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On Duality Theory in Multiobjective Programming[1]

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Abstract. In this paper, we study different vector-valued Lagrangian functions and we develop a duality theory based upon these functions for nonlinear multiobjective programming problems. The saddle-point theorem and the duality theorem are derived for these problems under appropriate convexity assumptions. We also give some relationships between multiobjective optimizations and scalarized problems. A duality theory obtained by using the concept of vector-valued conjugate functions is discussed.

Key Words. Lagrangian functions, M-convexity, saddle points, Slater's constraint qualification, dual functions, conjugate functions.

1. Introduction

Let X be a subset of R^n, and let $f, g_1, \ldots, g_m$ be functions defined on X, with values in R^1. For scalar programming problem, we mean the following problem:

$$\min f(x), \tag{1}$$

$$\text{s.t. } x \in X, \tag{2}$$

$$g_i(x) \leq 0, \qquad i = 1, \ldots, m. \tag{3}$$

Now, if function f takes values in a multidimensional space, say R^k, then (1) has no meaning until some order in R^k is defined. For this purpose, assume that a cone M is given in R^k, which specifies the domination structure as follows. Let y and z be vectors in R^k. We say that z dominates y, and we write $y \leq z$, with respect to M, if $z \in y + M$. For multiobjective

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programming problem, we mean the problem of finding a vector x in R^n satisfying (2) and (3) and such that $f(x)$ does not dominate any other vectors $f(y)$, with y satisfying (2), (3), and $f(x) \neq f(y)$. Further, suppose that a cone N is given in R^m. In this paper, we shall deal with a more general problem, denoted by Problem P: Find a vector x in R^n which satisfies

(i) $x \in X$;
(ii) $g(x) \in -N$;
(iii) there is no $y \in X$, such that $g(y) \in -N, f(x) \in f(y) + M, f(x) \neq f(y)$.

In order to study this problem, we shall construct different Lagrangian functions, depending on the variables considered. In addition to well-known vector variables (Tanino and Sawaragi, Refs. 1 and 2) and matrix variables (Bitran, Ref. 3; Corley, Ref. 4), we also deal with homogeneous function variables. Some properties of Lagrangian functions are examined, in particular their M-continuity, ensuring the existence of solutions to Problem P. The saddle-point theorem is established. According to Lagrangian functions, dual problems are obtained, together with the duality theorem. Weak solutions to Problem P, defined by Corley in Ref. 4, are also considered. Under adequate assumptions, to find a weak solution to Problem P, it is sufficient to work with a Lagrangian function of vector variables. This feature is important from the point view of solving practical problems. In Section 5, we study perturbation functions and discuss dual problems derived from vector-valued conjugate functions. Section 6 is devoted to the relationships between vector Lagrangian functions and scalar Lagrangian functions. These relationships describe situations where scalar programming can be effectively used in multiobjective optimization.

2. Preliminaries

Let Y and M denote, respectively, a set and a cone in R^k. The following definitions and properties concerning cones and functions are needed.

Definition 2.1. A point y in R^k is said to be a minimal [maximal] point of Y with respect to M, if $y \in Y$ and there is no $y' \in Y, y' \neq y$ such that $y \in y' + M[y' \in y + M$, respectively].

We shall denote by $\min(Y|M)$ the set of all minimal points of Y with respect to M. Similarly, the set of all maximal points of Y with respect to M shall be denoted by $\max(Y|M)$.

Let us introduce the symbols ∞_M and $-\infty_M$, by analogy with ∞ and $-\infty$ in the one-dimensional case. The point ∞_M dominates every point in R^k, and the point $-\infty_M$ is dominated by any other point of R^k. The operation rules for these symbols are the following:

$$a+\infty_M=\infty_M+a=\infty_M, \qquad \text{for } a\in R^k\cup\infty_M,$$

$$a-\infty_M=-\infty_M+a=-\infty_M, \qquad \text{for } a\in R^k\cup(-\infty_M),$$

$$\alpha\cdot\infty_M=\infty_M\cdot\alpha=\infty_M, \qquad \text{if } 0<\alpha\leqslant\infty,$$

$$\alpha\cdot\infty_M=\infty_M\cdot\alpha=-\infty_M, \qquad \text{if } 0>\alpha\geqslant-\infty,$$

$$0\cdot\infty_M=\infty_M\cdot 0=0=0\cdot-\infty_M=-\infty_M\cdot 0,$$

$$-(-\infty_M)=\infty_M,$$

$$\min(\phi\,|\,M)=\infty_M,$$

$$\max(\phi\,|\,M)=-\infty_M.$$

Denote

$$\bar{R}^k=R^k\cup(\infty_M)\cup(-\infty_M).$$

Definition 2.2. A set $Y\subset R^k$ is said to be M-convex if $Y+M$ is a convex set in R^k. A vector function f, defined on a convex set $X\subset R^k$, with values in R^k, is said to be an M-convex function if the epigraph $G(f)$ of f,

$$G(f)=\{(x,y)\in R^n\times R^k\colon x\in X, y\in f(x)+M\},$$

is a convex set in $R^n\times R^k$. If f is a $(-M)$-convex function, then we say that f is M-concave.

Relative to Definition 2.2, we have the following lemma, which can be verified readily.

Lemma 2.1. Suppose that $f\colon X\to R^k\cup(\infty_M)$, where X is a convex set in R^n. f is an M-convex function if and only if, for all $x, y\in X$ and all α, $0<\alpha<1$,

$$\alpha f(x)+(1-\alpha)f(y)\in f(\alpha x+(1-\alpha)y)+M. \tag{4}$$

Lemma 2.2. Let M_1 and M_2 be two arbitrary cones in R^k. For any subset Y in R^k, we have

$$\min(Y\,|\,M_1\cup M_2)=\min(Y\,|\,M_1)\cap\min(Y\,|\,M_2), \tag{5}$$

$$\max(Y\,|\,M_1\cup M_2)=\max(Y\,|\,M_1)\cap\max(Y\,|\,M_2). \tag{6}$$

Proof. For (5), suppose that

$$y\notin\min(Y\,|\,M_1\cup M_2).$$

By definition, either $y \notin Y$ or $y \in Y$ but there is $y' \in Y$, $y' \neq y$, such that

$$y \in y' + M_1 \cup M_2.$$

Hence, y cannot be in $\min(Y|M_1) \cap \min(Y|M_2)$. Conversely, if

$$y \notin \min(Y|M_1) \cap \min(Y|M_2),$$

for example

$$y \notin \min\,(Y|M_1),$$

then either $y \notin Y$ or there is $y' \in Y$ such that

$$y' \neq y, \qquad y \in y' + M_1 \subset y' + M_1 \cup M_2.$$

In both cases, y does not belong to $\min(Y|M_1 \cup M_2)$. Relation (6) is proved similarly. □

Definition 2.3. We say that a function $f: X \to R^k$ is M-continuous at point $x_0 \in X$ if, for any neighborhood U of $f(x_0)$ in R^k, there exists a neighborhood V of x_0 in R^n such that

$$f(x) \in U + M, \qquad \text{for all } x \in V \cap X.$$

We say that f is M-continuous on X if it is M-continuous at any point of X.

