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Optimal Control of a Rotary Crane[1]

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Abstract. This paper is concerned with the optimal control of a rotary crane, which makes two kinds of motion (rotation and hoisting) at the same time. The optimal control which transfers a load to a desired place as fast as possible and minimizes the swing of the load during the transfer, as well as the swing at the end of transfer, is calculated on the basis of a dynamic model. A new computational technique is employed for computing the optimal control, and several numerical results are presented.

Key Words. Optimal control, rotary crane, nonlinear systems, computational algorithms.

1. Introduction

Cranes may be classified into two types, according to their fundamental motion: one is the overhead traveling cranes, and the other is the rotary cranes. Optimal control of the overhead traveling cranes has been studied by many authors. For example, Martensson computed the optimal control, based on a torque control model as well as an acceleration control model (Ref. 1). However, although the rotary cranes are widely used, the optimal control has not yet been discussed so far to our knowledge.

The fundamental motion of rotary cranes is rotation, load hoisting, and boom hoisting. In this paper, we first derive the equations of coupled motion of rotation and load hoisting for the rotary cranes whose boom angle is kept constant. On the basis of the dynamical model derived, we

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calculate the optimal control which transfers a load to a desired place as fast as possible and at the same time minimizes the swing of the load during the transfer as well as the swing at the end of transfer. For the numerical computation, we employ a new computational method. Several numerical results indicate that the algorithm that we employ works out our problem very well and that the algorithm can be applied to solving a wide range of optimal control problems.

2. Dynamical Model of a Rotary Crane

For simplicity, we make the following assumptions.

(A1) The body of a crane and a load can be regarded as a rigid body and a material point, respectively.

(A2) Frictional torques which may exist in torque-transfer mechanisms can be neglected.

(A3) Boom angle and boom length are constant.

Strictly speaking, a rotary crane with load is not a rigid body, but an elastic body. In the case of installed rotary cranes, most of which are electrically driven, they are usually so designed that they have enough strength to be regarded as a rigid body. In the case of mobile cranes, since they are so designed that the body is not so heavy, they are more elastic than the installed cranes. Even in the case of the mobile cranes, according to experimental results given by Ito (Ref. 2), the elastic deformation of the boom of the crane is at most 1/100 of the length of the boom. In this paper, we consider the installed rotary cranes. Therefore, Assumption (A1) is reasonable for our control problem.

The following notations, which are shown in Fig. 1, are used in what follows:

(ξ, η, ζ) = coordinate of a load.

P_1 = lower end of a boom.

P_2 = upper end of a boom.

P_3 = projection of P_2 on the (ξ, η)-plane.

P_4 = projection of the load on the (ξ, η)-plane.

L = length of rope which is controlled by a hoisting motor.

H = constant height of point P_2.

R = constant radius of rotation.

θ = angle between OP_3 and the ξ-axis, which is controlled by a rotation motor.

α = angle between OP_3 and P_3P_4.

β = angle between rope and vertical line.

m = mass of the load.

F = tension of the rope.

D = distance between points P_3 and P_4, that is, swing of the load.

J_1 = moment of inertia of a hoisting drum as well as hoisting motor.

J_2 = moment of inertia of the rotary crane with respect to the ζ-axis.

ϕ = angle of rotation of the hoisting drum, which is equal to the angle of rotation of the hoisting motor.

b = radius of the drum.

T_1 = driving torque generated by the hoisting motor.

T_2 = driving torque generated by the rotation motor.

g = acceleration of gravity (9.81 m/sec^2).

τ = time.

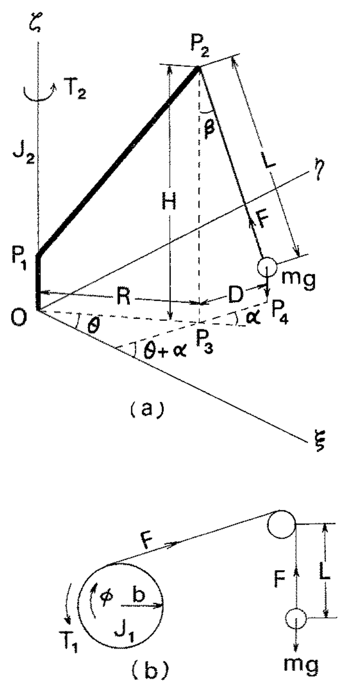

Fig. 1. Notations for a rotary crane.

We see from Fig. 1 that

$$\xi = R \cos \theta + L \sin \beta \cos(\theta + \alpha), \tag{1a}$$

$$\eta = R \sin \theta + L \sin \beta \sin(\theta + \alpha), \tag{1b}$$

$$\zeta = H - L \cos \beta. \tag{2}$$

The equations of motion of the load are given by

$$m(d^2\xi/d\tau^2) = -F \sin\beta \cos(\theta+\alpha), \tag{3}$$

$$m(d^2\eta/d\tau^2) = -F \sin\beta \sin(\theta+\alpha), \tag{4}$$

$$m(d^2\zeta/d\tau^2) = -mg + F\cos\beta. \tag{5}$$

The equations of motion for the rotation of the drum and the rotation of the crane are respectively given by

$$J_1(d^2\phi/d\tau^2) = bF - T_1, \tag{6}$$

$$J_2(d^2\theta/d\tau^2) = T_2 + RF\sin\alpha\sin\beta. \tag{7}$$

We assume that the angle β is so small that the approximations

$$\cos\beta \cong 1, \qquad \sin\beta \cong \beta$$

hold. Then, from (2) we see that

$$\zeta = H - L. \tag{8}$$

Moreover, using (5), (8), and the relation

$$b(d^2\phi/d\tau^2) = d^2L/d\tau^2,$$

we obtain

$$mb(d^2\phi/d\tau^2) = mg - F. \tag{9}$$

Eliminating $d^2\phi/d\tau^2$ from (6) and (9) gives

$$F = [mgJ_1/(J_1+mb^2)](1+bT_1/gJ_1). \tag{10}$$

By using (1), (2), (10), the equations of motion (3), (4), (5), (7) can be rewritten as

