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Theorems of the Alternative and Optimality Conditions

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Abstract. A theorem of the alternative is stated for generalized systems. It is shown how to deduce, from such a theorem, known optimality conditions like saddle-point conditions, regularity conditions, known theorems of the alternative, and new ones. Exterior and interior penalty approaches, weak and strong duality are viewed as weak and strong alternative, respectively.

Key Words. Theorems of the alternative, optimality conditions, regularity conditions, constraints qualification, separation theorems, Lagrangian functions, penalty functions, duality.

1. General Setting for a Theorem of the Alternative

Assume that we are given the positive integers n and ν, the nonempty sets $\mathcal{H} \subset \mathbb{R}^\nu$, $X \subseteq \mathbb{R}^n$, $Z \subset \mathbb{R}$, and the real-valued function $F: X \to \mathbb{R}^\nu$.

Definition 1.1. $w: \mathbb{R}^\nu \to \mathbb{R}$ is called the *weak separation function*, iff

$$\mathcal{H}^w \triangleq \{h \in \mathbb{R}^\nu: w(h) \notin Z\} \supseteq \mathcal{H}; \tag{1a}$$

$S: \mathbb{R}^\nu \to \mathbb{R}$ is called the *strong separation function*, iff

$$\mathcal{H}^s \triangleq \{h \in \mathbb{R}^\nu: s(h) \notin Z\} \subseteq \mathcal{H}. \tag{1b}$$

We want to study conditions for the generalized system

$$F(x) \in \mathcal{H}, \qquad x \in X, \tag{2}$$

to have (or not to have) solutions. We can prove the following theorem.

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Theorem 1.1. Let the sets $\mathscr{H}$, X, Z and the function F be given.

(i) The systems (2) and (3a),

$$w(F(x)) \in Z, \qquad \forall \in X, \tag{3a}$$

are not simultaneously possible, whatever the weak separation function w might be.

(ii) The systems (2) and (3b),

$$s(F(x)) \in Z, \qquad \forall x \in X, \tag{3b}$$

are not simultaneously impossible, whatever the strong separation function s might be.

Proof. (i) If (2) is possible, i.e., if $\exists \bar{x} \in X$ such that

$$\bar{h} \triangleq F(\bar{x}) \in \mathscr{H},$$

then (1a) implies

$$w(F(\bar{x})) = w(\bar{h}) \notin Z,$$

so that (3a) is false.

(ii) If (2) is impossible, i.e., if

$$h \triangleq F(x) \notin \mathscr{H}, \qquad \forall x \in X,$$

then(1b) implies

$$s(F(x)) = s(h) \in Z, \qquad \forall x \in X,$$

so that (3b) is true. This completes the proof. □

Denote by X^* the set of solutions of (2), and introduce the sets

$$X^w = \{x \in X: w(F(x)) \in Z\},$$

$$X^s = \{x \in X: s(F(x)) \in Z\}.$$

In Theorem 1.1, (i) can be written as

$$X^* \cap X^w = \varnothing$$

and (ii) as

$$X^* \cup X^s = X.$$

These two cases show *weak alternative* between (2) and (3a) and *strong alternative* between (2) and (3b), respectively. If, within a family of weak (strong) separation functions, it is possible to guarantee also

$$X^w = X, \qquad \text{when } X^* = \varnothing,$$

so that

$$X^* \cup X^w = X$$

holds (or

$$X^* = \varnothing, \qquad \text{when } X^s = X,$$

so that

$$X^* \cap X^s = \varnothing$$

holds), then we say that *alternative* holds between (2) and (3a), or (3b). To deepen this crucial aspect, it is useful to introduce the set

$$\mathcal{K} \triangleq \{h \in \mathbb{R}^\nu : h = F(x);\, x \in X\},$$

and note that (2) is impossible iff

$$\mathcal{H} \cap \mathcal{K} = \varnothing.$$

In this order of ideas, an important question consists in finding conditions on F and X under which a weak (strong) separation function guarantees alternative besides the weak (strong) one. In Section 3, this question will be analyzed for a wide class of systems. Now, let us show, by means of an important instance, how Theorem 1.1 can be used as a source for deriving theorems of the alternative or separation (even if they are not in the usual form); this is also a case where a weak separation function guarantees alternative.

Corollary 1.1. Let $C_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, p+q$, be convex sets, the first p, $p \geq 1$, of which are open. We have

$$\bigcap_{i=1}^{p+q} C_i = \varnothing \tag{4}$$

iff[2] there exist

$$\bar{\lambda}^i \in D_i \triangleq \{\lambda^i \in \mathbb{R}^n : \delta^*(\lambda^i; C_i) < +\infty\}, \qquad i=1, \ldots, p+q,$$

such that, for at least an index $i = 1, \ldots, p+q$, we have

$$\bar{\lambda}^i \neq 0, \qquad \langle \bar{\lambda}^i, h^i \rangle < \delta^*(\bar{\lambda}^i; C_i), \qquad \forall h^i \in C_i, \tag{5a}$$

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[2] Throughout the paper, $\langle \cdot , \cdot \rangle$ denotes scalar product; $\text{lev}_{\gtreqless \alpha} f$ denote the various level sets of f; $\delta^*(\lambda; C) \triangleq \sup_{y \in C} \langle \lambda, y \rangle$ is the support function of C at λ; $C^* \triangleq \{z : \langle y, z \rangle \geq 0, \forall y \in C\}$ is the polar of the convex cone C; and cl A, int A, ri A, frt A, $\sim A$, dim A denote closure, interior, relative interior, frontier, complement, dimension of the set A, respectively. 0 denotes both the origin of a space and the null vector; $\times$ marks Cartesian product.

and such that

$$\sum_{i=1}^{p+q} \bar{\lambda}^i = 0, \qquad \sum_{i=1}^{p+q} \delta^*(\bar{\lambda}^i; C_i) \leq 0. \tag{5b}$$

Proof. Set

$$\nu = (p+q)n, \qquad \mathcal{H} = C_1 \times \cdots \times C_{p+q}, \qquad X = \mathbb{R}^n,$$

$$Z =]-\infty, 0], \qquad F(x) = (x_1, \ldots, x_n, \ldots, x_1, \ldots, x_n).$$

$\mathcal{K}$ is now a particular linear manifold[3]. In place of w of Theorem 1.1, consider the function

$$w_1(h; \lambda) = \sum_{i=1}^{p+q} [-\langle \lambda^i, h^i \rangle + \delta^*(\lambda^i; C_i)],$$

where

$$h = (h^1, \ldots, h^{p+q}), \qquad \lambda = (\lambda^1, \ldots, \lambda^{p+q}).$$

Here, $\mathcal{H}^w$ is an open halfspace, which contains $\mathcal{H}$ iff

$$\lambda^i \in D_i, \qquad i = 1, \ldots, p+q,$$

and moreover, for at least an index $i = 1, \ldots, p+q$, we have

$$\lambda^i \neq 0, \qquad \langle \lambda^i, h^i \rangle < \delta^*(\lambda^i; C_i), \qquad \forall h^i \in C_i;$$

in the particular case where $q = 1$, the last condition can be replaced with $\lambda \neq 0$ only; in fact, such a condition is implied by $\lambda = 0$ and

$$\sim \mathcal{H}^w \supseteq \mathcal{K}.$$

Hence, w_1 fulfills (1a), and (i) of Theorem 1.1 can be applied. The convexity of $\mathcal{H}$ and $\mathcal{K}$ ensures alternative besides the weak one. Thus, (4), which is equivalent to the impossibility of (2) in the present case, holds iff there exist

$$\bar{\lambda}^i \in D_i, \qquad i = 1, \ldots, p+q,$$

which fulfill (5) and such that

$$\sum_{i=1}^{p+q} [-\langle \bar{\lambda}^i, x \rangle + \delta^*(\bar{\lambda}^i; C_i)] \leq 0, \qquad \forall x \in \mathbb{R}^n.$$

This inequality holds iff condition (5b) is consistent. This completes the proof. □

[3] Thus, the separation of $\mathcal{H}$ and $\mathcal{K}$ falls into the Hahn–Banach theorem.

In the particular case where every C_i is a cone with vertex at the origin, in Corollary 1.1 D_i can be replaced by $-C_i^*$ (as $D_i = -C_i^*$) and the inequality in (5b) is redundant. Corollary 1.1, with $q = 1$, is due to Dubovitskii and Milyutin (Ref. 1) and has been used to prove a general necessary condition for both finite and infinite-dimensional extremum problems.

Note that the separation of several sets, in the sense of Ref. 2, can be reduced to the separation of only two sets, by means of the device, which has been adopted in the proof of Corollary 1.1. In fact, the sets C_i are *separated* (Ref. 2), iff there exist linear functionals l_i and scalars α_i, satisfying the condition

$$\sum_i l_i \equiv 0, \qquad \sum_i \alpha_i \leq 0,$$

which is condition (5b), $\bar{\lambda}^i$ being the gradient of l_i, and such that

$$C_i \subseteq \{x \in \mathbb{R}^n: l_i(x) \leq \alpha_i\}.$$

Hence, Corollary 1.1 expresses a condition for separation of several sets.

Now, let us view as weak separation functions some well-known functions introduced in various contexts to handle optimization problems. With this aim, consider the following particular case:

$$\begin{aligned} &\nu = l + m, \qquad \mathscr{H} = \{(u, v) \in \mathbb{R}^l \times \mathbb{R}^m: u > 0; v \geq b\}, \qquad Z =]-\infty, 0], \\ &f: X \to R^l, \qquad g: X \to R^m, \qquad F(x) = (f(x), g(x) - b), \qquad h = (u, v - b), \end{aligned} \tag{6}$$

where the positive integers l and m, the m-vector b, and the functions f and g are given. Consider the function

$$w_2(u, v; \theta; \omega) \triangleq \langle \theta, u \rangle + \gamma_2(v; \omega) - \gamma_2(b; \omega), \qquad \theta \in \mathbb{R}_+^l, \omega \in \Omega,$$

where γ_2 is a nondecreasing function of v, $\forall \omega \in \Omega$, Ω being the domain of the parameter ω. If either $\theta = 0$ and γ_2 is increasing or $\theta \neq 0$, then w_2 satisfies (1a), and hence guarantees the weak alternative. The particular case of γ_2 affine is of special interest:

$$\gamma_2(v, \omega) = \langle \omega, v \rangle, \qquad \omega \in \Omega = \mathbb{R}_+^m.$$

We will see that most of the functions used in an optimization context are of the w_2-kind (Ref. 3).

