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On the Game of Two Cars

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Abstract. Necessary and sufficient conditions for the existence of the evasion strategy in the so-called game of two cars are given.

Key Words. Differential games, pursuit–evasion games, piecewise programming strategy, evasion strategy.

1. Introduction

Let us consider a well-known (Ref. 1) pursuit–evasion game in which the evader and the pursuer move according to the equations

$$\begin{aligned} x_1'(t) &= v_E \cos x_3(t), \\ x_2'(t) &= v_E \sin x_3(t), \\ x_3'(t) &= u(t), \end{aligned} \tag{1}$$

and

$$\begin{aligned} y_1'(t) &= v_p \cos y_3(t), \\ y_2'(t) &= v_p \sin y_3(t), \\ y_3'(t) &= w(t), \end{aligned} \tag{2}$$

respectively; here, u and w are measurable functions,

$$u: [0, \infty) \to [-a, a], \qquad 0 < a \in \mathbb{R},$$
$$w: [0, \infty) \to [-b, b], \qquad 0 < b \in \mathbb{R},$$

with v_E, v_p real, positive numbers.

Denote by U_t, $t \in [0, \infty)$, the set of all measurable functions $u: [t, \infty) \to [-a, a]$ and by W the set of all measurable functions $w: [0, \infty) \to [-b, b]$.

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Consider the set of all pairs of the form $(e, \{t_n\})$; here, e is a function defined on the set $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$, such that

$$e(t, x, y) \in U_t, \qquad (t, x, y) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3,$$

and $\{t_n\}$ denotes an increasing sequence of nonnegative numbers satisfying the following conditions:

$$0 = t_0, \qquad \lim_{n \to \infty} t_n = \infty.$$

This set is called the set of strategies of the evader and is denoted by $\mathscr{E}$.

It is not hard to see that $\mathscr{E}$ is the so-called set of piecewise programming strategies (see Ref. 2), in which decisions about the control are taken at times from the sequence $\{t_n\}$ only.

Also, consider the set of all functions $p: U_0 \to W$ such that, if

$$u, \tilde{u} \in U_0, \quad t \in [0, \infty), \qquad u|_{[0,t]} = \tilde{u}|_{[0,t]},$$

then

$$p(u)|_{[0,t]} = p(\tilde{u})|_{[0,t]}.$$

This set is called the set of strategies of the pursuer and is denoted by $\mathscr{P}$.

Lemma 1.1. Assume that

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3, \qquad y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$
$$(e, \{t_n\}) \in \mathscr{E}, \qquad p \in \mathscr{P}.$$

There exists exactly one pair of trajectories

$$x = (x_1, x_2, x_3), \qquad y = (y_1, y_2, y_3),$$

and exactly one pair of measurable functions $(u, w) \in U_0 \times W$, such that:

(i) $w = p(u)$;

(ii) x is a solution of (1) with the initial condition $x(0) = x$, and y is a solution of (2) with the initial condition $y(0) = y$;

(iii) $u|_{[t_n, t_{n+1}]} = e(t_n, x(t_n), y(t_n))|_{[t_n, t_n+1]}$, $n \in \mathbb{N}$.

Proof. It proceeds by induction.

We say that the trajectories from Lemma 1.1 are determined by the initial situation (x, y) and by the strategies $(e, \{t_n\})$ and p.

Definition 1.1. The pursuer wins for the initial condition $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$, if there exists a strategy $p \in \mathscr{P}$ such that, for any strategy $(e, \{t_n\}) \in \mathscr{E}$, one can find $t \in [0, \infty)$ for which

$$x_i(t) = y_i(t), \qquad i = 1, 2;$$

here, the pair (x, y) is determined by $(x, y), (e, \{t_n\}), p$.

Definition 1.2. The evader wins for any initial condition, if there exists a strategy $(e, \{t_n\}) \in \mathscr{E}$, such that

$$(x_1(t), x_2(t)) \neq (y_1(t), y_2(t)),$$

for any $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$, $(x_1, x_2) \neq (y_1, y_2)$, any strategy $p \in \mathscr{P}$, and every $t \in [0, \infty)$; here, the pair (x, y) is determined by $(x, y), (e, \{t_n\}), p$.