$$\begin{aligned}
d^2\xi/d\tau^2 &= [gJ_1/(J_1+mb^2)](1+bT_1/gJ_1)[(R\cos\theta-\xi)/L],\\
d^2\eta/d\tau^2 &= [gJ_1/(J_1+mb^2)](1+bT_1/gJ_1)[(R\sin\theta-\eta)/L],\\
d^2\zeta/d\tau^2 &= -d^2L/d\tau^2 = [gJ_1/(J_1+mb^2)](bT_1/gJ_1 - mb^2/J_1),\\
d^2\theta/d\tau^2 &= T_2/J_2 + (mR/J_2)[gJ_1/(J_1+mb^2)](1+bT_1/gJ_1)\\
&\qquad \times[(\eta\cos\theta - \xi\sin\theta)/L].
\end{aligned} \tag{11}$$

Now, we define the dimensionless variables

$$X = \xi/R, \qquad Y = \eta/R, \qquad Z = \zeta/R, \qquad l = L/R \tag{12}$$

and the parameters

$$h = H/R, \qquad \sigma = mb^2/J_1, \qquad \rho = mR^2/J_2. \tag{13}$$

In addition, we define the dimensionless time t by

$$t=\sqrt{K}\,\tau,$$

where

$$K=gJ_1/R(J_1+mb^2)=g/R(1+\sigma). \tag{14}$$

Furthermore, let us define the state variables and control variables as follows:

$$\begin{aligned} &x_1=X, \quad x_2=\dot{X}, \quad x_3=Y, \quad x_4=\dot{Y},\\ &x_5=Z, \quad x_6=\dot{Z}, \quad x_7=\theta, \quad x_8=\dot{\theta},\\ &u_1=(b/gJ_1)T_1, \quad u_2=(1/KJ_2)T_2, \end{aligned} \tag{15}$$

where the dot denotes the derivative with respect to t. Then, (11) can be rewritten as

$$\begin{aligned} \dot{x}_1&=x_2,\\ \dot{x}_2&=(1+u_1)(\cos x_7-x_1)/(h-x_5),\\ \dot{x}_3&=x_4,\\ \dot{x}_4&=(1+u_1)(\sin x_7-x_3)/(h-x_5),\\ \dot{x}_5&=x_6,\\ \dot{x}_6&=u_1-\sigma,\\ \dot{x}_7&=x_8,\\ \dot{x}_8&=u_2+\rho(1+u_1)[(x_3\cos x_7-x_1\sin x_7)/(h-x_5)]. \end{aligned} \tag{16}$$

Defining the state vector

$$x=(x_1,\ldots,x_8)^T$$

and the control vector

$$u=(u_1,u_2)^T,$$

where T denotes the transpose, we write (16) simply as

$$\dot{x}(t)=f(x(t),u(t)). \tag{17}$$

3. Control Problem

We now consider the following problem: Transfer a load which is initially on the surface of the Earth ($\theta=0$, $\zeta=0$) to the desired position in time t_1, so that the load is in complete rest at the end of transfer. Let the desired position be expressed by the angle of rotation θ_1 and the height H_1. Then, the initial condition

$$x(0) = x^I$$

and the final condition

$$x(t_1) = x^F$$

are respectively given by

$$\begin{aligned} x^I &= (1, 0, 0, 0, 0, 0, 0, 0)^T, \\ x^F &= (\cos\theta_1, 0, \sin\theta_1, 0, h_1, 0, \theta_1, 0)^T, \end{aligned} \tag{18}$$

where

$$h_1 = (H_1/R).$$

By defining a vector function

$$\psi(x) = x - x^F \in R^8,$$

the terminal constraint is given by

$$\psi(x(t_1)) = 0. \tag{19}$$

Since the torques of the driving motors are limited, it is natural to assume that

$$0 \le u_1(t) \le u_{1\,\max}, \qquad |u_2(t)| \le u_{2\,\max}, \tag{20}$$

where

$$u_{1\,\max} = (b/gJ_1)T_{1\,\max}, \qquad u_{2\,\max} = (1/KJ_2)T_{2\,\max}, \tag{21}$$

and $T_{1\,\max}$ and $T_{2\,\max}$ are the maximum torques, respectively. In the same way, the hoisting velocity is limited by its maximum value $V_{1\,\max}$, and the angular velocity of rotation is also limited by its maximum value $V_{2\,\max}$. Consequently, we assume that

$$|x_6(t)| \le v_{1\,\max}, \tag{22}$$

$$|x_8(t)| \le v_{2\,\max}, \tag{23}$$

where

$$v_{1\,\max} = V_{1\,\max}/(\sqrt{K}R), \qquad v_{2\,\max} = V_{2\,\max}/\sqrt{K}.$$

Let L_0 be the minimum rope length. Then, $x_5(t)$ must satisfy

$$0 \le x_5(t) \le x_{5\,\max}, \tag{24}$$

where

$$x_{5\,\max} = (H - L_0)/R. \tag{25}$$

Since it is difficult to include directly the state-variable constraints (22) to (24) in any computational scheme, we take the constraint (23) into consideration in the form of a penalty function in the cost functional, which will be stated later. As for (22) and (24), we apply the constraining hyperplane technique proposed by Martensson (Ref. 3). The basic idea in the constraining hyperplane technique is to approximate the state-variable constraint of the form

$$S(x(t)) \leq 0$$

by a mixed state-control-variable constraint of the form

$$g(x(t), u(t)) \leq 0.$$

In our case, let α_i, $i = 1, 2, 3$, be arbitrary positive numbers such that $\alpha_2 \neq \alpha_3$, and let

$$g(x, u) = \begin{bmatrix} u_1 - \sigma + \alpha_1(x_6 - v_{1\,\max}) \\ -u_1 + \sigma - \alpha_1(x_6 + v_{1\,\max}) \\ u_1 - \sigma + (\alpha_2 + \alpha_3)x_6 + \alpha_2\alpha_3(x_5 - x_{5\,\max}) \\ -u_1 + \sigma - (\alpha_2 + \alpha_3)x_6 - \alpha_2\alpha_3 x_5 \\ u_1 - u_{1\,\max} \\ -u_1 \\ u_2 - u_{2\,\max} \\ -u_2 - u_{2\,\max} \end{bmatrix}. \tag{26}$$