Finally, let us note that Theorem 1.1 generalizes Theorem 1 of Ref. 4.

2. Regularity

In order to deepen the analysis, we have to introduce some further concepts. This will be done in the case where[4]

$\mathcal{H}$ is a convex cone with vertex at the origin $0 \notin \mathcal{H}$. (7)

Consider the sets

$$\mathcal{E}(x) \triangleq \{h \in \mathbb{R}^\nu: F(x) - h \in \mathrm{cl}\,\mathcal{H}\}, x \in X; \qquad \mathcal{E} \triangleq \bigcup_{x \in X} \mathcal{E}(x).$$

The former is the convex cone with vertex at $F(x)$, obtained by translating $-\mathrm{cl}\,\mathcal{H}$; the latter is called the *conic extension* of $\mathcal{H}$ in respect of $-\mathrm{cl}\,\mathcal{H}$. $\mathcal{E}$ is a key set in further analysis, and hence its properties are important. First of all, note that $\mathcal{E}$ is not necessarily closed, as simple examples show. It is useful to introduce the tangent cone,[5] say $T(\bar{h})$, of $\mathcal{E}$ at $\bar{h} \in \mathrm{cl}\,\mathcal{E}$, and set

$$\mathcal{E}^0 \triangleq (\mathrm{cl}\,\mathcal{E}) \cap (\mathrm{cl}\,\mathcal{H}).$$

Now, we can get further insight about the weak alternative. When (2) is impossible, (3a) is not necessarily possible; however, something can be stated. An instance is offered by the following property.

Lemma 2.1. Let C be a face[6] of cl $\mathcal{H}$. If (2) is impossible and $\mathcal{E}^0 \neq \emptyset$, then we have

$$T(h) \cap \mathrm{int}\,\mathcal{H} = \emptyset, \qquad \forall h \in \mathcal{E}^\circ, \tag{8a}$$

$$0 \in \mathcal{E}^\circ \subseteq \mathrm{cl}\,\mathcal{H}^w, \tag{8b}$$

$$\mathcal{E}^\circ \cap \mathrm{ri}\, C \neq \emptyset \Rightarrow C \subseteq T(0), \tag{8c}$$

$$T(0) \cap \mathrm{ri}\, C \neq \emptyset \Rightarrow C \subseteq T(0). \tag{8d}$$

Proof. Assume that the cardinality of $T(h)$ is >1, otherwise (8a) is trivial. *Ab absurdo*, suppose that (8a) is false, so that $\exists \bar{h} \in \mathcal{E}^\circ$ such that

$$T(\bar{h}) \cap \mathrm{int}\,\mathcal{H} \neq \emptyset.$$

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[4] Most applications, even if not all, can be reduced to (7).

[5] The tangent cone is defined as the set of $\bar{h} + h$ for which there exist a sequence $\{h^r\} \subseteq \mathcal{E}$ such that

$$\lim_{r \to +\infty} h^r = \bar{h},$$

and a positive sequence $\{\alpha_r\} \subset \mathbb{R}$, such that

$$\lim_{r \to +\infty} \alpha_r(h^r - \bar{h}) = h$$

When the tangent cone reduces to a singleton, condition (8a) becomes meaningless. A more general definition of tangent cone can be achieved by replacing lim with a more general concept.

[6] Defined as the intersection between cl $\mathcal{H}$ and a supporting hyperplane for it.

Let

$$\tilde{h} \in T(\bar{h}) \cap \text{int } \mathcal{H}.$$

Then, there exist a sequence $\{h^r\} \subseteq \mathscr{E}$ and a positive sequence $\{\alpha_r\} \subset \mathbb{R}$, such that

$$\lim_{r \to +\infty} h^r = \bar{h}, \qquad \lim_{r \to +\infty} \alpha_r(h^r - \bar{h}) = \tilde{h} - \bar{h}. \tag{9}$$

Set

$$\bar{h}^r \triangleq \bar{h} + \alpha_r(h^r - \bar{h}).$$

The second part of (9) implies

$$\lim_{r \to +\infty} \bar{h}^r = \tilde{h},$$

so that $\exists r'$ such that

$$\bar{h}^r \in \text{int } \mathcal{H}, \qquad \forall r \geq r'.$$

From (9), we get now

$$\lim_{r \to +\infty} [(1/\alpha_r)(\bar{h}^r - \bar{h})] = \lim_{r \to +\infty} (h^r - \bar{h}) = 0;$$

hence, as

$$\lim_{r \to +\infty} (\bar{h}^r - \bar{h}) = \tilde{h} - \bar{h} \neq 0,$$

as the impossibility of (2) implies $\mathscr{E} \cap \mathcal{H} = \varnothing$ and

$$\mathscr{E} \cap \mathcal{H} = \varnothing \Rightarrow \mathscr{E}^\circ \cap \text{int } \mathcal{H} = \varnothing,$$

we deduce that

$$\lim_{r \to +\infty} \alpha_r = +\infty,$$

so that $\exists r''$ such that

$$h^r \in]\bar{h}, \bar{h}^r[, \qquad \forall r \geq r''.$$

Then, for all

$$r \geq \max\{r', r''\},$$

the convexity of $\mathcal{H}$, $\bar{h} \in \text{cl } \mathcal{H}$, and $\bar{h}^r \in \text{int } \mathcal{H}$ imply

$$h^r \in \text{int } \mathcal{H},$$

and hence

$$\mathscr{E} \cap \text{int } \mathcal{H} \neq 0.$$

This contradicts the impossibility of (2), and (8a) follows. To prove the first part of (8b), it is enough to show that $0 \in \text{cl}\, \mathscr{E}$ (as obviously $0 \in \text{cl}\, \mathscr{H}$). Let $\bar{h} \in \mathscr{E}^\circ$, so that $\bar{h} \in \text{frt}\, \mathscr{E}$, as the impossibility of (2) implies

$$\mathscr{E}^\circ = (\text{frt}\, \mathscr{E}) \cap (\text{frt}\, \mathscr{H}).$$

Then, there exists a sequence $\{h^r\} \subseteq \mathscr{E}$ which $\to \bar{h}$. Hence, there exists a sequence $\{x^r\} \subseteq X$ such that

$$h^r \in \mathscr{E}(x^r).$$

We have[7]

$$d(F(x^r), \text{frt}\, \mathscr{H}) \to 0;$$

thus,

$$d(F(x^r) - 0, \text{cl}\, \mathscr{H}) \to 0 \Rightarrow d(\mathscr{E}(x^r), 0) \to 0 \Rightarrow 0 \in \text{cl}\, \mathscr{E}.$$

To prove the second part of (8b), let $\bar{h} \in \text{cl}\, \mathscr{H}$, so that, $\forall \epsilon > 0$, $\exists \hat{h} \in \mathscr{H}$ such that

$$\|\hat{h} - \bar{h}\| < \varepsilon.$$

Then, (1a) implies

$$\hat{h} \in \mathscr{H}^w.$$

As ε is arbitrary,

$$\bar{h} \in \text{cl}\, \mathscr{H}^w.$$

Hence,

$$\text{cl}\, \mathscr{H} \subseteq \text{cl}\, \mathscr{H}^w$$

and, as obviously

$$\mathscr{E}^\circ \subseteq \text{cl}\, \mathscr{H},$$

the second part of (8b) follows. Now, let

$$\bar{h} \in \mathscr{E}^\circ \cap \text{ri}\, C,$$

and consider the translation of $-C$ with vertex at $\bar{h}$, namely,

$$\bar{C} \triangleq \{\bar{h}\} - C.$$

We have that

$$\bar{C} \subseteq \text{cl}\, \mathscr{E}$$

[7] $d(\cdot\,,\cdot)$ denotes the Euclidean distance.

and that $\bar{h} \in \text{ri}\, C$ implies

$$\{\beta h: h \in C \cap \bar{C};\, \beta > 0\} = C.$$

The proof of (8c) is now trivial. Finally, let

$$\tilde{h} \in T(0) \cap \text{ri}\, C.$$

Then, there exist a sequence $\{h^r\} \subseteq \mathscr{E}$ and a positive sequence $\{\alpha_r\} \subset \mathbb{R}$, such that

$$\lim_{r\to+\infty} h^r = 0, \qquad \lim_{r\to+\infty} \alpha_r h^r = \tilde{h}.$$

Set

$$\bar{h}^r = h^r - (1/\alpha_r)\tilde{h},$$

so that the sequence

$$\{\bar{h}^r\} \subseteq \text{cl}\, \mathscr{E},$$

and we have

$$\lim_{r\to+\infty} \bar{h}^r = 0, \qquad \lim_{r\to+\infty} \alpha_r \bar{h}^r = 0.$$

Consider the set

$$\tilde{C} \triangleq \{\tilde{h}\} - C,$$

which is a translation of $-C$, and

$$S \triangleq C \cap \tilde{C}, \qquad S^r \triangleq (\{\bar{h}^r\} + C) \cap (\{h^r\} - C).$$

Note that

$$\lim_{r\to+\infty} S^r = \{0\}, \qquad \lim_{r\to+\infty} \alpha_r S^r = S, \qquad S^r \subseteq \text{cl}\, \mathscr{E},$$

and that

$$\tilde{h} \in \text{ri}\, C \Rightarrow \{\beta h: h \in S;\, \beta > 0\} = C.$$

The fact that

$$C \subseteq T(0)$$

is now obvious. This completes the proof. □

Theorem 2.1. Let $\mathscr{E}$ be convex, (2) be impossible, and C be a face of cl $\mathscr{H}$. We have $C \subseteq P$, for every hyperplane P which separates $\mathscr{E}$ and $\mathscr{H}$, iff

$$C \subseteq T(0). \tag{10}$$

Proof. *Sufficiency. Ab absurdo,* suppose that a hyperplane, say $\bar{P}$ [it exists as $\mathscr{C}$ and $\mathscr{H}$ are convex and (2) is impossible], which separates $\mathscr{C}$ and $\mathscr{H}$, does not contain C, so that there exists a ray $\rho \subseteq C \subseteq \mathrm{cl}\,\mathscr{H}$ with $\rho \not\subseteq \bar{P}$ [(8b)$\Rightarrow 0 \in \bar{P}$]. As $\mathscr{C}$ is convex, $\bar{P}$ supports $T(0)$ at $h=0$; hence, as $\rho\backslash\{0\}$ and $T(0)$ belong to opposite and disjoint halfspaces (defined by $\bar{P}$), we have $\rho \not\subseteq T(0)$, which contradicts (10).