Some special cases are given below.

Case (*a*). If $M = \{0\}$, Definition 2.3 gives the continuity of f in the usual sense.

Case (*b*). If $k = 1$ and $M = R^1_+$ [$M = R^1_-$], then the M-continuity is the same as lower semicontinuity [upper semicontinuity].

Definitions of semicontinuity can be found in Ref. 5.

We shall use the following notations:

$$M^* = \{z \in R^k: \langle z, z' \rangle \geq 0, \text{ for every } z' \in M\},$$

$$\hat{M} = \{z \in R^k: \langle z, z' \rangle > 0, \text{ for every } z' \in M,\ z' \neq 0\}.$$

It is easy to see that $\hat{M}$ is nonempty, if M is a pointed closed convex cone, and in this case

$$\hat{M} = \operatorname{int} M^*.$$

Lemma 2.3. Assume that M is a convex cone. A function $f: X \to R^k$ is M-convex, M-continuous on a convex set X if and only if, for any $\mu \in M^*$, $\langle \mu, f(x) \rangle$ is a lower semicontinuous convex function on X.

Proof. It is simple to verify that f is M-convex if and only if $\langle\mu, f(x)\rangle$ is a convex function for any $\mu \in M^*$. Hence, it is sufficient to show that the M-continuity of f is equivalent to the lower semicontinuity of $\langle\mu, f(\cdot)\rangle$. Suppose that f is M-continuous on X and that $\{x_n\}$ is an arbitrary sequence of points in X converging to $x_0 \in X$. We must show that

$$\langle\mu, f(x_0)\rangle \leq \liminf_{n\to\infty}\langle\mu, f(x_n)\rangle. \tag{7}$$

By the M-continuity of f at x_0 for an ϵ-neighborhood $U(\varepsilon)$ of $f(x_0)$ in R^k,

$$U(\epsilon) = \{z \in R^k: \|f(x_0) - z\| < \epsilon\},$$

there exists a number n_0 such that

$$f(x_n) \in U(\epsilon) + M, \qquad \text{for all } n \geq n_0.$$

This means that

$$f(x_n) = f(x_0) + z_n + y_n, \qquad \text{for some } y_n \in M,\ z_n \in R^k,\ \|z_n\| < \epsilon.$$

Therefore,

$$\begin{aligned}\langle\mu, f(x_n)\rangle &\geq \langle\mu, f(x_n)\rangle - \langle\mu, y_n\rangle \\ &= \langle\mu, f(x_0)\rangle + \langle\mu, z_n\rangle \geq \langle\mu, f(x_0)\rangle - \epsilon\|\mu\|.\end{aligned}$$

This gives (7), for $\epsilon \to 0$.

Conversely, suppose that f is not M-continuous at some point $x_0 \in X$, that is, there is a neighborhood U of 0 in R^k such that

$$f(x_n) - f(x_0) \notin U + M,$$

for a sequence $\{x_n\}$ of points in X converging to x_0 as $n \to \infty$. By the convexity of M, there exists a polyhedron H which satisfies the following conditions:

(a) $H \subset U + M$;
(b) $B(\epsilon_0) + M \subset \operatorname{int} H$, where $B(\epsilon_0) = \{z \in R^k: \|z\| \leq \epsilon_0\}$, for some positive ϵ_0.

Since the number of faces of H is finite, one can find a hyperplane, defined by some vector $\mu \in R^k$ and a real number α,

$$\{z \in R^k: \langle z, \mu\rangle = \alpha\},$$

such that

$$\langle f(x_n) - f(x_0), \mu\rangle < \alpha,$$

for infinitely many n, and

$$\langle z, \mu\rangle \geq \alpha, \qquad \text{for all } z \in H. \tag{8}$$

From $M \subset \text{int } H$ and (8), it follows that

$$\mu \in M^*.$$

Furthermore,

$$\alpha \leqslant -\epsilon_0 \|\mu\|,$$

because

$$B(\epsilon_0) \subset H.$$

Combining this fact with (8), we arrive at the contradiction that $\langle \mu, f(\cdot) \rangle$ is not lower semicontinuous at x_0. The proof is completed. □

Corollary 2.1. Suppose that $\hat{M}$ is nonempty and that f is an M-convex, M-continuous function on a convex compact set $X \subset R^n$. Then, $\min[f(x)\colon x \in X \mid M]$ is nonempty.

Proof. Let μ be a vector of $\hat{M}$. By virtue of Lemma 2.3, the function $\langle \mu, f(\cdot) \rangle$ is convex and lower semicontinuous on a compact set X; hence, it has at least one minimum on X. It is obvious that the point minimizing $\langle \mu, f(\cdot) \rangle$ on X is a minimal point of $\{f(x)\colon x \in X\}$ with respect to M, and the corollary is proved. □

Definition 2.4. A set $A \subset R^k$ is said to be M-bounded if there exists a bounded set A_0 so that $A \subset A_0 + M$. The set A is said to be M-compact if it is M-bounded and $A + M$ is closed.

Note that, if int M is nonempty, for example $M = R^k_+$, then this definition is equivalent to the one in Ref. 2.

Lemma 2.4. Assume that M is a pointed closed convex cone and that A is a M-compact set. Then,

$$A \subset \min(A \mid M) + M.$$

Proof. The proof of this lemma is similar to the one of Lemma 2.2 in Ref. 2; so, we omit it. □

Lemma 2.5. Assume that M is a closed convex cone. If X is compact and f is M-continuous, then $f(X)$ is M-compact.

Proof. For every $y \in f(X)$, let U_y be a compact neighborhood of y in R^k. By the M-continuity of f, there exists a neighborhood V_x of x in R^n, where

$$x \in X, \qquad f(x) = y,$$

such that

$$f(V_x \cap X) \subset U_y + M.$$

By the compactness of X, there are points $x_1, \ldots, x_q$ in X, with

$$X \subset \bigcup_{i=1}^{q} V_{x_i}.$$

So, $f(X)$ is M-bounded, as

$$f(X) \subset \bigcup_{i=1}^{q} U_{y_i} + M,$$

where

$$y_i = f(x_i), \qquad i = 1, \ldots, q.$$

We now prove the closedness of $f(X) + M$. Let $\{x_i\}$ be a sequence of points in X which converges to $x_0 \in X$; and let $\{f(x_i) + a_i\}$ be a sequence of points in $f(X) + M$ which converges to $z_0 \in R^k$, $a_i \in M$. Does z_0 belong to $f(X) + M$? By virtue of the M-continuity of f, one can write

$$f(x_i) + a_i = f(x_0) + y_i + b_i + a_i,$$

where $b_i \in M$ and the sequence $\{y_i\}$, $y_i \in R^k$, converges to 0. Hence, the sequence $\{b_i + a_i\}$ is converging and its limit belongs to M, as M is convex and closed. Thus,

$$z_0 \in f(X) + M,$$

and the proof is completed. □

The multiobjective programming problem (Problem P) introduced in Section 1 can be written as follows: Find $x_0 \in X$ such that

$$f(x_0) \in \min[f(x): x \in X, g(x) \in -N \mid M].$$

By Lemma 2.2, we may assume, without loss of generality, that M is a pointed cone, that is, M contains no nontrivial subspace. Throughout this paper, we will also assume that M and N are closed convex cones, X is a nonempty compact set, f is an M-convex, M-continuous function, g is an N-convex, N-continuous function on X, although many results are valid without the convexity and continuity assumptions. We shall write simply $\min[f(x): x \in X]$, instead of $\min[f(x): x \in X \mid M]$, if there is no confusion.