Then, under suitable initial conditions which are usually satisfied, the condition

$$g(x(t), u(t)) \leq 0 \tag{27}$$

gives a sufficient condition for the state constraints (22) and (24) and the control constraint (20) to hold. The last four components in (27) represent the control constraint (20). If all the values of the positive parameters $\alpha_1, \alpha_2, \alpha_3$ tend to infinity, then the first four components in (27) tend to represent the state constraints (22) and (24). Therefore, we replace hereafter the conditions (20), (22), (24) by the condition (27), and we consider the optimal control under (27), so that computation is feasible.

It is clear that the condition (27) is rewritten as

$$\max\{0, \sigma-\alpha_1(x_6(t)+v_{1\max}), \sigma-(\alpha_2+\alpha_3)x_6(t)-\alpha_2\alpha_3x_5(t)\} \le u_1(t)$$
$$\le \min\{u_{1\max}, \sigma-\alpha_1(x_6(t)-v_{1\max}), \sigma-(\alpha_2+\alpha_3)x_6(t)$$
$$-\alpha_2\alpha_3(x_5(t)-x_{5\max})\},$$
$$|u_2(t)| \le u_{2\max},$$

and that the control region is given by a variable rectangular region as shown in Fig. 2. We denote by $U(x)$ the rectangular control region, namely,

$$U(x)=\{u: g(x, u)\le 0\}\subset R^2.$$

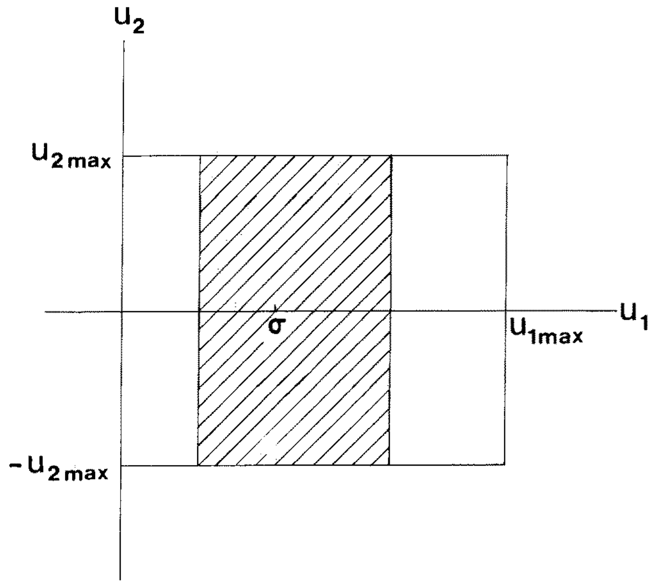

Fig. 2. Control region.

If we apply the constraining hyperplane technique (Ref. 3) to (23), then the control region cannot be a rectangle. That is why we consider the constraint (23) in the form of a penalty function in the cost functional.

Let us now consider the controllability of the system (17). Since (17) is nonlinear, we only consider local controllability at the end point x^F. The control

$$u^F = u(t_1)$$

which keeps the system at a standstill must satisfy

$$f(x^F, u^F) = 0.$$

Using (16) and (18), we obtain

$$u^F = (\sigma, 0).$$

Let us define the 8×8 matrix A and the 8×2 matrix B by

$$A = f_x(x^F, u^F) = [\partial f_i(x^F, u^F)/\partial x_j],$$

$$B = f_u(x^F, u^F) = [\partial f_i(x^F, u^F)/\partial u_j].$$

It can be easily seen that

$$\operatorname{rank}[B, AB, \ldots, A^7 B] = 6. \tag{28}$$

Therefore, the system (17) does not satisfy a sufficient condition for local controllability at the endpoint (Ref. 4). This fact tells us that it will not be easy to find the control which transfers a load to the desired final state x^F.

4. Optimal Control Problem and Necessary Conditions for Optimality

The problem now is to find the optimal control which transfers the initial state x^I of the system to the desired terminal state x^F as fast as possible and, at the same time, minimizes the swing of the load during the transfer. We define the swing of the load by

$$D(x) = [(\cos x_7 - x_1)^2 + (\sin x_7 - x_3)^2]^{1/2}. \tag{29}$$

Let $h[S]$ be the step function such that

$$h[S] = 1, \qquad \text{if } S \geq 0,$$

$$h[S] = 0, \qquad \text{if } S < 0.$$

Corresponding to the state constraint (23), let us define

$$S(x) = |x_8| - v_{2\,\max}(1 - \gamma),$$

where γ, $0 \leq \gamma < 1$, is a constant to be adjusted so that the constraint (23) holds. Now, the cost functional to be minimized is given by

$$J(u) = \int_0^{t_1} \tfrac{1}{2}[D^2(x(t)) + wh[S(x(t))]S^2(x(t))]\, dt, \tag{30}$$

where the state constraint (23) is incorporated as a penalty term and $w > 0$ is a penalty constant. Here, the terminal time t_1 is fixed, so that it is possible to reach the terminal state

$$\psi(x(t_1)) = 0$$

in time t_1 under the constraint (27).