Necessity. Ab absurdo, suppose that $C \not\subseteq T(0)$; hence,

$$C' \triangleq C \cap T(0)$$

is a subcone of C, such that

$$C\backslash C' \neq \varnothing \quad \text{and} \quad (C\backslash C') \cap T(0) = \varnothing.$$

Hence, by applying (8d), we have that C' must be a face of C.

Consider the set, say $\{P\}$, of hyperplanes, which separate the closed and convex cones $\mathrm{cl}\,\mathscr{H}$ and $T(0)$, and hence $\mathscr{C}$. Set

$$T_P = P \cap T(0);$$

obviously,

$$T_P \supseteq C'.$$

For no sequence $\{h^r\} \subset T(0)$, converging to an element $h \in T_P$, and for no sequence $\{\alpha_r\}$ of positive scalars, the sequence $\{\alpha_r(h^r - h)\}$ can converge to an element of $C\backslash C'$; otherwise, such an element would belong to the tangent cone of $T(0)$, and hence to $T(0)$. It follows that there exists a P' in the above set, such that

$$(C\backslash C') \cap P' = \varnothing,$$

and then the assumption is contradicted. This completes the proof. □

The preceding lemma suggests a definition. Let C be a face of $\mathrm{cl}\,\mathscr{H}$, and let P denote a separating hyperplane for $\mathscr{H}$ and $\mathscr{C}$.

Definition 2.1. When $\mathscr{C}$ is convex and (2) is impossible, we say that (2) is *k-irregular,* iff

$$\{C \subseteq P, \text{ for every } P\} \Rightarrow \dim C \leq k.$$

When $k=0$, we say that (2) is *regular* (or qualified).

3. Cone Functions

Now, we will consider the following important particular case of (7):

$$\nu = l+m, \qquad \mathscr{H} = \{(u, v) \in \mathbb{R}^l \times \mathbb{R}^m : u \in \operatorname{int} U; v \in V\}, \qquad Z =]-\infty, 0],$$
$$f: X \to \mathbb{R}^l, \qquad g: X \to \mathbb{R}^m, \qquad F(x) = (f(x), g(x)), \qquad h = (u, v), \tag{11}$$

where the positive integers l and m, the closed convex cones $U \subset \mathbb{R}^l$, $V \subset \mathbb{R}^m$, with $\operatorname{int} U \neq \varnothing$ (otherwise $\mathscr{H} = \varnothing$), and the functions f, g are given. In this case, it is easy to show that

$$w_3(u, v; \theta, \lambda) \triangleq \langle \theta, u \rangle + \langle \lambda, v \rangle, \qquad \theta \in U^*, \qquad \lambda \in V^*,$$

is a weak separation function. We may show that it guarantees alternative (besides the weak one). First of all, let us prove the following lemma.

Lemma 3.1. If X is convex and F is a $(\operatorname{cl} \mathscr{H})$-function,[8] then $\mathscr{E}$ is convex.

Proof. If the cardinality of $\mathscr{E}$ is ≤ 1, the thesis is trivial. Consider

$$h^i \in \mathscr{E}, \qquad i = 1, 2,$$

so that there exist

$$x^i \in X, \qquad i = 1, 2,$$

such that

$$h^i \in \mathscr{E}(x^i), \qquad i = 1, 2,$$

or

$$F(x^i) - h^i \in \operatorname{cl} \mathscr{H}, \qquad i = 1, 2.$$

Now, $\forall \alpha \in [0, 1]$, set

$$\hat{x} \triangleq (1-\alpha)x^1 + \alpha x^2 \in X, \qquad \hat{h} = (1-\alpha)h^1 + \alpha h^2,$$
$$\hat{F} \triangleq (1-\alpha)F(x^1) + \alpha F(x^2).$$

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[8] Let C be a convex cone with vertex at the origin. F is said to be a C-function on a convex set X, iff

$$F((1-\alpha)x^1 + \alpha x^2) - (1-\alpha)F(x^1) - \alpha F(x^2) \in C, \qquad \forall x^1, x^2 \in X, \forall \alpha \in]0, 1[.$$

Note that a $(\mathbb{R}^n_+)$-function is a concave function and a $(\mathbb{R}^n_-)$-function is a convex function.

The convexity of $\mathcal{H}$ implies

$$\hat{F} - \hat{h} \in \mathrm{cl}\, \mathcal{H};$$

this condition and

$$F(\hat{x}) - \hat{F} \in \mathrm{cl}\, \mathcal{H}$$

[as F is a (cl $\mathcal{H}$)-function] imply

$$F(\hat{x}) - \hat{h} \in \mathrm{cl}\, \mathcal{H}.$$

Hence, $\forall \alpha \in [0, 1]$,

$$\hat{h} \in \mathcal{E},$$

and the convexity of $\mathcal{E}$ follows. This completes the proof. □

We are now ready to prove the announced result.

Theorem 3.1. Let X be convex, f be a U-function, and g be a V-function. The system

$$f(x) \in \mathrm{int}\, U, \qquad g(x) \in V, \qquad x \in X, \tag{12}$$

is impossible, iff there exist $\bar{\theta} \in U^*$ and $\bar{\lambda} \in V^*$, with $(\bar{\theta}, \bar{\lambda}) \neq 0$, such that

$$\langle \bar{\theta}, f(x) \rangle + \langle \bar{\lambda}, g(x) \rangle \leq 0, \qquad \forall x \in X, \tag{13}$$

and moreover

$$\{x \in X: f(x) \in \mathrm{int}\, U;\, g(x) \in V;\, \langle \bar{\lambda}, g(x) \rangle = 0\} = \varnothing, \qquad \text{if } \bar{\theta} = 0.$$

Proof. First of all, let us prove that (12) is impossible iff

$$\mathcal{H} \cap \mathcal{E} = \varnothing.$$

The sufficiency is a straightforward consequence of

$$\mathcal{K} \subseteq \mathcal{E}.$$

To prove the necessity, assume that (12) is impossible and that

$$\mathcal{H} \cap \mathcal{E} \neq \varnothing.$$

Then, $\exists x' \in X$ such that

$$\mathcal{H} \cap \mathcal{E}(x') \neq \varnothing.$$

Hence, $\exists u', v'$ such that

$$f(x') - u' \in U, \qquad g(x') - v' \in V, \qquad (u', v') \in \mathcal{H}.$$

Taking into account the second part of (11), this implies

$$u' \in \text{int } U, \qquad v' \in V;$$

hence,

$$f(x') \in \text{int } U, \qquad g(x') \in V,$$

which contradicts the impossibility of (12). Now, we have to show that

$$\mathscr{H} \cap \mathscr{E} = \varnothing,$$

iff condition (13) holds. According to Lemma 3.1, $\mathscr{E}$ is convex; hence, cl $\mathscr{E}$ is the intersection of all its closed supporting halfspaces. It is easy to see that these coincide with the halfspaces of the kind

$$\langle \theta, u \rangle + \langle \lambda, v \rangle \leq k, \qquad \text{with } \theta \in U^*, \lambda \in V^*, (\theta, \lambda) \neq 0. \tag{14}$$

Because of the full dimensionality of U, the interior of the complement of such a halfspace contains $\mathscr{H}$, iff either $k=0$ and $\theta \neq 0$ or $k<0$. If

$$\mathscr{H} \cap \mathscr{E} = \varnothing,$$

$\exists \bar{\theta}, \bar{\lambda}$ satisfying (14) such that $\mathscr{E}$ is contained in the halfspace

$$\langle \bar{\theta}, u \rangle + \langle \bar{\lambda}, v \rangle \leq 0;$$

if

$$\bar{\theta} \neq 0,$$

$\mathscr{H}$ is included in the complement of such halfspace, and then (13) is evidently fulfilled; if

$$\bar{\theta} = 0,$$

the above inclusion of $\mathscr{H}$ does not happen; hence, we have to exclude the subset of $\mathscr{H}$ not contained in the above complement, i.e.,

$$\mathscr{H} \cap \{(u, v): \langle \bar{\lambda}, v \rangle = 0\}.$$

Viceversa, if

$$\mathscr{H} \cap \mathscr{E} \neq \varnothing,$$

condition (13) is easily contradicted. This completes the proof. □

When $U = \mathbb{R}^l_+$, Theorem 3.1 becomes Theorem 1 of Ref. 5; if, in addition, f and g are concave in the ordinary sense and $V = \mathbb{R}^m_+$, then Theorem 3.1 becomes Theorem 3 of Ref. 4. In the latter case, (i) of Theorem 1.1, at $w = w_1$, is a well-known statement (Ref. 6). When $\lambda = 0$, i.e., when (12) does not contain $g(x) \in V$, Theorem 3.1 becomes Theorem 3 of Ref. 7. A further instance of how theorems of the alternative can be derived from Theorem 3.1 is the following corollary usually derived from Corollary 1.1 (Ref. 8).

Corollary 3.1. Assume that the real $m \times n$ matrix A, the real n-vector c, and the closed convex cone X with vertex at the origin are given; let x denote a real n-vector. The system (15),

$$\langle c, x\rangle > 0, \qquad Ax \geq 0, \qquad x \in X, \tag{15}$$

is impossible, iff[9]

$$c \in -cl(X^* + \text{con } A). \tag{16}$$

Proof. In (11), set

$$l = 1, \qquad U = [0, +\infty[, \qquad V = \mathbb{R}^m_+, \qquad f(x) = \langle c, x\rangle, \qquad g(x) = Ax,$$

and identify (15) with (12). Theorem 3.1 can be applied; and, if $\bar{\theta} = 1$, (13) becomes:

$$\langle c + \bar{\lambda}A, x\rangle \leq 0, \qquad \forall x \in X,$$

or

$$-(c + \bar{\lambda}A) \in X^*,$$

so that (16) follows. If $\bar{\theta} = 0$, (13) leads to the statement:

$$x \in X \quad \text{and} \quad Ax \geq 0$$

imply

$$\langle c, x\rangle \leq 0,$$

or

$$\langle -c, x\rangle \geq 0, \qquad \forall x \in X \cap \text{con } A,$$

or

$$-c \in (X \cap \text{con } A)^* = \text{cl}(X^* + \text{con } A),$$

so that (16) follows. This completes the proof. □

In Theorem 3.1, the existence or nonexistence of $\bar{\theta} \neq 0$ introduces a partitioning of the set of systems (12) into two classes, which can be called, according to Definition 2.1, *regular* or *irregular*, respectively; irregularity can be further deepened into k-irregularity. A regularity condition for system (12) will now be studied.