2. Main Result

Theorem 2.1. The evader wins for any situation iff

$$v_P < v_E, \qquad bv_P \leq av_E, \tag{3a}$$

or

$$v_P = v_E, \qquad b < a. \tag{3b}$$

Without loss of generality, we may assume that

$$v_E = 1, \qquad v_P = v,$$

where $v \in \mathbb{R}$, $v > 0$; we will consider such a game from now on. In this case, the condition (3) assumes the form

$$v < 1, \qquad bv \leq a, \tag{4a}$$

or

$$v = 1, \qquad b < a. \tag{4b}$$

3. Necessity

(a) Let us assume that $v > 1$.

Lemma 3.1. Assume that the function $u: [0, \infty) \to [-a, a]$ is measurable and, for

$$t \in [0, a^{-2}b\sqrt{v^2 - 1}],$$

we have

$$w(t)=\left\{u(t)\sin\left[\left[\int_0^t u(s)\,ds\right]\right]\right\}\Big/\left\{v^2-\cos^2\left[\left[\int_0^t u(s)\,ds\right]\right]\right\}.$$

Then, w projects $[0, a^{-2}b\sqrt{v^2-1}]$ into $[-b, b]$ and it is a measurable function.

Lemma 3.2. Assume that the functions u and w are the same as in Lemma 3.1 and, for

$$t\in[0, a^{-2}b\sqrt{v^2-1}],$$

we have

$$x_2(t)=\int_0^t \sin\left[\pi/2+\int_0^s w(\tau)\,d\tau\right]ds,$$

$$y_2(t)=v\int_0^t \sin\left[\pi-\arcsin(v^{-1})+\int_0^s w(\tau)\,d\tau\right]ds.$$

Then,

$$x_2(t)=y_2(t), \qquad t\in[0, a^{-2}b\sqrt{v^2-1}].$$

Proof. Because

$$x_2(0)=y_2(0),$$

it is sufficient to prove that, for

$$t\in[0, a^{-2}b\sqrt{v^2-1}],$$

we have

$$x_2'(t)=y_2'(t).$$

Lemma 3.3. Assume that $a>b$, the functions u and w are the same as in Lemma 3.1, and

$$t_0=\min\{a^{-2}b\sqrt{v^2-1}, \sqrt{v-1}/a\sqrt{v+1}\},$$

$$d=\sqrt{v^2-1}\,t_0-(a/2)(1+v)t_0^2;$$

assume also that, for $t\in[0, t_0]$, we have

$$x_1(t)=\int_0^t \cos\left[\pi/2+\int_0^s u(\tau)\,d\tau\right]ds,$$

$$y_1(t)=d+v\int_0^t \cos\left[\pi-\arcsin(v^{-1})+\int_0^s w(\tau)\,d\tau\right]ds.$$

Then,

$$y_1(t_0) < x_1(t_0).$$

Proof. Let

$$f(t) = \sqrt{v^2-1}\,t - (a/2)(1+v)t^2.$$

Then,

$$f(t) > 0, \qquad t \in (0, 2\sqrt{v-1}/a\sqrt{v+1}).$$

Hence,

$$d = f(t_0) > 0.$$

From the inequality $b < a$, and since

$$\sin[\pi - \arcsin(v^{-1})] = v^{-1},$$

it results that

$$\begin{aligned} y_1(t) - x_1(t) &= y_1(0) - x_1(0) + t[y_1'(0) - x_1'(0)] \\ &\quad + \int_0^t \left\{ \int_0^s [y_1''(\tau) - x_1''(\tau)]\, d\tau \right\} ds \\ &\leq d + tv\cos[\pi - \arcsin(v^{-1})] + (a/2)(a+vb)t^2 \\ &= d - \sqrt{v^2-1}\,t + (a/2)(a+vb)t^2 < d - f(t). \end{aligned}$$

Therefore, for $t = t_0$, we have

$$y_1(t_0) - x_1(t_0) < d - f(t_0) = 0.$$

(b) Let now $v \leq 1$ and $a < bw$.