Let us define

$$\theta(x) = \tfrac{1}{2}\|\psi(x)\|_Q^2 = \tfrac{1}{2}\psi^T(x)Q\psi(x),$$

where Q is a positive-definite diagonal matrix of scaling factors and T denotes the transpose of a matrix. Then, the terminal constraint

$$\psi(x(t_1)) = 0$$

can be written as

$$\theta(x(t_1)) = 0. \tag{31}$$

Since the minimization of

$$\theta(x(t_1)) + \int_0^{t_1} \hat{L}(x(t), u(t))\, dt$$

is equivalent to the minimization of

$$\int_0^{t_1} [\hat{L}(x, u) + \theta_x(x)f(x, u)]\, dt,$$

where

$$\theta_x = (\partial\theta/\partial x_1, \ldots, \partial\theta/\partial x_n),$$

we include the terminal constraint (31) in the cost, and we consider the cost functional

$$J(u) = \int_0^{t_1} L(x(t), u(t))\, dt, \tag{32}$$

where

$$L(x, u) = \tfrac{1}{2}[D^2(x) + wh[S(x)]S^2(x)] + \psi^T(x)Qf(x, u). \tag{33}$$

By letting t_1 be as small as possible in the course of iteration of computation, we can obtain a satisfactory solution to the original optimal control problem.

Let $u^*(t)$ be the optimal control, and let $x^*(t)$ be the corresponding optimal trajectory satisfying (17), (19), (27). Then, it is necessary that there exist a continuous row-vector function

$$\lambda^*(t) = (\lambda_1^*(t), \ldots, \lambda_n^*(t)),$$

a piecewise continuous row-vector function

$$\mu^*(t) = (\mu_1^*(t), \ldots, \mu_m^*(t)),$$

where m is the dimension of the constraint vector g, and a constant n-dimensional row-vector b, such that the following conditions are fulfilled (Refs. 5 and 6):

$$\text{(i)}\quad d\lambda^*(t)/dt = -H_x(x^*(t), u^*(t), \lambda^*(t)) - \mu^*(t) g_x(x^*(t), u^*(t)), \tag{34}$$

$$\lambda^*(t_1) = b\psi_x(x^*(t_1)), \tag{35}$$

where

$$H(x, u, \lambda) = L(x, u, \lambda) + \lambda f(x, u). \tag{36}$$

$$\text{(ii)}\quad H_u(x^*(t), u^*(t), \lambda^*(t)) + \mu^*(t) g_u(x^*(t), u^*(t)) = 0. \tag{37}$$

$$\text{(iii)}\quad g_j(x^*(t), u^*(t)) \le 0, \qquad \mu_j^*(t) \ge 0, \qquad j = 1, \ldots, m, \tag{38}$$

$$\mu_j^*(t) g_j(x^*(t), u^*(t)) = 0, \qquad j = 1, \ldots, m. \tag{39}$$

(iv) For all $t \in [0, t_1]$, the function $H(x^*(t), u, \lambda^*(t))$ of the variable $u \in U(x^*(t))$ attains its minimum at the point

$$u = u^*(t);$$

namely, for any $v \in U(x^*(t))$,

$$H(x^*(t), u^*(t), \lambda^*(t)) \le H(x^*(t), v, \lambda^*(t)). \tag{40}$$

(v) Let $\hat{g}$ denote the vector formed from g by taking all the active components of g. Thus,

$$\hat{g}(x^*(t), u^*(t)) = 0.$$

For any two-dimensional vector ρ satisfying

$$\hat{g}_u(x^*(t), u^*(t))\rho = 0,$$

it follows that

$$\rho^T[H_{uu}(x^*(t), u^*(t), \lambda^*(t)) + \mu^*(t) g_{uu}(x^*(t), u^*(t))]\rho \ge 0. \tag{41}$$

Let d be the dimension of the active constraint vector $\hat{g}(x, u)$. Then, $d \le 2$, and the $d \times 2$ matrix $\hat{g}_u(x, u)$ has a full rank d. Since the $d \times d$ matrix $\hat{g}_u \hat{g}_u^T$ is nonsingular, we can define a $2 \times d$ matrix G by

$$G(x, u) = \hat{g}_u^T(x, u)[\hat{g}_u(x, u)\hat{g}_u^T(x, u)]^{-1}. \tag{42}$$

Let $\hat{\mu}^*(t)$ be a d-dimensional row-vector formed from $\mu^*(t)$ by taking the components corresponding to $\hat{g}(x, u)$. Then, from (37), we obtain

$$\hat{\mu}^*(t) = -H_u(x^*(t), u^*(t), \lambda^*(t)) G(x^*(t), u^*(t)). \tag{43}$$

By using (43), we see that (34) can be rewritten as

$$d\lambda^*(t)/dt = -H_x(x^*, u^*, \lambda^*) + H_u(x^*, u^*, \lambda^*)G(x^*, u^*)\hat{g}_x(x^*, u^*). \qquad (44)$$

5. Method of Computation

For the numerical computation, we employ a new algorithm (Ref. 7) which is similar to first-order differential dynamic programming (Ref. 8). To overcome the convergence difficulties, the convergence control parameters (CCP) technique proposed by Järmark (Ref. 9) and the method of multipliers originally proposed by Hestenes (Refs. 10 and 11) are combined with our algorithm. We seek the optimal control which minimizes the cost functional and at the same time satisfies the terminal condition

$$\psi(x(t_1)) = 0.$$

Thus, we have to solve two problems simultaneously; that is, one is the minimization problem, and the other is the two-point boundary-value problem. The CCP technique stabilizes the minimization process, and the method of multipliers works for satisfying the terminal condition.

