the corresponding q from (39), subject to the condition (5).

The optimum shape as a function of nondimensional x is shown in Fig. 3 for selected values of first natural frequency. Table 2 shows the increment in first fundamental frequency in comparison with that of a cantilever beam of rectangular cross section. It will be seen from the figures that the width goes on increasing toward the clamped end with the increase in λ (i.e., with a decrease in q), as was to be expected. Dotted lines indicate a cantilever of constant cross section and the same volume as the optimum cantilever. It may also be noted that

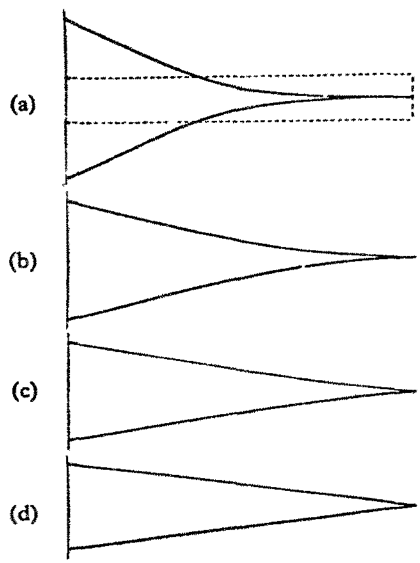

Fig. 3

Table 2. Ratio of $\sqrt{(\lambda/\lambda_c)}$ for various values of q ($p = 1$).

	$q = 0.0375$	$q = 0.2233$	$q = 1.1027$	$q = 4 \times 10^4$
$p = 1$	28.42 (Fig. 3a)	2.56 (Fig. 3b)	0.28 (Fig. 3c)	— (Fig. 3d)

the graph (Fig. 3d) corresponding to $q = 4 \times 10^4$ approximates almost exactly the classical optimum width shape of a massless cantilever with a heavy mass at its tip. The width function, which can be easily found in this case by considering the corresponding static problem, varies linearly from zero at the free end to a finite value at the clamped end (in our case equal to 2).

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