Lemma 3.4. Assume that the function $u: [0, \infty) \to [-a, a]$ is measurable and, for

$$t \in [0, \sqrt{b^2v^2 - a^2}/a\sqrt{b^2 - a^2}),$$

we have

$$w(t) = \left\{ -u(t)\cos\left[\left[\int_0^t u(s)\, ds\right]\right]\right\} \Big/ \sqrt{v^2 - \sin^2\left[\left[\int_0^t u(s)\, ds\right]\right]}.$$

Then, w projects $[0, \sqrt{b^2v^2 - a^2}/a\sqrt{b^2 - a^2})$ into $[-b, b]$ and it is measurable.

Lemma 3.5. Assume that the functions u and w are the same as in Lemma 3.4 and, for

$$t \in [0, \sqrt{b^2v^2 - a^2}/a\sqrt{b^2 - a^2}],$$

we have

$$x_2(t) = \int_0^t \sin\left[\int_0^s u(\tau)\, d\tau\right] ds,$$

$$y_2(t) = v \int_0^t \sin\left[\pi + \int_0^s w(\tau)\, d\tau\right] ds.$$

Then,

$$x_2(t) = y_2(t), \qquad t \in [0, \sqrt{b^2v^2 - a^2}/a\sqrt{b^2 - a^2}].$$

Proof. As before, it is sufficient to show that

$$x_2'(t) = y_2'(t).$$

Similarly as for the case $v > 1$, it is now possible to choose

$$d > 0, \qquad t_0 \in [0, \sqrt{b^2v^2 - a^2}/a\sqrt{b^2 - a^2}],$$

such that, if

$$x_1(t) = \int_0^t \cos\left[\int_0^s u(\tau)\, d\tau\right] ds,$$

$$y_1(t) = d + v \int_0^t \cos\left[\pi + \int_0^s w(\tau)\, d\tau\right] ds,$$

then

$$y_1(t_0) < x_1(t_0).$$

Thus, it now follows that, for $v \leq 1$ and $a < bv$, the pursuer can catch the evader also.

(c) In the case where $v = 1$ and $b = a$, it is sufficient to consider the same initial condition and the same argument for

$$w(t) = -u(t), \qquad t \in [0, \infty),$$

as in case (b).

In cases (a), (b), (c), there exists an initial condition (x, y), $(x_1, x_2) \neq (y_1, y_2)$, for which the pursuer wins. This can be obtained easily from Lemma 1.1 and the above considerations.

4. Sufficiency

Let us assume that $v < 1$ and $bv \leq a$. It is sufficient to prove the existence of the evasion strategy for the evader for $bv = a$ only. If $bv < a$, it is enough to use the strategy found for $a^* = bv$.

Let us fix arbitrarily $\alpha, \beta \in [0, \pi/2]$; assume that

$$\sin \alpha = v \sin \beta.$$

Lemma 4.1. The following conditions are satisfied:

$$0 < \arcsin v < \pi/2, \qquad \alpha \in [0, \arcsin v].$$

We introduce the notation

$$T = a^{-1}(\pi/2 - \alpha).$$

Lemma 4.2. The following inequality is true:

$$0 < a^{-1}(\pi/2 - \arcsin v) \leq T.$$

Lemma 4.3. We have that $\alpha \leq v\beta$ and, if $\beta > 0$, then $0 < \alpha < v\beta$.