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9 con A denotes here the convex hull of the cone generated by the rows of A.

Theorem 3.2. Assume that $\mathscr{E}^\circ \neq \varnothing$. Condition (13) can be fulfilled with $\bar{\theta} \neq 0$, iff

$$(\text{int } U) \cap T(u, v) = \varnothing, \qquad \forall (u, v) \in \mathscr{E}^\circ. \tag{17}$$

Proof. Denote by H the halfspace defined by

$$\langle \bar{\theta}, u \rangle + \langle \bar{\lambda}, v \rangle \leq 0.$$

Necessity. Ab absurdo, suppose that (17) is false, so that $\exists (\bar{u}, \bar{v}) \in \mathscr{E}^\circ$ such that

$$(\text{int } U) \cap T(\bar{u}, \bar{v}) \neq \varnothing.$$

Let $(\hat{u}, \hat{v})$ belong to such an intersection; of course, $\hat{v} = 0$. We have

$$(\hat{u}, \hat{v}) \in \mathscr{H},$$

as

$$\hat{u} \in \text{int } U \quad \text{and} \quad 0 = \hat{v} \in V;$$

and hence,

$$(\hat{u}, \hat{v}) \in \sim H,$$

as

$$\bar{\theta} \neq 0 \Rightarrow \mathscr{H} \subset \sim H.$$

On the other hand, H is a supporting halfspace of $\mathscr{E}$ at $(\bar{u}, \bar{v})$ as, from Lemma 3.1, $\mathscr{E}$ is convex; hence,

$$T(\bar{u}, \bar{v}) \subseteq H.$$

Then,

$$(\hat{u}, \hat{v}) \in T(\bar{u}, \bar{v}) \subseteq H,$$

and the contradiction is achieved.

Sufficiency. Ab absurdo, suppose that $\bar{\theta} = 0$ necessarily [namely, that every w_3 which fulfills (13) must have $\theta = 0$]. Then, considering (8b), Theorem 2.1 implies that

$$U \subseteq T(0, 0),$$

which contradicts (17). This completes the proof. □

A sufficient condition (due to Karlin, Ref. 6) for (17) to hold is that

$$\mathscr{M} \triangleq \{\lambda \in V^*: \langle \lambda, g(x) \rangle \leq 0, \forall x \in X\} = \varnothing.$$

In fact, $\mathscr{E}$ being convex, $\mathscr{M} \neq \varnothing$ is necessary, but not sufficient, to have $\bar{\theta} = 0$

necessarily in condition (13). Other sufficient conditions can be obtained as straightforward generalizations of known ones established for the ordinary convex case (Ref. 6). For instance, one consists in assuming, besides the hypothesis of Theorem 3.1, the existence of $\hat{x} \in X$, such that

$$g(\hat{x}) \in \text{int } V.$$

In this case, the halfspace defined by H (in the proof of Theorem 3.2), which contains $\mathscr{E}$, could not contain $(\hat{u} = f(\hat{x}),\ \hat{v} = g(\hat{x}))$ if $\bar{\theta} = 0$. More generally, starting with condition (10), deriving necessary and sufficient conditions for regularity (k-irregularity) of the system (12) in terms of its data is conceivable and useful.

Theorem 3.1 and Corollary 1.1 provide sufficient conditions for w_3 and w_1 to ensure alternative, respectively. In Ref. 4, there are two other conditions, whose extension to the generalized (in the sense of Theorem 3.1) concavity is conceivable. The question of other weak separation functions giving alternative (besides the weak one) and of strong separation functions giving alternative (besides the strong one) is open.

4. Asymptotic Weak Alternative

When a weak separation function is adopted, (i) of Theorem 1.1 does not enable one to claim anything about (2), if (3a) is impossible. However, in some cases, a claim is still possible, if the concept of the weak alternative is enlarged. This will be shown in the particular case where $Z =]-\infty, 0]$. With his aim, consider a family of weak separation functions, depending on a parameter[10]:

$$w(h; \omega), \qquad \omega \in \Omega, \tag{18}$$

and assume that a sequence, say $\{\omega^r, r = 1, 2, \ldots\}$, can be drawn out from Ω, such that[11]

$$w(h; \omega^r) \text{ is continuous in respect of } h,\ r = 1, 2, \ldots; \tag{19a}$$

$$\mathscr{H}^w(\omega^r) \supset \mathscr{H}^w(\omega^{r+1}), \qquad r = 1, 2, \ldots; \tag{19b}$$

$$\bigcap_{r=1}^{\infty} \mathscr{H}^w(\omega^r) = \mathscr{H}; \tag{19c}$$

[10] An instance is offered by w_3, where $\omega = (\theta, \lambda)$.

[11] Of course, $\mathscr{H}^w(\omega^r)$ denotes the set $\mathscr{H}^w$ corresponding to the particular $w(h; \omega^r)$.

$$\forall h \in \mathcal{H}, \exists k(h) > 0, \text{ such that } w(h; \omega^r) \geq k(h), \forall r = 1, 2, \ldots. \quad \text{(19d)}$$

Note that conditions (19) do not imply that $\mathcal{H}$ is closed.

Theorem 4.1. Let the weak separation function w fulfill condition (19). Then, (2) is impossible, iff

$$\inf_r \sup_{x \in X} w(F(x); \omega^r) \leq 0. \quad (20)$$

Proof. If (2) is possible, i.e., if $\exists \bar{x} \in X$ such that

$$\bar{h} \triangleq F(\bar{x}) \in \mathcal{H},$$

then (19d) implies

$$w(F(\bar{x}); \omega^r) = w(\bar{h}; \omega^r) \geq k(\bar{h}) > 0, \qquad \forall r = 1, 2, \ldots,$$

so that (20) is impossible. To show that (20) holds when (2) is impossible, i.e., when

$$h \triangleq F(x) \notin \mathcal{H}, \qquad \forall x \in X,$$

it is enough to prove that, $\forall \epsilon > 0$, $\exists \bar{r}$ such that

$$w(F(x): \omega^{\bar{r}}) < \epsilon, \qquad \forall x \in X. \quad (21)$$

Because of (19a), (19b), (19c), given $\delta > 0$, $\exists \bar{r}$ such that, $\forall h \in \mathcal{H}^w(\omega^{\bar{r}}) \backslash \mathcal{H}$, $\exists h'(h)$, with $w(h'(h), \omega^{\bar{r}}) = 0$, such that

$$\|h - h'(h)\| < \delta.$$

Hence, because of (19a), given any $\epsilon > 0$, by making a suitable choice of δ, we obtain

$$w(h; \omega^{\bar{r}}) = w(h; \omega^{\bar{r}}) - w(h'(h); \omega^{\bar{r}}) < \epsilon, \qquad \forall h \in \mathcal{H}^w(\omega^{\bar{r}}) \backslash \mathcal{H},$$

and thus (21) follows. This completes the proof. □

Theorem 4.1 can be interpreted in terms of a sequence of weak separation functions, whose level sets approximate $\mathcal{H}$, so that the weak alternative is obtained asymptotically. More precisely, because of (19), alternative is achieved. In the particular case where $\exists r^*$ such that $w(h; \omega^{r^*})$ guarantees weak alternative, then at $r = r^*$ the "if" part of Theorem 4.1 becomes (i) of Theorem 1.1.

If the class of functions (18) is enlarged by deleting (19c), then the "only if" part of Theorem 4.1 is obviously lost. By a suitable replacement of conditions (19), asymptotic alternative is conceivable in the strong case too.

5. Extremum Problems: Sufficient Conditions

Consider again the set $X \subseteq \mathbb{R}^n$, the real-valued function $\varphi: X \to \mathbb{R}$, and the following extremum problem:

$$\min \varphi(x), \qquad \text{s.t. } x \in R \triangleq \{x \in X: g(x) \geq 0\}. \tag{22}$$

We can see immediately that $\bar{x} \in R$ is an optimal solution of (22), iff system

$$f(x) \triangleq \varphi(\bar{x}) - \varphi(x) > 0, \qquad g(x) \geq 0, \qquad x \in X, \tag{23}$$

is impossible. By considering (11) in the more particular case

$$l = 1, \qquad U = [0, +\infty[, \qquad V = \mathbb{R}^m_+, \tag{24}$$

the system (23) can be identified with (2), so that Theorem 1.1 can be used to establish an optimality condition for (22). With this aim, consider the set of functions:

$$w_4(u, v; \theta, \omega) \triangleq \theta u + \gamma_4(v; \omega), \qquad \theta \geq 0, \qquad \omega \in \Omega,$$

where Ω is the domain of parameter ω, such that

$$\text{lev}_{\geq 0}\, \gamma_4 \supseteq \mathbb{R}^m_+, \qquad \bigcap_{\omega \in \Omega} \text{lev}_{\geq 0} \gamma_4(v; \omega) = \mathbb{R}^m_+, \tag{25a}$$

$$\gamma_4(\bar{v}; \bar{\omega}) > 0 \Rightarrow \exists \hat{\omega} \in \Omega \text{ such that } \gamma_4(\bar{v}; \hat{\omega}) < \gamma_4(\bar{v}; \bar{\omega}). \tag{25b}$$

If $\theta > 0$, w_4 fulfills (1a), and hence guarantees weak alternative. This may not happen if $\theta = 0$; however, instead of deleting the kind of functions (this would restrict the above set too much), we simply note that in the latter case weak alternative is still ensured under a further condition.

Lemma 5.1. (i) When $\theta > 0$ or $\theta = 0$ and $\text{lev}_{>0}\, \gamma_4 \supseteq \mathbb{R}^m_+$, the function w_4 guarantees weak alternative between (2) and (3a).