Let us define the function

$$K(x, u, \lambda; v, C) = H(x, u, \lambda) + \tfrac{1}{2}(u - v)^T C(u - v), \qquad (45)$$

where C is a nonnegative diagonal matrix, which is called the CCP matrix. For seeking the optimal pair $(x^*(t), u^*(t))$ satisfying the necessary conditions in the preceding section, we propose the following algorithm.

Step 0. Select a nominal control $u^0(t)$ and a corresponding nominal trajectory $x^0(t)$ that satisfy

$$g(x^0(t), u^0(t)) \leq 0$$

and

$$dx^0(t)/dt = f(x^0(t), u^0(t)), \qquad x^0(0) = x^I.$$

Then, solve the differential equation

$$d\lambda^0(t)/dt = -H_x(x^0, u^0, \lambda^0) + H_u(x^0, u^0, \lambda^0)G(x^0, u^0)\hat{g}_x(x^0, u^0), \qquad (46)$$

with the terminal condition

$$\lambda^0(t_1) = c\psi^T(x^0(t_1))Q\psi_x(x^0(t_1)), \qquad (47)$$

where c is a positive constant. Set $b^0 = 0$ and $i = 1$.

Step 1. Determine $x^i(t)$ and $u^i(t)$ that satisfy both

$$K(x^i(t), u^i(t), \lambda^{i-1}(t); u^{i-1}, C^i) = \min_{u \in U(x^i(t))} K(x^i(t), u, \lambda^{i-1}(t); u^{i-1}, C^i), \tag{48}$$

and the differential equations

$$dx^i(t)/dt = f(x^i(t), u^i(t)), \qquad x^i(0) = x^I. \tag{49}$$

This can be done by integrating (49) from $t = 0$ to $t = t_1$, while seeking $u^i(t)$ that minimizes K.

Step 2. Calculate

$$J(u^i) = \int_0^{t_1} L(x^i(t), u^i(t))\, dt. \tag{50}$$

If

$$J(u^i) > J(u^{i-1}) \quad \text{and} \quad C^i < C_{\max},$$

set

$$C^i := \min\{\alpha C^i, C_{\max}\},$$

where $\alpha > 1$ and $C_{\max}$ is the specified maximum value of the CCP matrix, and go to Step 1. If

$$J(u^i) > J(u^{i-1}) \quad \text{and} \quad C^i \geq C_{\max},$$

set

$$C^{i+1} = C_{\max},$$

and go to Step 3. If

$$J(u^i) \leq J(u^{i-1}),$$

set

$$C^{i+1} = \beta C^i,$$

where $0 < \beta < 1$, and go to Step 3.

Step 3. If both the conditions

$$\|\psi(x^i(t_1))\|_Q^2 < \varepsilon_1, \tag{51}$$

$$\int_0^{t_1} \|u^i(t) - u^{i-1}(t)\|\, dt < \varepsilon_2 \tag{52}$$

are satisfied, where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are given small numbers, then stop the computation. Otherwise, go to Step 4.

Step 4. If both the conditions

$$J(u^i) > J(u^{i-1}), \tag{53}$$

$$\|\psi(x^i(t_1))\|_Q^2 > \|\psi(x^{i-1}(t_1))\|_Q^2 \tag{54}$$

hold, then set

$$u^i := u^{i-1}, \qquad x^i := x^{i-1}, \qquad b^{i-1} = 0,$$

and go to Step 5. Otherwise, go to Step 5.

Step 5. Set

$$\hat{\mu}^i = -H_u(x^i, u^i, \lambda^i)G(x^i, u^i), \tag{55}$$

and solve the differential equation

$$-d\lambda^i(t)/dt = H_x(x^i, u^i, \lambda^i) + \hat{\mu}^i \hat{g}_x(x^i, u^i), \tag{56}$$

under the terminal condition

$$\lambda^i(t_1) = b^i \psi_x(x^i(t_1)), \tag{57}$$

where

$$b^i = b^{i-1} + c\psi^T(x^i(t_1))Q. \tag{58}$$

Set $i := i + 1$, and go to Step 1.

In Step 0 of the algorithm, (46) is the same as (44), and (47) is the terminal condition when the cost functional is given as

$$J(u) = \tfrac{1}{2}c\|\psi(x(t_1))\|_Q^2 + \int_0^{t_1} L(x(t), u(t))\, dt, \tag{59}$$

where the first term is added as a penalty function for the terminal condition

$$\psi(x(t_1)) = 0.$$

In Step 1,

$$K(x^i, u, \lambda^{i-1}; u^{i-1}, C^i) = H(x^i, u, \lambda^{i-1}) + \tfrac{1}{2}(u - u^{i-1})^T C^i (u - u^{i-1}) \tag{60}$$

is considered twice. Namely, it is included as a penalty term in the integral This idea is due to Järmark (Ref. 9). If the nonnegative matrix C^i is large, because of the quadratic penalty term in (60), the variation

$$\delta u^i = u^i - u^{i-1}$$

of the control is kept small, and the stability of the algorithm is ensured.

In Step 2, if the cost (50) increases compared with the foregoing value, then the matrix C^i is made larger and the computation is iterated until the cost decreases or the matrix C^i reaches the specified maximum value $C_{\max}$.

If the cost decreases compared with the foregoing value, then the next matrix C^{i+1} is set smaller.