Proof. Observe that

$$\sin \alpha = v \sin \beta, \qquad v \sin \beta \leq \sin(v\beta).$$

Therefore,

$$\alpha \leq v\beta.$$

If $\beta > 0$, then

$$\sin \alpha = v \sin \beta < \sin(v\beta);$$

and, because $v > 0$, we have

$$0 < \alpha < v\beta.$$

Lemma 4.4. Assume that, for $t \geq 0$,

$$x_2(t) = \int_0^t \sin(\alpha + as)\, ds,$$

$$y_2(t) = v \int_0^t \sin\left[\beta + \int_0^s w(\tau)\, d\tau\right] ds,$$

where $w: [0, \infty) \to [-b, b]$ is an arbitrary measurable function. Then, for $t \in (0, T]$, the following inequality is satisfied:

$$x_2(t) > y_2(t).$$

Proof. Let

$$t_1 = b^{-1}(\pi/2 - \beta).$$

Obviously, $t_1 \geq 0$. Because of Lemma 4.3, $\alpha \leq v\beta$; therefore,

$$\alpha + at_1 \leq v(\beta + bt_1) = v\pi/2 < \pi/2.$$

Thus, it can happen that

$$t_1 < T.$$

If $t_1 > 0$, then, for $t \in [0, t_1]$, we obtain that

$$\sin\left[\beta + \int_0^t w(s)\, ds\right] \leq \sin(\beta + bt).$$

The inequality

$$\alpha + at < \beta + bt$$

is satisfied for all $t > 0$. Observe that the cosine function is decreasing at $[0, \pi/2]$. Therefore, for $t \in (0, t_1]$, we have

$$\begin{aligned}[\sin(\alpha + at)]' &= a \cos(\alpha + at) > vb \cos(\beta + bt)\\ &= [v \sin(\beta + bt)]'.\end{aligned}$$

Thus, for $t \in (0, t_1]$, we obtain that

$$\begin{aligned}x_2'(t) &= \sin(\alpha + at) > v \sin(\beta + bt)\\ &\geq v \sin\left[\beta + \int_0^t w(s)\, ds\right] = y_2'(t).\end{aligned}$$

If we notice that

$$v \sin(\beta + bt_1) = v \sin(\pi/2) = v,$$

then, for any $t \geq 0$, the following must be true:

$$v \sin\left[\beta + \int_0^t w(s)\, ds\right] \leq v \sin(\beta + bt_1).$$

From the inequality

$$\sin(\alpha + at) > v \sin(\beta + bt), \qquad t \in (0, t_1],$$

it follows that

$$\sin(\alpha + at_1) > v \sin(\beta + bt_1).$$

Observe that the sine function is increasing at $[0, \pi/2]$. Therefore, for $t \in (t_1, T]$,

$$\sin(\alpha + at) > \sin(\alpha + at_1).$$

From the above, it follows that also, for $t \in (t_1, T]$,

$$x_2'(t) = \sin(\alpha + at) > \sin(\alpha + at_1)$$
$$> v \sin(\beta + bt_1) \geq v \sin\left[\beta + \int_0^t w(s)\, ds\right] = y_2'(t).$$

This ends the proof of the inequality

$$x_2(t) > y_2(t), \qquad t \in (0, T],$$

for $t_1 > 0$.

In the case where $t_1 = 0$, we use the fact that, for $t \in (0, T]$,

$$\sin(\alpha + at) > \sin(\alpha + at_1) = v \sin(\beta + bt_1)$$
$$= v \geq v \sin\left[\beta + \int_0^t w(s)\, ds\right].$$

Lemma 4.5. Assume that $w: [0, \infty) \to [-b, b]$ is a measurable function. Then, for $t \in [0, b^{-1}\pi/2]$, the following inequality is fulfilled:

$$\sin\left[\beta + \int_0^t w(s)\, ds\right] \geq \sin(\beta - bt).$$

Lemma 4.6. Assume that, for $t \geq 0$,

$$x_2(t) = \int_0^t \sin(\alpha - as)\, ds,$$

$$y_2(t) = v \int_0^t \sin\left[\beta + \int_0^s w(\tau)\, d\tau\right] ds,$$

where $w: [0, \infty[-b, b]$ is an arbitrary measurable function. Then, for $t \in (0, b^{-1}\pi/2]$, the following inequality is fulfilled:

$$x_2(t) < y_2(t).$$

Proof. It is sufficient to prove that, for $t \in (0, b^{-1}\pi/2]$,

$$x_2'(t) < y_2'(t).$$

Because of Lemma 4.5, it is enough to show that

$$\sin(\alpha - at) < v \sin(\beta - bt), \qquad t \in (0, b^{-1}\pi/2].$$

Let

$$f(t) = \sin(\alpha - at), \quad g(t) = v \sin(\beta - bt), \qquad t \geq 0.$$

Therefore, for $t \geq 0$, we obtain

$$\begin{aligned} f'(t) &= -vb\cos(|\alpha - vbt|), \\ g'(t) &= -vb\cos(|\beta - bt|). \end{aligned} \tag{5}$$