3. Lagrangian Functions

Let Y be the space of all continuous positively homogeneous functions from R^m into R^k; let Y_1 be the space of all linear functions from R^m into

R^k; and let Y_e be the space of functions s from R^m into R^k which are defined as follows:

$$s(z) = e\langle \tilde{s}, z\rangle, \qquad \text{for } z \in R^m;$$

here, $\tilde{s}$ is a vector in R^m determining s, and e is a fixed vector of M. Define

$$Y^+ = \{s \in Y: s \text{ is } M\text{-convex and monotonic, in the sense that } x \in y + N \text{ implies } s(x) \in s(y) + M\},$$

$$Y_1^+ = \{s \in Y_1: s(N) \subseteq M\},$$

$$Y_e^+ = \{s \in Y_e: \tilde{s} \in N^*\}.$$

It is obvious that Y, Y_1, Y_e are linear spaces and that Y^+, Y_1^+, Y_e^+ are convex cones. Moreover,

$$Y_e \subset Y_1 \subset Y, \qquad Y_e^+ \subset Y_1^+ \subset Y^+.$$

According to these spaces, we define Lagrangian functions L, L_1, L_e by the relations below. The function

$$L: R^n \times Y \to \bar{R}^k$$

is defined by

$$L(x, s) = \begin{cases} f(x) + sg(x), & \text{if } x \in X, s \in Y^+, \\ -\infty_M, & \text{if } x \in X, s \notin Y^+, \\ +\infty_M, & \text{if } x \notin X. \end{cases}$$

The functions

$$L_1: R^n \times Y_1 \to \bar{R}^k, \; L_e: R^n \times Y_e \to \bar{R}^k$$

are defined similarly. Note that L_e coincides with the restriction of L_1 on Y_e and that L_1 coincides with the restriction of L on Y_1.

Proposition 3.1. The function $L(x, s)$ is M-convex in x for every fixed $s \in Y$ and is M-concave in s for every fixed $x \in R^n$.

Proof. First, we prove that, for any fixed $s \in Y$, $L(x, s)$ is M-convex in x. By definition, we must verify the convexity of the epigraph

$$G = \{(x, z) \in R^n \times R^k: z \in L(x, s) + M\},$$

in $R^n \times R^k$. Suppose that

$$(x_i, z_i) \in G, \qquad i = 1, 2.$$

We shall prove that, for all α, $0 < \alpha < 1$,

$$\alpha(x_1, z_1) + (1 - \alpha)(x_2, z_2) \in G. \tag{9}$$

If $s \in Y^+$, $x_i \in X$, $i = 1, 2$, then we have

$$L(x_i, s) = f(x_i) + sg(x_i), \qquad i = 1, 2. \tag{10}$$

From the convexity of f and Lemma 2.1, it follows that

$$\alpha f(x_1) + (1-\alpha) f(x_2) \in f(\alpha x_1 + (1-\alpha)x_2) + M. \tag{11}$$

Since $s \in Y^+$ and g is N-convex, we also have

$$\alpha sg(x_1) + (1-\alpha) sg(x_2) - sg(\alpha x_1 + (1-\alpha)x_2) \in M. \tag{12}$$

Combining $(10) \to (12)$, we obtain

$$\alpha L(x_1, s) + (1-\alpha) L(x_2, s) \in L(\alpha x_1 + (1-\alpha)x_2, s) + M.$$

This shows that (9) is satisfied. If $s \in Y^+$ and $x_1 \notin X$ (or $x_2 \notin X$), then

$$L(x_1, s) = +\infty_M \text{ and } (x_1, z) \in G, \qquad \text{for no } z \in R^k.$$

Now, suppose that $s \notin Y^+$. We have two cases to consider.

Case (a). x_1 and x_2 are in X.

Case (b). $x_1 \notin X$ or $x_2 \notin X$.

Case (b) is impossible, as we have just noted. For case (a), $(x, z) \in G$, for every $x \in X$ and $z \in R^k$. Therefore, (9) is always satisfied. Thus, G is a convex set, and the first part of the proposition is proved. For the second part, let

$$G' = \{(s, z) \in Y \times R^k : z \in L(x, s) - M\}.$$

If $x \notin X$, it is obvious that L is M-concave in s. If $x \in X$ and $(s_i, z_i) \in G'$, $i = 1, 2$, then two cases must be examined.

Case (a). s_1 and s_2 are in Y^+.

Case (b). $s_1 \notin Y^+$ or $s_2 \notin Y^+$.

In both cases, we have

$$\alpha(s_1, z_1) + (1-\alpha)(s_2, z_2) \in G', \qquad \text{for all } \alpha,\ 0 < \alpha < 1.$$

This completes the proof. □

Definition 3.1. A point $(x_0, s_0) \in R^n \times Y$ is said to be a saddle point for the function $L(x, s)$ if:

(i) $L(x_0, s_0) \in \min[L(x, s_0) : x \in R^n]$;
(ii) $L(x_0, s_0) \in \max[L(x_0, s) : s \in Y]$.

The definitions of saddle points for the functions L_1 and L_e are similar.

Theorem 3.1. A point (x_0, s_0) is a saddle point for the function L if and only if:

(a) $L(x_0, s_0) \in \min[L(x, s_0): x \in R^n]$;
(b) $g(x_0) \in -N$;
(c) $s_0 g(x_0) = 0$.