In Step 5, (55) is formally the same as (43), and (56) is formally the same as (44). The terminal conditions (57) and (58) are due to the method of multipliers of Hestenes (Ref. 10) and also due to the balance function method (Ref. 11). The validity of (57) and (58) can be explained as follows. Suppose that the cost functional in the ith iteration is given by

$$J(u) = \tfrac{1}{2}c\|\psi(x(t_1))\|_Q^2 + b^{i-1}\psi(x(t_1)) + \int_0^{t_1} [L(x(t), u(t)) \\ + \tfrac{1}{2}(u(t) - u^{i-1}(t))^T C^i (u(t) - u^{i-1}(t))]\, dt, \tag{61}$$

where b^{i-1} is the Lagrange multiplier row-vector for the constraint

$$\psi(x(t_1)) = 0.$$

Then, the terminal condition for $\lambda(t)$ is given by

$$\lambda(t_1) = [b^{i-1} + c\psi^T(x(t_1))Q]\psi_x(x(t_1)), \tag{62}$$

which gives (57). By modifying the multiplier vector b properly at each step of the iteration, as in (58), it is expected that the rate of convergence of $\psi(x(t_1))$ to zero will be improved.

In the algorithm, the terminal condition

$$\psi(x(t_1)) = 0$$

is considered twice. Namely, it is included as a penalty term in the integral cost functional (32). It is once again included in (61) as a terminal cost, which gives the terminal condition (62) for $\lambda(t)$.

In Step 4, if both the conditions (53) and (54) hold and the computation comes to a deadlock, we regard the last control and trajectory as a new nominal solution and start the computation again by setting

$$b^{i-1} = 0.$$

6. Numerical Results

We assume a monotower crane, which is often used for construction of buildings. We assume the parameter values as follows:

$$R = 30\text{ m}, \qquad H = 50\text{ m}, \qquad L_0 = 5\text{ m}, \qquad m = 10\text{ tons},$$

$$V_{1\max} = 0.9165\text{ m/sec}, \qquad V_{2\max} = 1.46\text{ rpm} = 0.153\text{ rad/sec}.$$

We assume that the hoisting motor plus the drum is equivalently of cylindrical shape with radius b and mass m_0. Then, the moment of inertia J_1 is given by

$$J_1 = m_0 b^2/2.$$

Assuming that

$$m_0 = 8 \text{ tons},$$

from (13) we obtain

$$\sigma = mb^2/J_1 = 2m/m_0 = 2.5.$$

The symbol

$$\rho = mR^2/J_2$$

represents the ratio of the moment of inertia of the load to that of the crane with respect to the ζ-axis. We assume that $\rho = 0.2$. Since

$$K = 0.0933$$

from (14), $v_{1\,\max}$ and $v_{2\,\max}$ can be calculated as follows:

$$v_{1\,\max} = V_{1\,\max}/\sqrt{K}R = 0.1, \qquad v_{2\,\max} = V_{2\,\max}/\sqrt{K} = 0.5.$$

Let

$$T_{1\,\max} = Mgb,$$

where M is the maximum weight of the load. Then, assuming that

$$M = 20 \text{ tons},$$

from (21) we obtain

$$u_{1\,\max} = bT_{1\,\max}/gJ_1 = 2M/m_0 = 5.$$

We assume that

$$u_{2\,\max} = T_{2\,\max}/KJ_2 = V_{2\,\max}/KT,$$

where T is the time for acceleration from standstill to the maximum speed $V_{2\,\max}$. Assuming that

$$T = 6.55 \text{ sec},$$

we obtain

$$u_{2\,\max} = 0.25.$$

The parameter values used in the computation are as follows:

$$\alpha_1 = \alpha_2 = 10, \qquad \alpha_2 = 12,$$
$$Q = \mathrm{diag}(10, 50, 10, 50, 10, 50, 10, 50),$$
$$c = 0.1, \qquad w = 10, \qquad \gamma = 0.2,$$
$$\alpha = 2, \qquad \beta = 2^{-0.2} = 0.87,$$
$$C_{\max} = 20{,}000\, I,$$

where I denotes the identity matrix. If the cost value decreases consecutively five times with an initial CCP matrix C, then the CCP matrix at the fifth iteration is equal to $C/2$, because

$$\beta = 2^{-0.2}.$$

Furthermore, if the cost value increases in the next (sixth) iteration, then the CCP matrix jumps to the initial value C again, because

$$\alpha = 2.$$

The convergence of the algorithm depends strongly on the fitness of the nominal trajectory initially chosen. We selected a pair of nominal controls, such that

$$\begin{aligned} u_1(t) - \sigma &= (\pi/t_1) \min\{v_{1\,\max}, \pi h_1/2t_1\} \cos(\pi t/t_1), \\ u_2(t) &= (\pi/t_1) \min\{v_{2\,\max}, \pi\theta_1/2t_1\} \cos(\pi t/t_1). \end{aligned} \tag{63}$$

It is clear from (16) and (18) that the nominal trajectory satisfies

$$|x_6(t)| \le v_{1\,\max}, \qquad x_6(t_1) = 0.$$

Moreover,

$$x_5(t_1) = h_1, \qquad \text{if } v_{1\,\max} \ge \pi h_1/2t_1.$$

If $\rho = 0$, the nominal trajectory satisfies the same conditions as above, i.e.,

$$|x_8(t)| \le v_{2\,\max}, \qquad x_8(t_1) = 0.$$

Numerical solutions were obtained for three cases. Figure 3 indicates the nominal controls and the corresponding nominal solution in the case where

$$\theta_1 = \pi, \qquad H_1 = 30\text{ m}, \qquad t_1 = 14$$

(actual time $\tau_1 = 45.8$ sec), and Fig. 4 indicates the optimal controls and the optimal solution obtained by the computation. In Figs. 3 to 8, the quantities $u_1 - \sigma$, u_2, Z, $\dot{Z}$, θ, $\dot{\theta}$, $\dot{X}$, $\dot{Y}$ are plotted against the dimensionless