(ii) When $\theta = 0$ and $\text{lev}_{>0}\, \gamma_4 \not\supseteq \mathbb{R}^m_+$, w_4 guarantees weak alternative between (2) and (3a) under the condition (26),

$$X^0 \triangleq X \cap (\text{lev}_{>0} f) \cap (\text{lev}_{\geq 0}\, g) \cap (\text{lev}_{=0}\, \gamma_4) = \varnothing. \tag{26}$$

Proof. (i) (1a) becomes

$$\text{lev}_{>0}\, w_4 \supseteq]0, +\infty[\times \mathbb{R}^m_+$$

or

$$(u, v) \in]0, +\infty[\times \mathbb{R}^m_+ \Rightarrow \theta u + \gamma_4(v; \omega) > 0.$$

This relationship holds as now we have either

$$\theta > 0 \quad \text{and} \quad \text{lev}_{\geq 0}\gamma_4 \supseteq \mathbb{R}^m_+,$$

or

$$\theta = 0 \quad \text{and} \quad \text{lev}_{>0}\gamma_4 \supseteq \mathbb{R}^m_+.$$

Thus, the thesis follows from (i) of Theorem 1.1.

(ii) In the present case (11)–(24), because of condition $X^0 = \varnothing$, which ensures that no element of X is sent into int U, we can replace $Z =]-\infty, 0]$ with $Z =]-\infty, 0[$. Then, (1a) becomes

$$\text{lev}_{\geq 0} w_4 \supseteq [0, +\infty[\times \mathbb{R}^m_+,$$

or

$$(u, v) \in \mathbb{R}_+ \times \mathbb{R}^m_+ \Rightarrow \theta u + \gamma_4(v; \omega) \geq 0.$$

Now, $\theta = 0$; hence, this relationship holds as it is the first part of (25a). Again, weak alternative follows from (i) of Theorem 1.1. This completes the proof. □

Note that, over $\text{lev}_{>0} f$, the possibility of (3a) is crucial to show the impossibility of (2), while it is redundant over $\text{lev}_{\leq 0} f$.

Taking into account Lemma 5.1, it is easy to interpret (i) of Theorem 1.1 as a sufficient optimality condition for (22); this is contained in the following corollary.

Corollary 5.1. Assume that $\bar{x} \in \mathbb{R}^n$ fulfills these conditions: (i) $\bar{x} \in R$; (ii) there exist $\bar{\theta} \in \mathbb{R}_+$ and $\bar{\omega} \in \Omega$, such that

$$\bar{\theta}[\varphi(\bar{x}) - \varphi(x)] + \gamma_4(g(x); \bar{\omega}) \leq 0, \qquad \forall x \in X, \tag{27}$$

and moreover

$$\{x \in X: \varphi(x) < \varphi(\bar{x}); g(x) \geq 0; \gamma_4(g(x); \bar{\omega}) = 0\} = \varnothing,$$

if

$$\bar{\theta} = 0 \quad \text{and} \quad \text{lev}_{>0}\gamma_4 \not\supseteq \mathbb{R}^m_+.$$

Then, $\bar{x}$ is a global minimum point of (22).

Now, introduce the function

$$\mathscr{L}(x; \theta, \omega) \triangleq \theta\varphi(x) - \gamma_4(g(x); \omega),$$

and let us prove the following theorem.

Theorem 5.1. Conditions (i) and (ii) of Corollary 5.1 are equivalent to the following one: there exist $\bar{x} \in X$, $\bar{\theta} \in \mathbb{R}_+$, and $\bar{\omega} \in \Omega$, such that

$$\mathscr{L}(\bar{x}; \bar{\theta}, \omega) \leq \mathscr{L}(\bar{x}; \bar{\theta}, \bar{\omega}) \leq \mathscr{L}(x; \bar{\theta}, \bar{\omega}), \qquad \forall x \in X, \forall \omega \in \Omega, \tag{28}$$

and moreover

$$\{x \in X: \varphi(x) < \varphi(\bar{x}); g(x) \geq 0; \gamma_4(g(x); \bar{\omega}) = 0\} = \varnothing,$$

if

$$\bar{\theta} = 0 \quad \text{and} \quad \text{lev}_{>0} \supsetneq \mathbb{R}^m_+.$$

Proof. Let us prove that (i)–(ii) of Corollary 5.1⇒(28). $\bar{\omega} \in \Omega$ and $\bar{x} \in R$ imply that

$$\gamma_4(g(\bar{x}); \bar{\omega}) \geq 0,$$

as

$$\text{lev}_{\geq 0}\, \gamma_4 \supseteq \mathbb{R}^m_+;$$

at $x = \bar{x}$, (27) implies

$$\gamma_4(g(\bar{x}); \bar{\omega}) \leq 0;$$

it follows that

$$\gamma_4(g(\bar{x}); \bar{\omega}) = 0.$$

Hence, (27) is equivalent to the second part of (28). Now, note that

$$g(\bar{x}) \geq 0$$

is equivalent to

$$\gamma_4(g(\bar{x}); \omega) \geq 0, \qquad \forall \omega \in \Omega,$$

as

$$\bigcap_{\omega \in \Omega} \text{lev}_{\geq 0}\, \gamma_4 = \mathbb{R}^m_+;$$

it follows that

$$\gamma_4(g(\bar{x}); \bar{\omega}) \leq \gamma_4(g(\bar{x}); \omega), \qquad \forall \omega \in \Omega, \tag{29}$$

which is equivalent to the first part of (28). In order to prove that (28)⇒(i)–(ii) of Corollary 5.1, note that the first part of (28) is equivalent to (29), and this implies

$$g(\bar{x}) \geq 0;$$

note that $g(\bar{x}) \not\geq 0 \Rightarrow \exists \hat{\omega} \in \Omega$ such that $\gamma_4(g(\bar{x}), \hat{\omega}) < 0$ and, hence, as $\omega_0 \gamma_4$ fulfills (25), $\forall \omega_0 > 0$, if γ_4 does, $\gamma_4(g(\bar{x}); \omega)$ is not bounded from below and (29) is contradicted; and hence, because of (25a),

$$\gamma_4(g(\bar{x}); \bar{\omega}) \geq 0.$$

Condition (i) of Corollary 5.1 is proven. Assume that

$$\gamma_4(g(x); \omega) > 0.$$

Then, because of (25b), the first part of (28) is contradicted. Thus,

$$\gamma_4(g(\bar{x}); \bar{\omega}) = 0$$

is achieved. Account taken of this equality, it is easy to show that the second part of (28) implies (27). This completes the proof. □

Note that, when (27) or (28) holds, from the proof of Theorem 5.1, we have that $(\bar{x}, \bar{\omega})$ fulfills the generalized complementarity condition

$$\gamma_4(g(x); \omega) = 0,$$

which collapses to the well-known ordinary one, when γ_4 is linear and hence $\Omega = \mathbb{R}^m_+$. Even if the two conditions are obviously equivalent, the former can be useful. To show this, set

$$\lambda = (\lambda_1, \ldots, \lambda_m), \qquad \mu = (\mu_1, \ldots, \mu_m),$$

$$\omega = (\lambda, \mu), \qquad \Omega = \mathbb{R}^m_+ \times \mathbb{R}^m_+,$$

and assume that

$$\gamma_4(v; \omega) = \sum_{i=1}^{m} \lambda_i T_i(v_i; \mu_i),$$

where

$$T_i(v_i; \mu_i) \gtreqless 0, \qquad \forall \mu_i \geq 0,$$

according to

$$v_i \gtreqless 0,$$

respectively. In this case, if we set $T = (T_1, \ldots, T_m)$, from Theorem 5.1 we get the *generalized complementarity system*

$$T(g(x); \mu) \geq 0, \qquad \lambda \geq 0, \qquad \langle T(g(x); \mu), \lambda \rangle = 0,$$

which collapses to the well-known one, when

$$T(v; \mu) = v.$$

Now, consider the particular case where

$$T_i(v_i; \mu_i) = v_i \exp(-\mu_i v_i);$$

T_i represents an exponential transformation of v_i. It is immediate to show that the function

$$w_5(u, v; \theta, \lambda, \mu) = \theta u + \langle \lambda, e(v; \mu) \rangle, \qquad \theta \in \mathbb{R}, \qquad \lambda, \mu \in \mathbb{R}^m_+,$$

where

$$e(v; \mu) = (v_i \exp(-\mu_i v_i), i = 1, \ldots, m),$$

and which contains the linear function at $\mu = 0$, is a particular case of w_4. By means of transformation T in the complementarity system, $T(g(x); \mu)$ may have a certain property, for instance concavity, differentiability, which does not hold for g.

Another particular case is obtained by requiring γ_4 to be nondecreasing with respect to v, i.e., the case of w_2 of Section 1, if in (22) we replace $g(x) \geq 0$ with[12] $g(x) \geq b$; an instance is given by

$$\gamma_4(v; \lambda, \mu) = \sum_{i=1}^{m} \lambda[1 - \exp(-\mu_i v_i)],$$

already studied, even if from a different point of view (Ref. 9). A further particular case of the weak separation function is

$$w_6(u, v) = u - \gamma_6(v) - \bar{\gamma},$$

where

$$\gamma_6(v) \triangleq \sup_{(u,v) \in \mathcal{K}} (u), \qquad \bar{\gamma} = \sup_{v \geq 0} [\gamma_6(v) + |\gamma_6(v)|]/2.$$

It turns out that

$$\gamma_6(v) = \sup_{\substack{g(x)=v \\ x \in X}} [\varphi(\bar{x}) - \varphi(x)] = \varphi(\bar{x}) - \psi(v),$$

where

$$\psi(v) \triangleq \inf_{\substack{g(x)=v \\ x \in X}} \varphi(x)$$

is the so-called *perturbation function* (Refs. 10–12). w_6 has the disadvantage of depending on $\mathcal{K}$.

Now, note that (28) can be regarded as a generalized saddle-point condition, and $\mathcal{L}$ as a generalized Lagrangian function. When we adopt w_5 at $\mu = 0$, (28) becomes the well-known John saddle-point condition and $\mathcal{L}$ the classic Lagrangian function (Ref. 6). When we adopt w_5, (28) can be viewed also as an ordinary saddle-point condition for an exponential transformation of the constraining functions.