Proof. First, suppose that (x_0, s_0) is a saddle point for L. From Definition 3.1, it follows that (a) holds and there is no $s \in Y$ which satisfies

$$L(x_0, s) \in L(x_0, s_0) + M, \qquad L(x_0, s) \neq L(x_0, s_0). \tag{13}$$

It is clear that

$$x_0 \in X \quad \text{and} \quad s_0 \in Y^+;$$

otherwise,

$$L(x_0, s_0) = -\infty_M \text{ or } +\infty_M.$$

Hence,

$$L(x_0, s_0) = f(x_0) + s_0 g(x_0).$$

For $s \notin Y^+$, relation (13) is impossible, so we need only to consider the case $s \in Y^+$, which gives

$$L(x_0, s) = f(x_0) + s g(x_0);$$

and (ii) in Definition 3.1 is equivalent to the fact that there is no $s \in Y^+$ such that

$$(s - s_0) g(x_0) \in M \backslash 0.$$

Suppose, to the contrary, that

$$g(x_0) \notin -N.$$

Applying a separation theorem (Ref. 5) to $-N$ and the compact set $\{g(x_0)\}$ yields the existence of a vector $y \in N^*$ such that

$$\langle y, g(x_0)\rangle > 0 \geqslant \langle y, z\rangle, \qquad \text{for all } z \in -N.$$

Let e be a vector of M, $e \neq 0$. We construct a function $s \in Y^+$ by the following relation:

$$s(z) = e\langle y, z\rangle + s_0(z).$$

It is obvious that

$$(s - s_0) g(x_0) \neq 0, \qquad (s - s_0) g(x_0) \in M.$$

This contradiction shows that (b) holds. From (b), it is easy to see that

$$s_0 g(x_0) = 0;$$

otherwise,

$$s_0 g(x_0) \in -M;$$

and, setting $s = s_0/2$, we arrive at a contradiction to (ii).

Now, suppose that (x_0, s_0) satisfies (a), (b), (c). Condition (a) is the same as (i). From (b) and (c), one has

$$f(x_0) + sg(x_0) \in f(x_0) + s_0 g(x_0) - M, \qquad \text{for all } s \in Y^+.$$

Thus,

$$L(x_0, s_0) \in \max[L(x_0, s): s \in Y],$$

and the proof is completed. □

Corollary 3.1. If $(x_0, s_0) \in R^n \times Y$ is a saddle point for the function $L(x, y)$, then x_0 is a solution to Problem P.

Proof. We have shown in Theorem 3.1 that

$$x_0 \in X \quad \text{and} \quad g(x_0) \in -N.$$

Suppose that x_0 is not a solution to Problem P; that is, there exists a vector $x \in X$, with

$$g(x) \in -N, \qquad f(x) \neq f(x_0), \qquad f(x_0) \in f(x) + M.$$

Since

$$s_0 \in Y^+, \qquad s_0 g(x) \in -M,$$

we have

$$f(x_0) + s_0 g(x_0) \in f(x) + s_0 g(x) + M,$$

where

$$f(x_0) \neq f(x) + s_0 g(x).$$

This contradicts the relation

$$L(x_0, s_0) \in \min[L(x, s_0): x \in R^n],$$

and the proof is completed. □

Corollary 3.2. Every saddle point of the functions L_e and L_l is a saddle point of the functions L_l and L, respectively.

Proof. It is obvious from the equivalence between (ii) in Definition 3.1 and (b), (c) in Theorem 3.1. □

Let S denote the set of all solutions to Problem P; let SY, SY_1, SY_e denote the sets of all $x \in X$, such that (x, s), (x, s_1), (x, s_e) are saddle points of L, L_1, L_e, respectively, for some vectors $s \in Y$, $s_1 \in Y_1$, $s_e \in Y_e$.

Corollary 3.3. The following inclusions hold:

$$SY_e \subseteq SY_1 \subseteq SY \subseteq S.$$

Proof. The proof of this corollary is obvious. □

Definition 3.2. (*Ref. 1*). A point $x_0 \in X$, with $g(x_0) \in -N$, is said to be a proper solution to Problem P if the closure of the set $\{f(x) - f(x_0) + M: x \in X,\ g(x) \in -N\}$ and the cone $(-M)$ intersect each other only at $\{0\}$.

Definition 3.3. We say that Slater's constraint qualification is satisfied if N is a closed convex cone with nonempty interior and there exists $x \in X$ such that $g(x) \in -\text{int } N$.

Theorem 3.2. Suppose that x_0 is a proper solution to Problem P and that Slater's constraint qualification is satisfied. Then, there exists $\tilde{s}_0 \in N^*$ such that (x_0, s_0) is a saddle point of L_e.

Proof. By Lemma 2.4 in Ref. 1, there exists a vector

$$\mu \in \text{int } M^*,$$

which is nonempty, by the assumption on M, such that

$$\langle \mu, f(x_0) \rangle = \min\{\langle \mu, f(x) \rangle: x \in X,\ g(x) \in -N\}.$$

Since

$$\mu \in \text{int } M^*, \qquad e \in M,\ e \neq 0,$$

we have

$$\langle \mu, e \rangle > 0,$$

and hence,

$$\langle \mu, f(x_0) \rangle = \min\{\langle \mu, f(x) \rangle: x \in X,\ \langle \mu, e \rangle g(x) \in -N\}.$$

By a standard Lagrange multiplier theorem in Ref. 6, there exists $\tilde{s}_0 \in N^*$ such that

$$\langle \tilde{s}_0, g(x_0) \rangle = 0$$

and

$$\langle \mu, f(x_0)\rangle + \langle \mu, e\rangle \cdot \langle \tilde{s}_0, g(x_0)\rangle$$
$$\leq \langle \mu, f(x_0)\rangle + \langle \mu, e\rangle \cdot \langle \tilde{s}_0, g(x)\rangle, \qquad \text{for all } x \in X. \tag{14}$$

We claim that

$$f(x_0) \in \min[f(x) + e\langle \tilde{s}_0, g(x)\rangle : x \in X].$$

Indeed, if this is not true, there exists $x \in X$, with

$$f(x_0) \neq f(x) + e\langle \tilde{s}_0, g(x)\rangle,$$
$$f(x_0) - f(x) - e\langle \tilde{s}_0, g(x)\rangle \in M.$$

Hence,

$$\langle \mu, f(x_0)\rangle > \langle \mu, f(x)\rangle + \langle \mu, e\rangle\langle g(x), \tilde{s}_0\rangle,$$

contradicting (14). Now, the theorem follows from Theorem 3.1. □

Definition 3.4. (*Ref. 4*). Let A be a subset of R^k. A vector $x \in A$ is said to be a weak minimal vector of A with respect to M, and we write $x \in W \min A$, if there is no $y \in A$, $y \neq x$, such that $x \in y + \operatorname{int} M$.

Note that it is assumed that $\operatorname{int} M$ is nonempty.

The definitions of weak maximal vectors, weak solutions to Problem P, and weak saddle point are similar.

Theorem 3.3. Suppose that $e \in \operatorname{int} M$ and that Slater's constraint qualification is satisfied. Then, a vector $x_0 \in X$ is a weak solution to Problem P if and only if there is $\tilde{s}_0 \in N^*$ such that (x_0, s_0) is a weak saddle point of L_e.