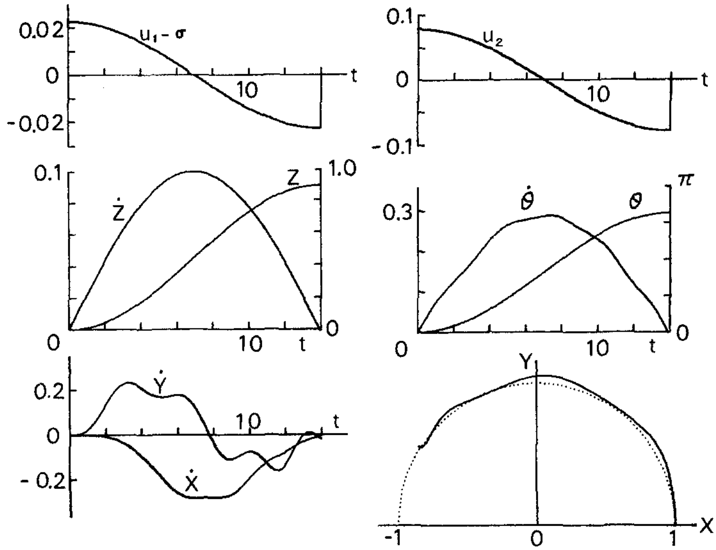

Fig. 3. Nominal trajectory ($\theta_1 = 180°$, $h_1 = 1$).

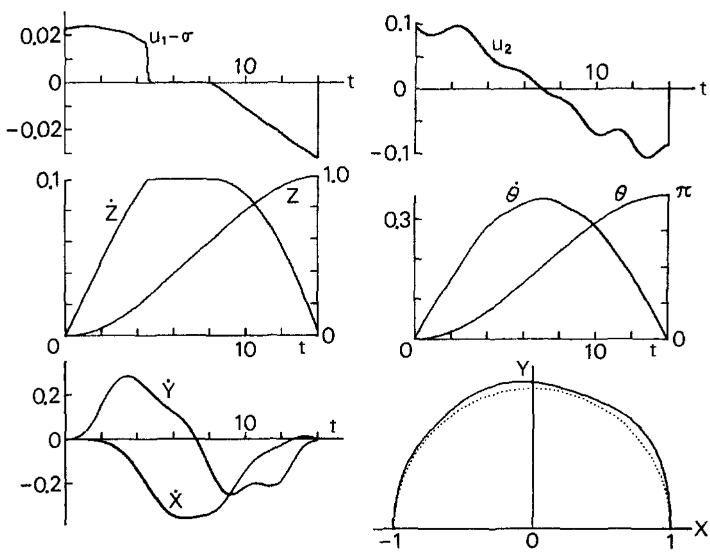

Fig. 4. Optimal solution ($\theta_1 = 180°$, $h_1 = 1$).

time t, and the locus of the load in the (X, Y)-plane is shown. In Fig. 3, the terminal conditions for $X(t_1)$, $Y(t_1)$, $Z(t_1)$, $\theta(t_1)$ are not satisfied. However, in Fig. 4 all terminal conditions are satisfied, and the swing of the load during the transfer is satisfactorily small.

Figure 5 indicates the nominal controls and the nominal solution in the case where

$$\theta_1 = \pi, \qquad H_1 = 15 \text{ m}, \qquad t_1 = 14$$

(actual time $\tau_1 = 45.8$ sec); and Fig. 6 indicates the optimal solution obtained by the computation. Figures 7 and 8 respectively indicate the nominal trajectory and the optimal solution, in the case where

$$\theta_1 = \pi/2, \qquad H_1 = 15 \text{ m}, \qquad t_1 = 7$$

(actual time $\tau_1 = 22.9$ sec). Figures 6 and 8 also give satisfactory results, respectively. In Figures 4 and 8, the hoisting velocity reaches its maximum, and in Fig. 8 the torque of the rotation motor reaches its maximum.

In these computations, the differential equations (49) and (56) were solved by using the method of Heun (Ref. 13) with a uniform steplength $\Delta = t_1/N$, where N was taken as $N = 200$. In particular,

$$u^i(k) \equiv u^i(k\Delta) \quad \text{and} \quad x^i(k) \equiv x^i(k\Delta)$$

in Step 1 of the algorithm were determined as follows.

(i) Set $k = 0$.

(ii) Given $x^i(k)$, determine $u^i(k)$ via (48).

(iii) Using the approximation

$$\hat{x}^i(k+1) = x^i(k) + f(x^i(k), u^i(k))\Delta \tag{64}$$

to $x^i(k+1)$, calculate $x^i(k+1)$ by

$$x^i(k+1) = x^i(k) + \tfrac{1}{2}\Delta[f(x^i(k), u^i(k)) + f(\hat{x}^i(k+1), u^i(k))]. \tag{65}$$

(iv) Set $k := k+1$, and go to (ii).

All computations were done by using the ACOS System 900 of Osaka University. The CPU time needed to obtain the optimal solution was about 70 sec to 150 sec, and the number of iterations was 120 to 250. The values of the CCP matrix ranged from about 1,000I to 20,000I. In general, in the case where some state-dependent control constraint is active, a much larger number of iteration was needed. If the time t_1 is set larger, then the computation is easier, and the swing of the load during the transfer becomes smaller.

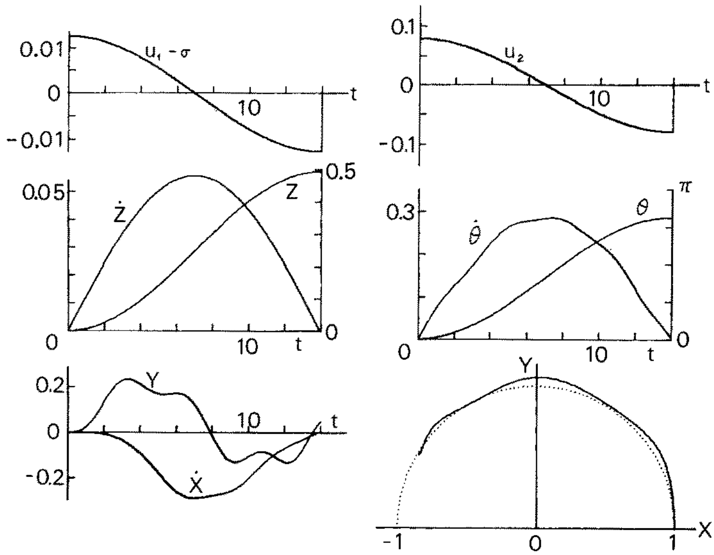

Fig. 5. Nominal trajectory ($\theta_1 = 180°$, $h_1 = 0.5$).