Now, assume that $\beta > 0$. Thus, from Lemma 4.3, we obtain that $0 < \alpha < v\beta$. Let

$$\tilde{t} = \alpha/vb, \qquad t_1 = (\beta + \alpha)/b(1+v), \qquad t_2 = (\beta - \alpha)/b(1-v).$$

Therefore, we have $t_2 > t_1 > \tilde{t}$, and also

$$|\alpha - vbt_1| = |\beta - bt_1|, \qquad i = 1, 2.$$

We can see that

$$\begin{aligned} |\alpha - vbt| &< |\beta - bt|, \qquad t \in [0, t_1) \cup (t_2, \infty), \\ |\alpha - vbt| &> |\beta - bt|, \qquad t \in (t_1, t_2). \end{aligned} \tag{6}$$

Moreover,

$$t_1 = (\beta + \alpha)/b(1+v) < (\beta + v\beta)/b(1+v) = b^{-1}\beta \leq b^{-1}\pi/2.$$

If

$$b^{-1}\pi/2 \leq t_2,$$

then, from (5) and (6), it results that

$$f'(t) < g'(t), \qquad t \in (0, t_1);$$

and, from

$$f(0) = g(0),$$

we also get

$$f(t) < g(t), \qquad t \in (0, t_1].$$

From (5) and (6), it follows that, for

$$t \in (t_1, b^{-1}\pi/2),$$

the following inequality is satisfied:

$$f'(t) > g'(t).$$

In addition (see proof of Lemma 4.3),

$$f(b^{-1}\pi/2) < \sin(v\beta - v\pi/2) \leq v\sin(\beta - \pi/2) = g(b^{-1}\pi/2).$$

Therefore,

$$f(t) < g(t), \qquad t \in (t_1, b^{-1}\pi/2].$$

For

$$t_2 < b^{-1}\pi/2,$$

using an analogous argument, it is sufficient to prove that

$$f(t_2) < g(t_2);$$

this is obvious because

$$-\pi/2 \leq \alpha - v\beta < 0;$$

therefore,

$$f(t_2) = \sin[(\alpha - v\beta)/(1-v)] < v \sin[(\alpha - v\beta)/(1-v)] = g(t_2).$$

Thus, in the case where $\beta > 0$, our lemma is proven. If $\beta = 0$, then $\alpha = 0$; and, for

$$t \in (0, b^{-1}\pi),$$

we have

$$f(t) = \sin(\alpha - vbt) < v \sin(\beta - bt) = g(t).$$

Thus,

$$f(t) < g(t), \qquad t \in (0, b^{-1}\pi/2],$$

and this ends the proof of Lemma 4.6.

Lemma 4.7. Let

$$\tilde{\alpha} \in [\alpha, \pi/2].$$

Assume that, for

$$t \in [0, a^{-1}(\pi/2 - \arcsin v)/2],$$

we have

$$\tilde{x}_2(t) = \int_0^t \sin(\tilde{\alpha} + as)\, ds,$$

and the function $x_2(t)$ is the same as in Lemma 4.4. Then,

$$\tilde{x}_2(t) \geq x_2(t).$$

Proof. From Lemma 4.1, it follows that, for

$$t \in [0, a^{-1}(\pi/2 - \arcsin v)/2],$$

we have

$$\tilde{x}_2'(t) \geq x_2'(t).$$

Lemma 4.8. Let

$$\tilde{\alpha} \in [0, \pi/2], \qquad \sin \tilde{\alpha} \geq v \sin \beta.$$

Assume that, for $t \geq 0$,

$$\tilde{x}_2(t) = \int_0^t \sin(\tilde{\alpha} + as)\, ds,$$

and the function $y_2(t)$ is the same as in Lemma 4.4. Then, for

$$t \in (0, a^{-1}(\pi/2 - \arcsin v)/2],$$

the following inequality is satisfied:

$$\tilde{x}_2(t) > y_2(t).$$

Proof. Take the function $x_2(t)$, considered in Lemma 4.4. From Lemmas 4.7, 4.4, 4.2, it follows that, for

$$t \in [0, a^{-1}(\pi/2 - \arcsin v)/2,$$

we have

$$\tilde{x}_2(t) \geq x_2(t) > y_2(t).$$

Lemma 4.9. Let

$$\tilde{\beta} \in [-\pi/2, 0].$$

For

$$t \in [0, b^{-1}\pi/4],$$

assume that

$$y_2(t) = v \int_0^t \sin\left[\beta + \int_0^s w(\tau)\, d\tau\right] ds,$$

$$\tilde{y}_2(t) = v \int_0^t \sin\left[\tilde{\beta} + \int_0^s w(\tau)\, d\tau\right] ds,$$

where w: $[0, \infty) \to [-b, b]$ is an arbitrary measurable function. Then,

$$\tilde{y}_2(t) \leq y_2(t).$$

Proof. It can be shown that

$$y_2'(t) - \tilde{y}_2'(t) \geq 0, \qquad t \in [0, b^{-1}\pi/4].$$

Conclusion 4.1. Let

$$\tilde{\alpha} \in [0, \pi/2], \qquad \tilde{\beta} \in [-\pi/2, \pi/2], \qquad \sin \tilde{\alpha} \geq v \sin \tilde{\beta}.$$

For $t \geq 0$, assume that

$$\tilde{x}_2(t) = \int_0^t \sin(\tilde{\alpha} + as)\, ds,$$

$$\tilde{y}_2(t) = v \int_0^t \sin\left[\tilde{\beta} + \int_0^s w(\tau)\, d\tau\right] ds,$$

where $w: [0, \infty[-b, b]$ is an arbitrary measurable function. Then, for

$$t \in (0, \min\{a^{-1}(\pi/2 - \arcsin v)/2, b^{-1}\pi/4\}],$$

the following inequality is fulfilled:

$$\tilde{x}_2(t) > \tilde{y}_2(t).$$

Proof. Let us consider

$$\beta^* = \max\{\tilde{\beta}, 0\}, \qquad \alpha^* = \arcsin(v \sin \beta^*).$$

Let us also note that values of α and β, with $\alpha, \beta \in [0, \pi/2]$, satisfying the equality

$$\sin \alpha = v \sin \beta,$$

have been chosen quite freely. In particular, we can assume that

$$\alpha = \alpha^*, \qquad \beta = \beta^*.$$

If $\tilde{\beta} \geq 0$, then

$$\sin \tilde{\alpha} \geq v \sin \tilde{\beta} = v \sin \beta^*;$$

and, from Lemma 4.8 and with

$$\beta = \beta^* = \tilde{\beta},$$

we obtain

$$\tilde{x}_2(t) > \tilde{y}_2(t), \qquad t \in (0, a^{-1}(\pi/2 - \arcsin v)/2].$$

If $\tilde{\beta} < 0$, then

$$\sin \tilde{\alpha} \geq 0 = v \sin 0 = v \sin \beta^*.$$

Therefore, from Lemma 4.8 and for $\beta=\beta^*$, and if

$$y_2^*(t)=v\int_0^t \sin\left[\beta^*+\int_0^s w(\tau)\,d\tau\right]ds,$$

then

$$\tilde{x}_2(t)>y_2^*(t), \qquad t\in(0, a^{-1}(\pi/2-\arcsin v)/2].$$

Because of Lemma 4.9, for $\beta=\beta^*$ as well,

$$y_2^*(t)\geqslant \tilde{y}_2(t), \qquad t\in[0, b^{-1}\pi/4].$$

Therefore, for

$$t\in(0, \min\{a^{-1}(\pi/2-\arcsin v)/2, b^{-1}\pi/4\}],$$

we have

$$\tilde{x}_2(t)>y_2^*(t)\geqslant\tilde{y}_2(t).$$

Using Lemma 4.6 instead of Lemma 4.4 and applying an argument analogous to that presented above, we obtain the following conclusion.