Proof. Theorem 2 in Ref. 4 assures the existence of $\tilde{s}_0$ such that (x_0, s_0) is a weak saddle point of L_e if x_0 is a weak solution to Problem P. Now, let (x_0, s_0) be a weak saddle point of L_e, for some $\tilde{s}_0 \in N^*$. By definition,

$$L_e(x_0, s_0) \in W \max[L_e(x_0, s) : s \in Y_e].$$

This means that there exists no $\tilde{s} \in N^*$ such that

$$e\langle \tilde{s} - \tilde{s}_0, g(x_0)\rangle \in \operatorname{int} M,$$

or equivalently,

$$\langle \tilde{s} - \tilde{s}_0, g(x_0)\rangle \leq 0, \qquad \text{for all } \tilde{s} \in N^*.$$

From this, it follows that $g(x_0) \in -N$; otherwise, one can find $\tilde{s} \in N^*$ such that

$$\langle \tilde{s}, g(x_0)\rangle > \langle \tilde{s}_0, g(x_0)\rangle;$$

it also follows that

$$\langle \tilde{s}_0, g(x_0)\rangle = 0.$$

Suppose, to the contrary, that x_0 is not a weak solution to Problem P. There is $x \in X$ such that

$$g(x) \in -N, \qquad f(x_0) \in f(x) + \text{int } M.$$

We have

$$\begin{aligned} f(x_0) &= f(x_0) + e\langle \tilde{s}_0, g(x_0)\rangle \\ &\in f(x) + \text{int } M \subset f(x) + e\langle \tilde{s}_0, g(x)\rangle - e\langle \tilde{s}_0, g(x)\rangle + \text{int } M \\ &\subset f(x) + e\langle \tilde{s}_0, g(x)\rangle + \text{int } M. \end{aligned}$$

This contradicts

$$L_e(x_0, s_0) \in W \min[L_e(x, s_0): x \in R^n].$$

Thus, the proof is completed. □

We complete this section by proving the M-continuity of L. For this purpose, we introduce the following definition of norm in the space Y:

$$\|s\| = \max_{\substack{z \in R^m \\ \|z\| \leq 1}} \|s(z)\|, \qquad \text{for } s \in Y.$$

It is clear that this definition of norm is correct and, in this way, Y is a normed space.

Proposition 3.2. $L(x, s)$ is an M-continuous function on $X \times Y^+$.

Proof. Let $(x_0, s_0) \in X \times Y^+$; and let U be an arbitrary neighborhood of $L(x_0, s_0)$ in R^k. Our aim is to find neighborhoods V of x_0 in R^n and W of s_0 in Y such that

$$L(x, s) \in U + M, \qquad \text{for } (x, s) \in (V \times W) \cap (X \times Y^+).$$

First, we note that there exist neighborhoods U_1 of $f(x_0)$ and U_2 of $s_0 g(x_0)$ in R^k, so that

$$U_1 + U_2 \subset M + U.$$

By the continuity of s_0, one can find a neighborhood Q of $g(x_0)$ in R^m such that

$$s_0(z) \in U_2, \qquad \text{for all } z \in Q.$$

Moreover, there exists a neighborhood W of s_0 in Y, with

$$s(Q) \subset U_2, \qquad \text{for all } s \in W.$$

The M-continuity of f and the N-continuity of g yield the existence of neighborhoods V_1 and V_2 of x_0 in R^n such that

$$f(x) \in U_1 + M, \qquad \text{for every } x \in V_1,$$

$$g(x) \in Q + N, \qquad \text{for every } x \in V_2.$$

Set

$$V = V_1 \cap V_2$$

to obtain

$$V \times W \subset R^n \times Y,$$

which is to be found. Indeed, if

$$(x, s) \in (V \times W) \cap (X \times Y^+),$$

then

$$L(x, s) = f(x) + sg(x) \in U_1 + M + s(z + N),$$

where z is in Q. Since $s \in Y^+$, we have

$$s(z + N) \subset s(z) + M.$$

Consequently,

$$s(z + N) \subset U_2 + M;$$

therefore,

$$L(x, s) \in U_1 + M + U_2 + M \subset U + M + M + M \subset U + M.$$

The proposition is proved. □

4. Primal and Dual Functions

According to the Lagrangian functions defined in Section 3, we get the following functions called primal and dual:

$$P(x) = \max[L(x, s): s \in Y],$$

$$D(s) = \min[L(x, s): x \in R^n].$$

The functions $P_1(x)$ and $P_e(x)$ as well as $D_1(x)$ and $D_e(x)$ are defined similarly.

The primal problem (Problem P) can be expressed as follows: Find x_0 in R^n such that

$$P(x_0) \cap \min[P(x): x \in R^n] \neq \varnothing.$$

The dual problem (Problem D) can be expressed as follows: Find s_0 in Y such that

$$D(s_0) \cap \max[D(s): s \in Y] \neq \varnothing.$$

Before giving some features relative to dual and primal problems, observe that the primal problems P, P_1, P_e are the same as the problem introduced in Section 2. Indeed, by the definition of L, we see that

$$\max[L(x, s): s \in Y] = +\infty_M, \qquad \text{if } x \notin X,$$

$$\max[L(x, s): s \in Y] = \max[f(x) + sg(x): s \in Y^+], \qquad \text{if } x \in X.$$

Moreover, if $g(x)$ does not belong to $-N$, then there is $s \in Y_e^+$ such that

$$sg(x) \in M, \qquad sg(x) \neq 0.$$

Hence,

$$\alpha sg(x) \to \infty_M, \qquad \text{as } \alpha \to \infty,$$

$$\max[f(x) + sg(x): s \in Y^+] = +\infty_M.$$

If $g(x)$ is in $-N$, then

$$sg(x) \in -M, \qquad \text{for } s \in Y^+,$$

$$\max[f(x) + sg(x): s \in Y^+] = f(x).$$

In this way,

$$P(x) = \begin{cases} f(x), & \text{if } x \in X \text{ and } g(x) \in -N, \\ \infty_M, & \text{otherwise.} \end{cases}$$

By analogy,

$$P(x) = P_1(x) = P_e(x).$$

Proposition 4.1. For $D(s)$ defined as above, the following statements hold:

(i) $D(s)$ is an M-convex set in R^k, for every $s \in Y$;
(ii) $D(\cdot)$ is an M-concave set-valued function, namely,

$$D(\alpha s_1 + (1-\alpha)s_2) \subset \alpha D(s_1) + (1-\alpha)D(s_2) + M,$$

for all s_1, $s_2 \in Y$ and all α, $0 < \alpha < 1$.