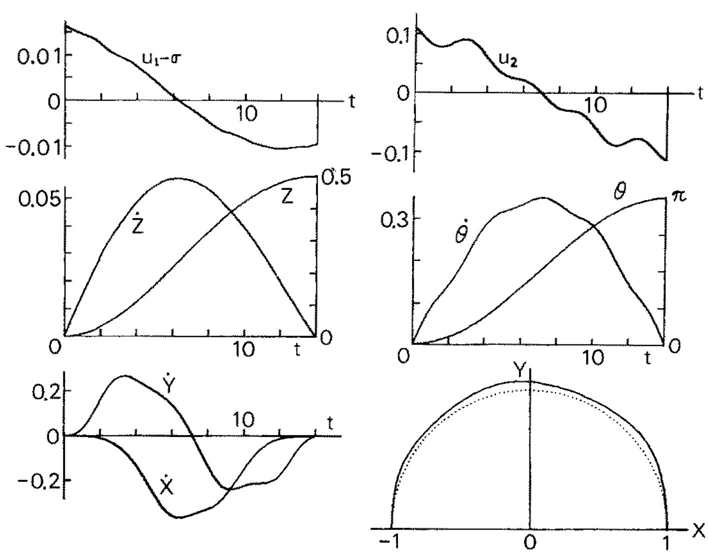

Fig. 6. Optimal solution ($\theta_1 = 180°$, $h_1 = 0.5$).

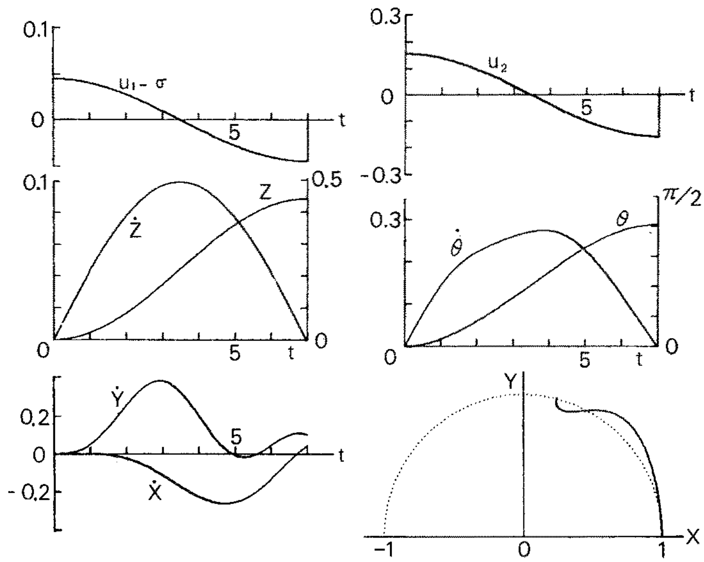

Fig. 7. Nominal trajectory ($\theta_1 = 90°$, $h_1 = 0.5$).

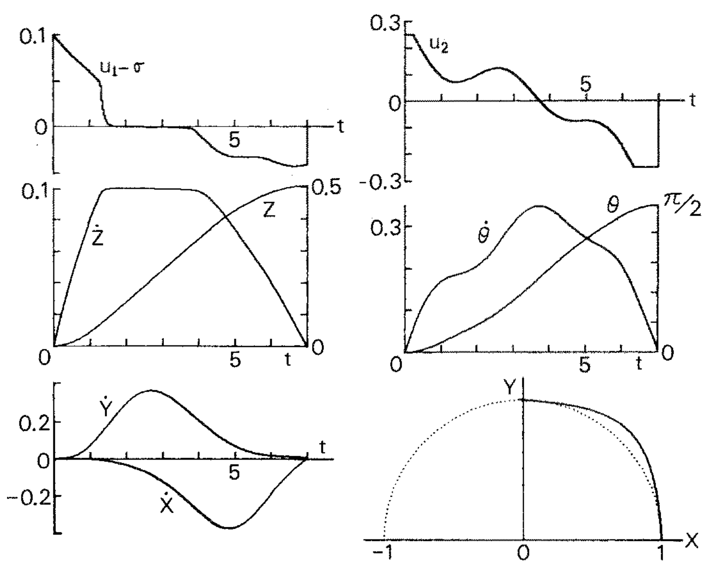

Fig. 8. Optimal solution ($\theta_1 = 90°$, $h_1 = 0.5$).

7. Concluding Remarks

We applied a new algorithm, combined with the CCP technique and the method of multipliers, to the numerical solution of the optimal control of a rotary crane. The numerical results obtained indicate that the algorithm is suitable for our problem; also, this algorithm can be applied widely to various problems of optimization.

In order that the load be in complete rest at the end of the transfer and that the swinging of the load during the transfer be minimized, the torque of the motors should be controlled in the entire course of the transfer of the load. Thus, the computed optimal control scheme becomes inevitably complicated, as shown in Figs. 4, 6, and 8, respectively. Since the algorithm used is simple, compared with the second-order DDP method (Ref. 1), it might be possible to install a microcomputer in the crane, compute the optimal control between two successive transfers of loads, and control the crane following the computed optimal scheme. At present, most rotary cranes are controlled by crane operators manually. If the optimal control pattern is calculated beforehand, the operating load of the crane operator can be reduced considerably.

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