Conclusion 4.2. Let

$$\tilde{\alpha}\in[0,\pi/2], \qquad \tilde{\beta}\in[0,\pi/2], \qquad \sin\tilde{\alpha}<v\sin\tilde{\beta}.$$

For $t\geqslant 0$, assume that

$$\tilde{x}_2(t)=\int_0^t \sin(\tilde{\alpha}-as)\,ds,$$

$$\tilde{y}_2(t)=v\int_0^t \sin\left[\tilde{\beta}+\int_0^s w(\tau)\,d\tau\right]ds,$$

where $w:[0,\infty)\to[-b,b]$ is an arbitrary measurable function. Then, for

$$t\in(0, \min\{a^{-1}(\pi/2-\arcsin v)/2, b^{-1}\pi/4\}],$$

the following inequality is fulfilled:

$$\tilde{x}_2(t)<\tilde{y}_2(t).$$

Conclusion 4.3. Let

$$\tilde{\alpha}\in[0,\pi/2], \qquad \tilde{\beta}\in[\pi/2,\pi], \qquad \sin\tilde{\alpha}\geqslant v\sin\tilde{\beta}.$$

For $t \geq 0$, assume that

$$\tilde{x}_2(t) = \int_0^t \sin(\tilde{\alpha} + as)\, ds,$$

$$\tilde{y}_2(t) = v \int_0^t \sin\left[\tilde{\beta} + \int_0^s w(\tau)\, d\tau\right] ds,$$

where $w: [0, \infty) \to [-b, b]$ is an arbitrary measurable function. Then, for

$$t \in (0, \min\{a^{-1}(\pi/2 - \arcsin v)/2, b^{-1}\pi/4\}],$$

the following inequality is fulfilled:

$$\tilde{x}_2(t) > \tilde{y}_2(t).$$

Proof. For $t \geq 0$,

$$\tilde{y}_2(t) = v \int_0^t \sin\left[\pi - \tilde{\beta} + \int_0^s (-w(\tau))\, d\tau\right] ds;$$

and, because

$$\beta^* = \pi - \tilde{\beta} \in [0, \pi/2],$$

it is sufficient to use Conclusion 4.1.

Similarly, considering the remaining cases, we obtain that there is $\delta > 0$ such that, for $\tilde{\alpha}, \tilde{\beta} \in [0, 2\pi]$ and $d > 0$, and for the initial situation of the game,

$$((0, 0, \tilde{\alpha}), (d, 0, \tilde{\beta})),$$

the evader can escape the pursuer at least for $t \in [0, \delta]$. Thus, there exists a function e and a sequence $t_n = n\delta$, $n \in \mathbb{N}$, such that the strategy $(e, \{t_n\})$ allows the evader to win for any situation.

Similarly, when $v = 1$ and $b < a$.

5. Remarks

(a) In Section 3, it was shown that the condition (3) is sufficient. By analogous methods, we can prove that there exist $R, T > 0$ such that, for any initial condition, the evader not only wins, but also, from the time T, he keeps his distance from the pursuer at not less than R.

(b) The known sufficient conditions for the existence of the evasion strategy presented in Refs. 3 and 4 cannot be applied to the game described above.

(c) The problem described in this paper has been partially solved in Ref. 5.

(d) In Ref. 6, similar necessary and sufficient conditions for the pursuer to effect capture starting from any initial state are presented. From point (a) of Section 2 of our paper and Theorem 2 of Ref. 6, it follows that, if $v_E < v_P$ and $bv_P < av_E$, then there exist initial conditions for which the pursuer wins and there exist initial conditions for which the evader can avoid capture.

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