Proof. Statement (i) is obvious. For (ii), we mention that, if A, B, C are arbitrary M-compact sets in R^k, with $A \subset B + C$, then

$$\min A \subset B + C \subset \min B + \min C + M.$$

We apply Lemma 2.4 and Lemma 2.5 to obtain

$$\begin{aligned} D(\alpha s_1 + (1-\alpha)s_2) &\\ &= \min[L(x, \alpha s_1 + (1-\alpha)s_2): x \in R^n] \\ &= \min[\alpha L(x, s_1) + (1-\alpha)L(x, s_2): x \in R^n] \\ &\subset \alpha \min[L(x, s_1): x \in R^n] + (1-\alpha) \min[L(x, s_2): x \in R^n] + M \\ &= \alpha D(s_1) + (1-\alpha)D(s_2) + M. \end{aligned}$$

The proof is completed. □

Proposition 4.2. *Weak Duality.* For any function $s \in Y$ and any vector $x \in R^n$, we have

$$z - P(x) \notin M\backslash 0, \qquad \text{for all } z \in D(s).$$

Proof. Suppose, to the contrary, that there are x, s, z, with

$$z - P(x) \in M, \qquad z \neq P(x), \qquad z \in D(s). \tag{15}$$

It is obvious from (15) that

$$x \in X, \qquad g(x) \in -N, \qquad s \in Y^+.$$

In this case,

$$P(x) = f(x), \qquad sg(x) \in -M.$$

Therefore,

$$\begin{aligned} z \in f(x) + M &\subset f(x) + sg(x) - sg(x) + M \\ &\subset f(x) + sg(x) + M. \end{aligned} \tag{16}$$

Since $z \neq f(x)$, it follows that

$$z \neq f(x) + sg(x).$$

Relation (16) contradicts the assumption $z \in D(s)$. □

Definition 4.1. We say that (x_0, s_0) is a pair of dual solutions to Problems P and D, if x_0 solves Problem P and

$$f(x_0) \in D(s_0) \cap \max[D(s): s \in Y].$$

The last relation shows that s_0 solves Problem D.

Theorem 4.1. For (x_0, s_0) to be a saddle point of L, it is sufficient and necessary that it be a pair of dual solutions to Problems P and D.

Proof. Suppose that (x_0, s_0) is a saddle point for L. From Corollary 3.1, we see that x_0 is a solution to Problem P. To prove that (x_0, s_0) is a pair of dual solutions, we must show that

$$f(x_0) \in D(s_0) \cap \max[D(s): s \in Y]. \tag{17}$$

For this purpose, notice that

$$D(s_0) = \min[L(x, s_0): x \in R^n], \qquad s_0 \in Y^+;$$

hence,

$$f(x_0) \in D(s_0).$$

If

$$f(x_0) \notin \max[D(s): s \in Y],$$

then there exists $s \in Y$ such that, for some $z \in D(s)$,

$$z \neq f(x_0), \qquad z \in f(x_0) + M. \tag{18}$$

It is obvious that $s \in Y^+$ and

$$D(s) = \min[f(x) + sg(x): x \in R^n].$$

Hence,

$$z = f(x) + sg(x), \qquad \text{for some } x \in X,$$

and (18) yields

$$f(x) + sg(x) \in f(x_0) + M. \tag{19}$$

Since

$$sg(x_0) \in -M,$$

(19) can be rewritten as follows:

$$\begin{aligned} f(x) + sg(x) &\in f(x_0) + sg(x_0) - sg(x_0) + M \\ &\subset f(x_0) + sg(x_0) + M. \end{aligned} \tag{20}$$

As M is a pointed convex cone, it is easy to see that

$$f(x) + sg(x) \neq f(x_0) + sg(x_0).$$

This, together with (20), shows that

$$z = f(x) + sg(x)$$

cannot be in $D(s)$. Thus,

$$f(x_0) \in \max[D(s): s \in Y],$$

and (17) follows. Now, assume that x_0 solve Problem P and that (17) holds. We have to show that (x_0, s_0) is a saddle point of L. Suppose that

$$s_0 g(x_0) = 0.$$

Then,

$$L(x_0, s_0) = f(x_0) = f(x_0) + s_0 g(x_0) \in D(s_0) = \min[L(x, s_0): x \in R^n],$$

and clearly,

$$g(x_0) \in -N.$$

From this, it follows that (x_0, s_0) is a saddle point of L (Theorem 3.1). To complete the proof, we have only to show that

$$s_0 g(x_0) = 0.$$

If this does not hold, i.e.,

$$s_0 g(x_0) \neq 0,$$

obviously $s_0 \in Y^+$; hence,

$$s_0 g(x_0) \in -M \backslash 0.$$

Therefore,

$$f(x_0) \neq f(x_0) + s_0 g(x_0),$$

$$f(x_0) \in f(x_0) + s_0 g(x_0) + M,$$

contradicting (17), and the theorem is proved. □

Remark 4.1. All the results presented above are valid if, instead of D and L, we consider D_1 and L_1 or D_e and L_e.

Corollary 4.1. Suppose that x_0 is a proper solution to Problem P and that Slater's constraint qualification is satisfied. Then,

$$f(x_0) \in \max[D_e(s): s \in Y_e].$$

Proof. Theorem 3.2 assures that there exists $\tilde{s}_0 \in N^*$ such that (x_0, s_0) is a saddle point of L_e. Now, the corollary follows directly from Theorem 4.1. □

5. Perturbation Function

Let us define an M-convex function $F(x)$ on R^n by means of the relation

$$F(x)=\begin{cases} f(x), & \text{if } x\in X \text{ and } g(x)\in -N,\\ +\infty_M, & \text{otherwise.}\end{cases}$$

As we have noticed in Section 4,

$$F(x)=P(x),$$

and Problem P can be written equivalently as the following unconstrained minimization problem for the function F: Find $x_0\in R^n$ such that

$$F(x_0)\in \min[F(x)\colon x\in R^n].$$

Let us denote the perturbation vector u, $u\in R^m$. We can generalize $F(x)$ to $F(x, u)$,

$$F(x,u)=\begin{cases} f(x), & \text{if } x\in X \text{ and } g(x)\in -u-N,\\ +\infty_M, & \text{otherwise.}\end{cases}$$

Lemma 5.1. $F(x, u)$ is an M-convex function on $R^n\times R^m$.

Proof. We have to prove the convexity of the epigraph

$$G=\{(x, u, z)\in R^n\times R^m\times R^k\colon z\in F(x, u)+M\},$$

in $R^n\times R^m\times R^k$. For this purpose, let

$$(x_i, u_i, z_i)\in G, \qquad i=1,2, \qquad 0<\alpha<1.$$

It is easy to see that

$$x_i\in X, \quad g(x_i)\in -u_i-N, \qquad i=1,2.$$

Hence,

$$\alpha x_1+(1-\alpha)x_2\in X,$$

by the convexity of X, and

$$g(\alpha x_1+(1-\alpha)x_2)\in \alpha g(x_1)+(1-\alpha)g(x_2)-N\subset -\alpha u_1-(1-\alpha)u_2-N,$$

by the N-convexity of g. Therefore,

$$F(\alpha x_1+(1-\alpha)x_2, \alpha u_1+(1-\alpha)u_2)=f(\alpha x_1+(1-\alpha)x_2);$$

consequently,

$$\begin{aligned}\alpha z_1+(1-\alpha)z_2&\in \alpha F(x_1, u_1)+(1-\alpha)F(x_2, u_2)+M\\ &\subset \alpha f(x_1)+(1-\alpha)f(x_2)+M\subset f(\alpha x_1+(1-\alpha)x_2)+M,\end{aligned}$$

by the M-convexity of f. Thus, G is convex, and the lemma is proved. □

Theorem 5.1. Let the functions L_e and F be as above. These functions are conjugate, in the sense made precise by the following relations:

(i) $L_e(x, s) = \min[F(x, u) - e\langle \tilde{s}, u\rangle : u \in R^m]$,
(ii) $F(x, u) = \max[L_e(x, s) + e\langle \tilde{s}, u\rangle : \tilde{s} \in R^m]$.