When X is convex and φ, g are concave, (27) and (28) become necessary too. This is an obvious consequence of the fact that now any w_4 guarantees alternative, according to Theorem 3.1, at

$$U = \mathbb{R}_+, \qquad V = \mathbb{R}_+^m, \qquad f(x) = \varphi(\bar{x}) - \varphi(x).$$

[12] In this case, with the notation of Section 1, (27) becomes obviously

$$\theta[\varphi(\bar{x}) - \varphi(x)] + \gamma_2(g(x); \omega) - \gamma_2(b; \omega) \leq 0, \qquad \forall x \in X.$$

In this case, it is of course convenient to adopt the linear function (w_5 at $\mu = 0$); thus, Theorem 5.1 becomes the well-known John saddle-point (necessary and sufficient) condition for convex programs (Ref. 6). Iff condition (17), or at least if $\mathcal{M} = \emptyset$, is satisfied, then in (27) and (28) we can set $\bar{\theta} = 1$, and problem (22) is called *regular*.

6. Necessary Conditions

In the preceding section, it has been shown that a sufficient optimality condition for (22) can be derived from a separation function which guarantees weak alternative. Now, it is obvious to expect a necessary condition from a strong separation function. With this aim, consider again problem (22), and let $\bar{\mathcal{K}} \subseteq \mathbb{R}^{1+m}$ be such that[13] $\mathcal{K} \subseteq \bar{\mathcal{K}}$, namely,

$$(\varphi(\bar{x}) - \varphi(x), g(x)) \in \bar{\mathcal{K}}, \qquad \forall x \in X. \tag{30}$$

$\bar{\mathcal{K}}$ trivially exists, as (30) is satisfied by at least $\bar{\mathcal{K}} = \mathbb{R} \times \mathbb{R}^m$. If there exists a positive real ρ, such that

$$\|\varphi(x)\| \leq \rho/2; \|g(x)\| \leq \rho, \qquad x \in X,$$

then, we can set

$$\bar{\mathcal{K}} = [-\rho, \rho]^{1+m}.$$

Consider the following set of functions[14]:

$$s_1(u, v; \omega) = u - \delta_1(v; \omega), \qquad \omega \in \Omega,$$

where Ω is the domain of parameter ω, such that

$$\bar{\mathcal{K}} \cap \operatorname{lev}_{>0} s_1 \subseteq \mathcal{H}; \qquad \operatorname{cl} \bigcup_{\omega \in \Omega} \bar{\mathcal{K}} \cap \operatorname{lev}_{>0} s_1(u, v; \omega) = \operatorname{cl}(\bar{\mathcal{K}} \cap \mathcal{H}). \tag{31}$$

When $\bar{\mathcal{K}}$ has a particular form, for instance,

$$\bar{\mathcal{K}} = \mathbb{R}^{1+m} \quad \text{or} \quad \bar{\mathcal{K}} = [-\rho, \rho]^{1+m},$$

then (31) can be expressed in terms of δ_1, as it happens in (25); remembering this, a lemma like lemma 5.1 is quite obvious, since the first part of (31) says that s_1 fulfills (1b) on $\bar{\mathcal{K}}$, and hence it is a strong separation function on this domain. We are now able to derive a necessary condition from Theorem 1.1.

---

[13] Recall that we are in case (24).

[14] u might be multiplied by θ; here, we assume $\theta = 1$; in general, the case $\theta = 0$ has no interest.

Corollary 6.1. Let φ, g fulfill condition (30), and let s_1 be defined by (31). If $\bar{x} \in \mathbb{R}^n$ is a global minimum point of (22), then,

$$\varphi(\bar{x}) - \varphi(x) - \delta_1(g(x); \omega) \leq 0, \qquad \forall x \in X, \tag{32}$$

whatever $\omega \in \Omega$ might be.

Proof. The optimality of $\bar{x}$ implies the impossibility of (23). As assumption (31) makes s_1 a strong separation function, the thesis follows from (ii) of Theorem 1.1. □

Two instances of functions of type s_1 are the following ones:

$$s_2(u, v; \lambda, \mu) = u - \sum_{i-1}^{m} \lambda_i \exp(-\mu_i v_i), \qquad \lambda_i, \mu_i \in \mathbb{R}_+, \qquad \lambda_i \geq \rho,$$

$$s_3(u, v; \mu) = u - \rho \sum_{i=1}^{m} (1 - v_i/\rho)^{2\mu_i}, \qquad \mu > 0,$$

where

$$\lambda = (\lambda_1, \ldots, \lambda_m), \qquad \mu = (\mu_1, \ldots, \mu_m).$$

The classic necessary conditions of the Lagrange–Karush–Kuhn–Tucker type do not come from the strong alternative, as might seem. On the contrary, they are a consequence of a further analysis of the weak alternative, like Lemma 2.1, and are based on local arguments. In this sense, they are not in a symmetric logical position with respect to saddle-point conditions. This is shown by the following property, which generalizes the above conditions.

Corollary 6.2. Let w_4 be the weak separation function of Section 5. If $\bar{x} \in \mathbb{R}^n$ is a global minimum point of (22), then there exists θ and ω such that $\bar{x}$ is a stationary point of problem

$$\max_{x \in X} w_4(\varphi(\bar{x}) - \varphi(x), g(x); \theta, \omega). \tag{33}$$

Proof. The thesis consists in proving that there exist $\bar{\theta} \geq 0$ and $\bar{\omega} \in \Omega$, such that, for a suitable neighborhood N of $\bar{x}$, we have

$$\limsup_{x \to \bar{x}} \frac{w_4(f(x), g(x); \bar{\theta}, \bar{\omega}) - w_4(f(\bar{x}), g(\bar{x}); \bar{\theta}, \bar{\omega})}{\|x - \bar{x}\|} \leq 0, \qquad x \in X \cap N. \tag{34}$$

Ab absurdo, suppose that (34) does not hold. Hence, there exists a neighborhood N' of $\bar{x}$, such that, for $x \neq \bar{x}$, we have[15]:

$$\frac{w_4(f(x), g(x); \theta, \omega) - w_4(0, g(\bar{x}); \theta, \omega)}{\|x-\bar{x}\|} > 0,$$

$$\forall x \in N' \cap X, \forall \theta \geq 0, \forall \omega \in \Omega.$$

Taking into account the fact that

$$g(\bar{x}) \geq 0,$$

and then

$$w_4(0, g(\bar{x}); \theta, \omega) \geq 0, \qquad \forall \theta \geq 0, \forall \omega \in \Omega,$$

it follows that

$$w_4(f(x), g(x); \theta, \omega) > w_4(0, g(\bar{x}); \theta, \omega) \geq 0,$$

$$\forall x \in N' \cap X \backslash \{\bar{x}\}, \forall \theta \geq 0, \forall \omega \in \Omega,$$

and hence[16]

$$(f(x), g(x)) \in \mathscr{H}^w(\theta, \omega), \qquad \forall x \in N' \cap (X \backslash \{\bar{x}\}), \forall \theta \geq 0, \forall \omega \in \Omega.$$

Now, remembering (25a), it follows that

$$(f(x), g(x)) \in \operatorname{int} \mathscr{H}, \qquad \forall x \in N' \cap (X \backslash \{\bar{x}\}),$$

which contradicts (8a). This completes the proof. □

When φ and g are differentiable, $X = \mathbb{R}^n$, and w_4 is linear (that is, w_4 becomes w_5 at $\mu = 0$), then (33) is equivalent to

$$\min_{x \in \mathbb{R}^n} [\theta\varphi(x) - \langle \lambda, g(x) \rangle],$$

and the thesis of Corollary 6.2 [that is, (34)] becomes the well-known equation

$$\nabla_x[\theta\varphi(x) - \langle \lambda, g(x) \rangle] = 0,$$

which represents the so-called weak Lagrangian principle. If φ and g are differentiable on X, then (34) takes the still well-known form of directional derivative.

Note that, as the thesis of Corollary 6.2 holds whatever the weak separation function w_4 might be, it is obvious to think of the simplest one,

[15] Recall that $f(x) = \varphi(\bar{x}) - \varphi(x)$, so that $f(\bar{x}) = 0$.

[16] Now, $\mathscr{H}^w(\theta, \omega)$ denotes the set $\mathscr{H}^w$ corresponding to $w_4(u, v; \theta, \omega)$.

namely, the linear one. However, it may be useful to adopt a nonlinear w_4, so that (33) has some suitable property. For instance, problem (33) can turn out to be differentiable, even if (22) is not. With this aim, consider the following example, where, for the sake of simplicity, we assume that the constraints in (22) are of type $g(x)=0$; we need only to define

$$\mathcal{H}=\{(u, v)\in\mathbb{R}\times\mathbb{R}^m: u>0; v=0\}.$$

Set

$$n=1, \qquad m=1, \qquad X=\mathbb{R}, \qquad \varphi(x)=x^2, \qquad g(x)=x\sin(1/x).$$

If we consider the separation function

$$w_4(u, v; \theta, \omega)=u-\omega v^2, \qquad \omega>0,$$

(33) is equivalent to

$$\min_{x\in\mathbb{R}}[x^2+\omega x^2\sin^2(1/x)],$$

which is differentiable, notwithstanding that (22) is not.

Note that the thesis of Corollary 6.2 generalizes only a part of the well-known stationary conditions; in fact, it does not contain the so-called complementarity condition

$$\langle\bar{\omega}, g(\bar{x})\rangle=0,$$

which follows by noting that, if

$$\bar{h}\triangleq(\bar{u}=f(\bar{x}), \bar{v}=g(\bar{x}))\in\mathbb{R}_+\times\mathbb{R}_+^m\backslash\{0\},$$

then

$$\langle\bar{\omega}, g(\bar{x})\rangle>0$$

contradicts (8a).

Lastly, note that sufficient conditions, based on the Hessian matrix, are a further local analysis; hence, even if they appear in the context of weak separation function, they must not be considered as coming from weak separation function, like saddle-point ones.