Proof. If x is not in X, then (i) holds trivially. Suppose that x belongs to X. We have

$$\begin{aligned}
&\min[F(x, u) - e\langle \tilde{s}, u\rangle : u \in R^m] \\
&\quad = \min[\{F(x, u) - e\langle \tilde{s}, u\rangle : -u \in g(x) + N\} \\
&\qquad \cup \{F(x, u) - e\langle \tilde{s}, u\rangle : -u \notin g(x) + N\}] \\
&\quad = \min[\{F(x, u) - e\langle \tilde{s}, u\rangle : -u \in g(x) + N\} \cup \infty_M] \\
&\quad = \begin{cases} f(x) + e\langle \tilde{s}, g(x)\rangle, & \text{if } \tilde{s} \in N^*, \\ -\infty_M, & \text{otherwise}, \end{cases} \\
&\quad = L_e(x, s).
\end{aligned}$$

Similarly, if x is not in X, then (ii) holds trivially. Suppose that $x \in X$. We have

$$\begin{aligned}
&\max[L_e(x, s) + e\langle \tilde{s}, u\rangle : \tilde{s} \in R^m] \\
&\quad = \max[\{L_e(x, s) + e\langle \tilde{s}, u\rangle : \tilde{s} \in N^*\} \\
&\qquad \cup \{L_e(x, s) + e\langle \tilde{s}, u\rangle : \tilde{s} \notin N^*\}] \\
&\quad = \max[\{L_e(x, s) + e\langle \tilde{s}, u\rangle : \tilde{s} \in N^*\} \cup \{-\infty_M\}] \\
&\quad = \max[f(x) + e\langle \tilde{s}, g(x)\rangle + e\langle \tilde{s}, u\rangle : \tilde{s} \in N^*] \\
&\quad = \begin{cases} f(x), & \text{if } g(x) \in -u - N, \\ +\infty_M, & \text{otherwise}, \end{cases} \\
&\quad = F(x, u).
\end{aligned}$$

This completes the proof. □

Remark 5.1. By writing $L_e(x, s) = \min[\ldots]$ and $F(x, u) = \max[\ldots]$, we say that the sets $\min[\ldots]$ and $\max[\ldots]$ are points and they coincide with $L_e(x, s)$ and $F(x, u)$, respectively.

Let Z be the space of all continuous positively homogeneous functions from R^n into R^k; let Z_1 and Z_e be defined by analogy with Y_1 and Y_e. The conjugate functions F_e^*, F_1^*, F^* of F, with vector variable, matrix variable, and function variable are respectively the following:

$$F_e^* : Z_e \times Y_e \to R^k, \qquad F_1^* : Z_1 \times Y_1 \to R^k, \qquad F^* : Z \times Y \to R^k.$$

The function F^* is defined as follows:

$$F^*(t, s) = \max[tx + su - F(x, u): (x, u) \in R^n \times R^m].$$

The functions F_1^* and F_e^* are defined similarly.

Note that the notion max in the definition of F^* is not very correct. We shall interpret it as max of the closure of the set

$$\{tx + su - F(x, u): (x, u) \in R^n \times R^m\}.$$

Proposition 5.1. F_e^*, F_1^*, F^* are set-valued M-convex functions, namely,

$$F^*(\alpha t_1 + (1-\alpha)t_2, \alpha s_1 + (1-\alpha)s_2) \subset \alpha F(t_1, s_1) + (1-\alpha)F(t_2, s_2) - M.$$

Proof. To prove this proposition, it is sufficient to remark that for arbitrary $(-M)$-compact sets A, B, C in R^k, with $A \subset B + C$, one has

$$\max A \subset B + C \subset \max B + \max C - M. \qquad \square$$

The dual problem (Problem D') is: Find $s_0 \in Y$ such that

$$(-F^*(0, s_0)) \cap \max[-F^*(0, s): s \in Y] \neq \varnothing.$$

Problems D_e' and D_1' are defined similarly.

Proposition 5.2. Problems D_e' and D_e coincide.

Proof. By definition of F_e^*, we have

$$\begin{aligned}
-F_e^*(0, s) &= -\max[e\langle \tilde{s}, u\rangle - F(x, u): (x, u) \in R^n \times R^m] \\
&= \min[F(x, u) - e\langle \tilde{s}, u\rangle: (x, u) \in R^n \times R^m] \\
&= \min[f(x) - e\langle \tilde{s}, u\rangle: x \in X, u \in -g(x) - N] \\
&= \begin{cases} \min[f(x) + e\langle \tilde{s}, g(x)\rangle: x \in X], & \text{if } \tilde{s} \in N^*, \\ -\infty_M, & \text{otherwise}, \end{cases} \\
&= \min[L_e(x, s): x \in R^n] \\
&= D_e(s).
\end{aligned}$$

The proof is completed. $\square$

It is interesting to note that Problems D$'$ and D_1' are different from Problems D and D_1, respectively. However, if we define the Lagrangian functions relative to perturbation functions, then we can ensure the required coincidence. Indeed, let

$$\bar{L}(x, u, s): R^n \times R^m \times Y \to R^k$$

be defined as follows:

$$\bar{L}(x, u, s) = \begin{cases} f(x) + su, & \text{if } x \in X,\ u \in g(x) + N, \\ +\infty_M, & \text{otherwise.} \end{cases}$$

The primal and dual functions will be:

$$\bar{P}(x) = \max[\bar{L}(x, u, s): s \in Y],$$

$$\bar{D}(s) = \min[\bar{L}(x, u, s): (x, u) \in R^n \times R^m].$$

It must be noted that

$$\bar{P}(x) = +\infty_M, \qquad \text{when } u \notin g(x) + N,$$

and $\bar{P}(x)$ does not depend on u, if

$$u \in g(x) + N.$$

The more precise statement will be clear in Proposition 5.3.

The primal problem (Problem P) can be expressed as follows: Find $x_0 \in R^n$ such that

$$\bar{P}(x_0) \cap \min[\bar{P}(x): x \in R^n] \neq \varnothing.$$

The dual problem (Problem $\bar{\mathrm{D}}$) can be expressed as follows: Find $s_0 \in Y$ such that

$$\bar{D}(s_0) \cap \max[\bar{D}(s): s \in Y] \neq \varnothing.$$

Proposition 5.3. Problems $\bar{\mathrm{D}}$ and D′ coincide. The same result is valid for Problems $\bar{\mathrm{P}}$ and P.