7. Lagrangian Penalty Approaches

Penalty approaches are a natural extension of the original Lagrangian method and aim at reaching an optimal solution of a constrained extremum problem by solving a sequence of unconstrained ones. It will be shown how these approaches can be viewed in terms of weak and strong separation functions. With this aim, consider again problem (22), with $X = \mathbb{R}^n$, and the continuous functions $p_r: \mathbb{R}^m \to \mathbb{R}$, $r = 1, 2, \ldots$, such that

$$p_r(v) = 0, \text{ if } v \geq 0, \qquad p_r(v) > 0, \text{ if } v \ngeq 0,$$
$$p_{r+1}(v) > p_r(v), \qquad \lim_{r \to +\infty} p_r(v) = +\infty, \qquad v \ngeq 0. \tag{35}$$

Consider the particular case (11)–(24). It is easily seen that the function

$$w_7(u, v; r) = u - p_r(v) \tag{36}$$

is of the w_4-kind and fulfills condition (19). In fact, now $\omega^r = r$, and $\mathscr{H}^w(r)$ is the positivity level set of (36). Moreover, (36) is a weak separation function. Hence, (i) of Theorem 1.1 can be applied and (3a) becomes

$$\varphi(\bar{x}) - \varphi(x) - p_r(g(x)) \leq 0, \qquad \forall x \in \mathbb{R}^n, \tag{37}$$

and is a sufficient condition for the feasible $\bar{x}$ to be optimal. Such a condition can be weakened by applying Theorem 4.1; (20) becomes

$$\lim_{r \to +\infty} \inf_{x \in \mathbb{R}^n} [\varphi(x) + p_r(g(x))] \geq \varphi(\bar{x}) \tag{38}$$

and is a sufficient condition which is weaker than (37). Denote by Φ_r the infimum in (38). From (35), we deduce that

$$\Phi_1 \leq \Phi_2 \leq \cdots \leq \Phi \triangleq \inf_{x \in \mathbb{R}^n} \varphi(x). \tag{39}$$

Assume that $\exists \bar{r}$ such that $\Phi_{\bar{r}} > -\infty$, and that there is a proper $x^r \in \mathbb{R}^n$ such that

$$\varphi(x^r) = \Phi_r, \qquad \forall r \geq \bar{r}.$$

If $\bar{x}$ is any limit point of sequence $\{x^r\}$, then condition (38) is fulfilled and Theorem 4.1 gives the optimality of $\bar{x}$. The construction of sequence $\{x^r\}$ by solving the infimum problems in (38) is the well-known *exterior penalty method* (Ref. 12) and p_r is called a *penalty function*; if the above convergence can be ensured after a finite number of steps, i.e., if $\exists \bar{r}$ such that (37) is fulfilled at $r = \bar{r}$, then p_r is called *exact penalty function* (Ref. 13). Hence, the conditions for a penalty function to be exact can be regarded as conditions which ensure (37) instead of (38).

A particular case of (36), corresponding to a well-known penalty function, is

$$w_8(u, v; r, \alpha) = u - r \sum_{i=1}^{m} (-\min\{0, v_i\})^\alpha, \qquad \alpha \geq 1.$$

A more general class of functions satisfying (19) is contained in Ref. 13, where the case of both equality and inequality constraints is considered.

The latter requires only formal changes in the above reasoning. In fact, if the constraints of (22) are $g(x)=0$, it is enough to replace $V=\mathbb{R}_+^m$ with $V=\{0\}$ in (24), so that now

$$\mathscr{H}=\{(u,v)\in\mathbb{R}\times\mathbb{R}^m\colon u>0;\ v=0\}.$$

In such a case, a weak separation function is, for instance, the following one:

$$w_9(u,v;\lambda,r)=u+\langle\lambda,v\rangle-r\langle v,v\rangle,\qquad \text{with } \lambda\in\mathbb{R}_+^m,\ r\in\mathbb{R}_+,$$

which corresponds to the so-called *augmented Lagrangian approach* (Ref. 12). At $\lambda=0$, w_9 corresponds to one of the first penalty functions which have been considered.

It follows that the *exterior penalty approach* can be formulated in terms of weak separation. Among other things, this enables one to extend the penalty approach to solve systems, as shown by Theorem 2.1.

Now, it is easy to say that the *interior penalty approach* (Ref. 14) can be formulated in terms of strong separation. For instance, it is easy to show that a strong separation function for (22) is

$$s_4(u,v;r)=u-p(v;r),\qquad \text{with } r>0,$$

and where

$$p(v;r)=r\sum_{i=1}^{m}(1/v_i),\qquad \text{if } v>0,$$
$$p(v;r)=+\infty,\qquad \text{if } v\not> 0,$$

is a well-known interior penalty function.

8. Duality

When X is convex and $-\varphi$, g are concave, so that problem (22) is convex, a new problem, called dual, is associated to it (Ref. 6). Here, it is shown that the dual problem naturally arises when optimality is studied through alternative. In this way, some generalizations are easily achieved.

Assume that problem (22) is regular, so that in (27) we can set $\theta=1$. Note that (27) holds iff $\exists\bar\omega\in\Omega$ such that

$$\Phi(\bar\omega)\geq\varphi(\bar x),$$

where[17]

$$\Phi(\omega)\triangleq\min_{x\in X}[\varphi(x)-\gamma_4(g(x);\omega)\triangleq\Lambda(x;\omega)].\tag{40}$$

[17] The symbols min and max must be replaced by inf and sup, respectively, if necessary. Note that

$$\Lambda(x;\omega)=\mathscr{L}(x;1,\omega).$$

Thus, the fulfillment of (27) leads one to consider the following problem:

$$\max_{\omega\in\Omega} \Phi(\omega), \tag{41}$$

which is called the *weak dual problem* of (22). The fact that (41) comes from a weak separation function shows that its extremum is less than or equal to the minimum of (22). This is a straightforward consequence of a recent result (Ref. 15).

Theorem 8.1. *Weak duality.* We have

$$\max_{\omega\in\Omega} \Phi(\omega) \leq \min_{x\in R} \varphi(x). \tag{42}$$

Proof. As γ_4 fulfills (25), $u+\gamma_4(v;\omega)$ is a weak separation function, so that both systems,

$$\varphi(\bar{x})-\varphi(x)>0, \qquad x\in R,$$

and

$$\Phi(\omega)\geq\varphi(\bar{x}), \qquad \omega\in\Omega,$$

cannot be possible simultaneously. As this statement does not depend on the value of $\varphi(\bar{x})$, it remains true if $\varphi(\bar{x})$ is replaced with any real α. Hence, from Lemma 1 of Ref. 15, (42) holds. This completes the proof. □

When we can ensure that a stationary point of the problem in (40) is also a global minimum point (this happens, for instance, when X and Λ are convex), then problem (41) can be equivalently written as

$$\max \Lambda(x;\omega), \tag{43a}$$

subject to

$$\liminf_{y\to x} \frac{\Lambda(y;\omega)-\Lambda(x;\omega)}{\|y-x\|}\geq 0, \qquad y\in X\cap N, \qquad \omega\in\Omega, \tag{43b}$$

where N is a neighborhood of x.

When $X=\mathbb{R}^n$, $-\varphi$ and g are concave and differentiable, so that we can set

$$\omega=\lambda, \qquad \gamma_4(v;\omega)=\langle\lambda, v\rangle,$$

i.e., we can adopt w_5 at $\theta=1$, $\mu=0$, (43) takes the more familiar form (Ref. 6)

$$\max L(x;\lambda), \qquad \text{s.t. } \nabla_x L(x;\lambda)=0, \qquad \lambda\geq 0, \tag{44}$$

where

$$L(x;\lambda)\triangleq\varphi(x)-\langle\lambda, g(x)\rangle$$

is the ordinary Lagrangian function.

If w_4 is the second exponential function of Section 5, or

$$w_4(v;\lambda,\mu)=u+\sum_{i=1}^{m}\lambda_i[1-\exp(-\mu_i v_i)],$$

then a crucial point is to ensure the convexity (with respect to x) of the exponentially transformed constraining function $\exp[-\mu_i g_i(x)]$. If this happens, (43) can be set up, even if (22) is not convex (note that it is not restrictive to assume φ to be linear).

Another interesting case is the one where $-\varphi$, g are concave over a cone.

Now, consider the case where, in (22),

$$g(x)\geq 0$$

is replaced by

$$g(x)\geq b;$$

namely, (22) is replaced by

$$\min \varphi(x), \qquad \text{s.t. } x\in R_b\triangleq\{x\in X: g(x)\geq b\}. \tag{45}$$

It has been noted that w_2 of Section 1 is a weak separation function. Now, assume that problem (45) is regular, so that we can set $\theta=1$. As (45) is of type (22) and as, at $\theta=1$, w_2 is a particular case of the weak separation function considered in (40), Theorem 8.1 obviously holds here too.[18] Now, note that the existence of $\Phi(\omega)$, which here becomes

$$\Phi(\omega)=\gamma_2(b;\omega)+\min_{x\in X}[\varphi(x)-\gamma_2(g(x);\omega)],$$

implies the existence of a constant, say k, such that

$$\varphi(x)-\gamma_2(g(x);\omega)\geq k, \qquad \forall x\in X,$$

or

$$\varphi(x)\geq k+\gamma_2(g(x);\omega), \qquad \forall x\in X.$$

As k can be embedded into γ_2 itself (as γ_2 remains nondecreasing, or we can assume $k=0$), to ensure the existence of such a minimum we have to

[18] As a matter of fact, w_2 guarantees alternative (besides the weak one), so that (42) is verified as equality.

restrict ourselves to those γ_2 (i.e., to those ω) such that

$$\gamma_2(g(x); \omega) \leq \varphi(x), \qquad \forall x \in X. \tag{46a}$$

When this inequality has been fulfilled, (41) becomes

$$\max \gamma_2(b; \omega), \tag{46b}$$

subject to (46a). Problems (45) and (46) obviously satisfy Ineq. (42). The dual (46) generalizes some forms recently introduced (Ref. 10, 11, 16), and it is studied in Ref. 3.

The difference between the right-hand side and the left-hand side in (42) is called the *weak duality gap.* When weak alternative ensures alternative, it is zero, as happens in the convex case. Two classes of nonconvex problems are shown in Ref. 4 for which the gap equals zero.