Proof. Let us calculate $-F^*(0, s)$ as follows:

$$\begin{aligned} -F^*(0, s) &= -\max[su - F(x, u): (x, u) \in R^n \times R^m] \\ &= \min[F(x, u) - su: (x, u) \in R^n \times R^m] \\ &= \min[f(x) - su: x \in X, u \in -g(x) - N] \\ &= \min[f(x) + su: x \in X, u \in g(x) + N] \\ &= \min[\bar{L}(x, u, s): (x, u) \in R^n \times R^m] \\ &= \bar{D}(s). \end{aligned}$$

For Problems $\bar{\mathrm{P}}$ and P, we see that, if

$$u \in g(x) + N, \qquad u \neq 0,$$

then there are $s_i \in Y$ such that

$$s_i u \to \infty_M, \qquad \text{as } i \to \infty.$$

This means that

$$\bar{P}(x)=\begin{cases} f(x), & \text{if } x\in X \text{ and } 0\in g(x)+N,\\ +\infty_M, & \text{otherwise.}\end{cases}$$

This completes the proof. □

Let $\bar{Y}^+$ denote the set of all $s\in Y$, such that

$$s(-N)\cap M=\{0\}.$$

We say that $(x_0, s_0)\in R^n\times Y$ is a saddle point of $\bar{L}$ if:

$$\bar{L}(x_0, g(x_0), s_0)\in \min[\bar{L}(x, u, s_0): (x, u)\in R^n\times R^m],$$

$$\bar{L}(x_0, g(x_0), s_0)\in \max[\bar{L}(x_0, g(x_0), s): s\in Y^+].$$

It can be proved that many results in Sections 3 and 4 are valid for $\bar{L}$.

6. Scalar Lagrangian Function

Let l' be a fixed nonzero vector of M. Corresponding to this vector, we define a scalar Lagrangian function for Problem P as follows:

$$l(x, \tilde{s})=\begin{cases} \langle l', f(x)\rangle+\langle \tilde{s}, g(x)\rangle, & \text{if } x\in X,\ \tilde{s}\in N^*,\\ -\infty, & \text{if } x\in X,\ \tilde{s}\notin N^*,\\ +\infty, & \text{if } x\notin X.\end{cases}$$

It is easy to see that $l(x, \tilde{s})$ is convex in x, for any fixed $\tilde{s}$, and is concave in $\tilde{s}$, for any fixed x. Furthermore, it is a lower semicontinuous function on $X\times N^*$.

Let us recall some relationships between the solutions of Problem P and the solutions of the scalarized problem. Denote

$S(\lambda)=\{x_0\in X: x_0$ minimizes $\langle\lambda, f(x)\rangle$ on X, under the constraint $g(x)\in -N\}$.

Lemma 6.1. The following inclusions hold:

$$\{S(\lambda): \lambda\in \hat{M}, \|\lambda\|=1\}\subseteq S\subseteq \{S(\lambda): \lambda\in M^*, \|\lambda\|=1\}.$$

Proof. The proof of this lemma was presented in Ref. 7. It can be verified directly, without any difficulty. □

Proposition 6.1. Assume that x_0 is a solution to Problem P and that Slater's constraint qualification is satisfied. Then, there exist a vector $l'\in M^*$, $l'\neq 0$, and a vector $\tilde{s}_0\in N^*$ such that $(x_0, \tilde{s}_0)$ is a saddle point of $l(x, \tilde{s})$.

Proof. Apply Lemma 6.1 and the proof of Theorem 3.2 to get l' and $\tilde{s}_0$. □

Corollary 6.1. If $(x_0, \tilde{s}_0)$ is a saddle point of $l(x, \tilde{s})$ for some $l' \in \text{int } M^*$, then x_0 is a solution to Problem P.

Proof. The proof of this corollary is evident. □

Proposition 6.2. If (x_0, s_0) is a saddle point of $L_e(x, s)$, defined in Section 3, then there exists a vector l' in M^* such that either $(x_0, \tilde{s}_0)$ or $(x_0, 0)$ is a saddle point of $l(x, \tilde{s})$.

Proof. This proposition follows at once from Theorem 3.1 and Lemma 6.1. □

Corollary 6.2. If, in addition to the assumptions concerning M, we assume that int M is nonempty, then $(x_0, \tilde{s}_0)$ in Proposition 6.2 will be a saddle point of $l(x, \tilde{s})$.

Proof. This is the case when

$$\langle \lambda, e \rangle \neq 0.$$

Therefore, $(x_0, \tilde{s}_0)$ is a saddle point of $l(x, \tilde{s})$, with

$$l' = \lambda / \langle \lambda, e \rangle.$$ □

Proposition 6.3. If $(x_0, \tilde{s}_0)$ is a saddle point of $l(x, \tilde{s})$, for some $l' \in \text{int } M^*$, then (x_0, s_0) is a saddle point of the Lagrangian function $L_e(x, s)$, with the vector $e/\langle l', e \rangle$ replacing the vector e.

Proof. This proposition follows at once from Theorem 3.1 and Lemma 6.1. □

According to the Lagrangian $l(x, s)$, we define a primal problem (Problem p) and a dual problem (Problem d) as follows:

$$\min p(x),$$

$$\text{s.t. } x \in R^n, \text{ where } p(x) = \max\{l(x, \tilde{s}): \tilde{s} \in R^m\};$$

$$\max d(\tilde{s}),$$

$$\text{s.t. } \tilde{s} \in R^m, \text{ where } d(\tilde{s}) = \min\{l(x, \tilde{s}): x \in R^n\}.$$

We say that $(x_0, \tilde{s}_0)$ is a pair of dual solutions to Problems p and d, if x_0 solves Problem p, $\tilde{s}_0$ solves Problem d, and of course

$$d(\tilde{s}_0) = p(x_0).$$

It is clear that $p(x)$ is a convex function and $d(\tilde{s})$ is a concave function. It is also known from the general theory (see Ref. 5) that, for $(x_0, \tilde{s}_0)$ to be a pair of dual solutions, it is sufficient and necessary that $(x_0, \tilde{s}_0)$ be a saddle point of $l(x, \tilde{s})$.

Proposition 6.4. Suppose that (x_0, s_0) is a pair of dual solutions to Problems P_e and D_e, defined in Section 4. Then, there exists a vector $l' \in M^*$, such that either $(x_0, \tilde{s}_0)$ or $(x_0, 0)$ is a pair of dual solutions to Problems p and d.

Proof. This proposition follows directly from Theorem 4.1, Proposition 6.2, and the remarks made above. □

Proposition 6.5. Suppose that $(x_0, \tilde{s}_0)$ is a pair of dual solutions to Problems p and d, for some $l' \in \operatorname{int} M^*$. Then, (x_0, s_0) is a pair of dual solutions to Problems P_e and D_e, with $e/\langle l', e\rangle$ replacing e.

Proof. This is an immediate consequence of Theorem 4.1 and Proposition 6.3. □

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