Now, consider condition (32), which can be equivalently written as

$$\Psi(\omega) \triangleq \min_{x \in X} [\varphi(x) + \delta_1(g(x); \omega)] \geq \varphi(\bar{x}), \qquad \forall \omega \in \Omega. \tag{47}$$

Thus, the fulfillment of (32) leads one to consider the following problem:

$$\min_{\omega \in \Omega} \Psi(\omega), \tag{48}$$

which will be called the *strong dual problem* of (22). The fact that (48) comes from a strong separation function leads one to expect that its extremum might be greater than or equal to the minimum of (22).

Theorem 8.2. *Strong Duality.* We have

$$\min_{\omega \in \Omega} \Psi(\omega) \geq \min_{x \in R} \varphi(x). \tag{49}$$

Proof. The proof is a straightforward consequence of Corollary 6.1. □

An obvious consequence of (49) is Lemma 3 of Ref. 15, which now states the equivalence between these conditions:

(i) $\exists \omega \in \Omega$, such that $\Psi(\omega) \geq \min_{x \in R} \varphi(x)$; (50a)

(ii) both systems (50b),

$$\varphi(\bar{x}) - \varphi(x) > 0,\ x \in R, \qquad \text{and} \qquad \Psi(\omega) \geq \varphi(\bar{x}),\ \omega \in \Omega, \tag{50b}$$

cannot be impossible simultaneously.

Now, assume that (41) comes from a weak separation function which guarantees alternative too. Then both systems, which appear in the proof of Theorem 8.1, cannot be possible simultaneously (because of the weak alternative), so that (42) holds, and both cannot be impossible simultaneously (because of alternative) so that (because of Lemma 2 of Ref. 15) we get the equality in (42).

The difference between the left-hand side and the right-hand side in (49) is called the *strong duality gap*. When it is zero, there is a sequence of minimization problems, namely, the ones in (47) at some ω's, whose extrema converge to the one of (22). This corresponds to the interior penalty approach of Section 7, just as the sequence of problems extracted from (40) corresponds to exterior penalty approach.

If a weak separation function w is adopted in (40), which is not of the form $w = u + \gamma(v; \omega)$, the definition of the dual problem should require an implicit function approach. When

$$\gamma(v; \omega) = \sum_{i=1}^{m} \gamma_i(v_i; \omega_i),$$

i.e., if γ is separable, then γ_i receives an interpretation as multiplier function. Similar questions arise in the strong case.

9. Concluding Remarks

Theorems of the alternative can be considered as a general framework within which optimality conditions and related topics can be studied.

More precisely, it is shown that saddle-point sufficient conditions, weak duality, and exterior penalty schemes correspond to generalized weak alternative or separation. For symmetry reasons, strong alternative is considered, and it is shown that it produces necessary conditions (not of the stationary type; these, on the contrary, turn out to be a further deepening of weak analysis), strong duality (embedding the known strong duality theorem), and interior penalty schemes.

In what is developed in the preceding sections, the underlying concept is the image of a constrained extremum problem. Taking into account the notation of Section 2, problem (22) can be equivalently[19] formulated as follows:

$$\varphi(\bar{x}) - \max_{(u,v)\in \mathscr{K}^\circ} (u). \tag{51}$$

Note that problem (51), which can be called the *image* of problem (22), is a real-valued problem, even if X is a subset of a suitable functional space

[19] In the sense that the minimum of (22) equals (51), when they exist.

to which many of the preceding results can be generalized. The approach to some optimization topics, which is understood in the preceding sections, consists in introducing the image problem (51), in studying a certain question on it, and then, when a result has been obtained on the image problem, to obtain its counterimage, namely the corresponding result in the space where x runs. For instance, in the case of regularity, the former part is represented by Theorem 3.2. The development of the latter part, as well as the analysis of other topics, like uniqueness and stability of optimal solutions, converse duality, Fenchel duality, should produce useful results. For instance, uniqueness of optimal solutions to (22) can be reduced to the scheme of Section 1 in the following way: $\bar{x} \in R$ is a unique optimal solution to (22) iff, $\forall \gamma \in \mathbb{R}^n$, the system

$$\langle \gamma, x-\bar{x}\rangle > 0, \qquad \varphi(\bar{x})-\varphi(x) \geq 0, \qquad g(x) \geq 0,$$

is impossible.

Another instance is represented by Fenchel duality, which can be deduced by the general approach contained in Sections 5 and 8, without any enlargement of the space where x runs, as happens in recent results (Ref. 18). Further investigation in this direction should produce results in the knowledge of the so-called perturbation (or optimal value) function, and hence in some related topics, like reciprocal problems.

A special important application of the above scheme might be to combinatorial problems (an instance, related to the integer Farkas theorem, is contained in Ref. 4) and to discrete optimization problems, namely, the case where X is a subset of the discrete space $\mathbb{Z}^n$. In (22), assume that

$$X = \bar{X} \cap \mathbb{Z}^n_+, \qquad z \in \mathbb{Z}^n \Rightarrow \varphi(x) \in \mathbb{Z}, \qquad g(x) \in \mathbb{Z}^m, \tag{52}$$

where $\bar{X} \subset \mathbb{R}^n$ is compact. In this case, it is not restrictive to set

$$\mathcal{H} = [1, +\infty[\times \mathbb{R}^m_+,$$

so that strict separation of $\mathcal{H}$ and $\mathcal{K}$ can be achieved, when they are disjoint, i.e., when (23) is impossible. In this case, a function of type w_4, in particular w_5, enables one to get weak dual and, with the addition of a suitable constant, also a strong dual, so as to get lower and upper bounds, respectively.

Problem (22), case (52), can be reduced to the above scheme in another way. For instance, the constraint $x_j \in \mathbb{Z}$ can be equivalently replaced by

$$q_j(x_j) \triangleq \sin^2(\pi x_j) = 0$$

or

$$q_j(x_j) \triangleq \rho \prod_{s=0}^{L_j} (x_j - s)^2 = 0,$$

where ρ is a large enough positive real number and L_j an upper bound for x_j. As a consequence, the discrete problem becomes a continuous one, as $\mathbb{Z}^n$ can be replaced by $\mathbb{R}^n$.

Another kind of problem, which can be reduced to the above scheme, is a variational inequality. Consider a real-valued function $F: X \to \mathbb{R}^n$, a convex cone V, and the problem which consists in finding

$$\bar{x} \in R \triangleq \{x \in X: g(x) \in V\},$$

such that

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \qquad \forall x \in R. \tag{53}$$

This is a general setting for a finite-dimensional variational inequality. In Ref. 4, (53) has been reduced to the alternative scheme by means of the obvious remark that $\bar{x} \in R$ is a solution of (53) iff the system

$$f(x) \triangleq \langle F(\bar{x}), \bar{x} - x \rangle > 0, \qquad g(x) \in V, \qquad x \in X,$$

is impossible. If g is a V-function and X convex, then Theorem 3.1 offers a necessary and sufficient condition for $\bar{x}$ to be a solution of (53). It is useful to know conditions on g, X under which the condition of Theorem 3.1 holds even if it is not true that g is a V-function and X is convex; one of them can be derived from Corollary 1 of Ref. 4.

References

1. DUBOVITSKII, A. Y., and MILYUTIN, A. A., *The Extremum Problem in the Presence of Constraints*, Doklady Akademii Nauk SSSR, Vol. 149, pp. 759–762, 1963.
2. BAIR, J., *Sur la Séparation de Familles Fines d'Ensembles Convexes*, Bulletin de la Societé Royal des Sciences, Liège, Vol. 41, pp. 281–291, 1972.
3. TIND, J., and WOLSEY, L. A., *An Elementary Survey of General Duality Theory*, Mathematical Programming, Vol. 21, pp. 241–261, 1981.
4. GIANNESSI, F., *Theorems of Alternative, Quadratic Programs, and Complementarity Problems*, Variational Inequalities and Complementarity Problems, Edited by R. W. Cottle, F. Giannessi, and J. L. Lions, John Wiley and Sons, New York, New York, pp. 151–186, 1980.
5. BAZARAA, M. S., *A Theorem of the Alternative with Applications to Convex Programming: Optimality, Duality, and Stability*, Journal of Mathematical Analysis and Applications, Vol. 41, pp. 701–715, 1973.
6. MANGASARIAN, O. L., *Nonlinear Programming*, McGraw-Hill, New York, New York, 1969.
7. EISENBERG, E., *On Cone Functions*, Recent Advances in Mathematical Programming, Edited by P. Wolfe and R. L. Graves, McGraw-Hill, New York, New York, pp. 27–33, 1963.

8. BEN-ISRAEL, A., BEN-TAL, A., and ZLOBEC, S., *Optimality in Nonlinear Programming. A Feasible Directions Approach*, John Wiley and Sons, New York, New York, 1981.
9. GOULD, F. J., *Nonlinear Pricing: Applications to Concave Programming*, Operations Research, Vol. 19, pp. 1026–1035, 1971.
10. GOULD, F. J., *Extensions of Lagrange Multipliers in Nonlinear Programming*, SIAM Journal on Applied Mathematics, Vol. 17, pp. 1280–1297, 1969.
11. ROCKAFELLAR, R. T., *Augmented Lagrange Multiplier Functions and Duality in Nonconvex Programming*, SIAM Journal on Control, Vol. 12, pp. 268–283, 1974.
12. ROCKAFELLAR, R. T., *Penalty Methods and Augmented Lagrangians in Nonlinear Programming*, Proceedings of the 5th Conference on Optimization Techniques, Springer-Verlag, Berlin, Germany, pp. 418–425, 1973.
13. HAN, S. P., and MANGASARIAN, O. L., *Exact Penalty Functions in Nonlinear Programming*, Mathematical Programming, Vol. 17, pp. 251–269, 1979.
14. FIACCO, A. V., and MCCORMICK, G. P., *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, John Wiley and Sons, New York, New York, 1968.
15. MCLINDEN, L., *Duality Theorems and Theorems of the Alternative*, Proceedings of Annals of Mathematical Society, Vol. 53, pp. 172–175, 1975.
16. EVERETT, H., *Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources*, Operations Research, Vol. 11, pp. 399–417, 1963.
17. BEN-ISRAEL, A., CHARNES, A. and KORTANEK, K. O., *Asymptotic Duality over Closed Convex Sets*, Journal of Mathematical Analysis and Applications, Vol. 35, pp. 677–691, 1981.
18. MAGNANTI, T. L., *Fenchel and Lagrange Duality Are Equivalent*, Mathematical