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Compromise Solutions, Domination Structures, and Salukvadze's Solution[1]

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Abstract. We outline the concepts of compromise solutions and domination structures in such a way that the underlying assumptions and their implications concerning the solution concept suggested by Salukvadze may be clearer. An example is solved to illustrate our discussion.

Key Words. Vector-valued optimization, vector performance indexes, multicriteria optimization, compromise solutions, domination structures, nondominated solutions, N-person games, decision theory.

1. Introduction

In daily decision making, we deal quite often with problems involving not only a single criterion. Rather, we may have multiple objective decision problems or decision problems involving more than one decision maker. For these types of decision problems, there is not as yet a universally accepted solution concept, even though there exist quite a few (see Refs. 1–38). Although we may use a recently derived concept of domination structures (Refs. 25, 27) to study the assumptions which underly each solution concept, there is no apparent reason to believe that a certain solution concept is superior to others. Although we, as researchers, consultants, or decision makers, can use the concept of domination structures to suggest a limitation of decisions to the

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set of nondominated solutions (which may contain more than one alternative), the final decision must depend on the judgment and/or the relative strength in negotiation or bargaining of the decision makers.

In the search for solution concepts, one often starts with static decision problems (one-stage decision problems) rather than with the much more complicated cases of dynamic decision problems. As a consequence, there is a natural gap among different researchers. Those dealing with static problems may be unaware of the implications of solution concepts in dynamic cases. On the other hand, those interested in dynamic problems may be unaware of current developments in solution concepts.

Recently, there appeared two interesting papers (Refs. 32–33) by Salukvadze, in which he considers dynamic decision problems with multiple objective functionals. He proposes a solution concept and works out the computational details. However, the discussion of the solution concept is not complete. The underlying assumptions as well as their implications are not entirely clear. The solution concept proposed by Salukvadze has been discussed independently in Ref. 24 and 34; in fact, it is a special case of the compromise solutions in Ref. 24.

In this note, we outline some underlying assumptions and their implications concerning the solution concept suggested by Salukvadze in order that its applicability may be clearer. In particular, in Section 2 we define the decision space and criteria space for both static and dynamic decision problems. We then focus on the concept of compromise solutions and list their known properties. Since Salukvadze's solution is a special case of compromise solutions, all properties described are applicable to his solution. In Section 3, we introduce the concept of domination structure and show how strong the assumption is that underlies compromise solutions (and hence Salukvadze's solution). Finally, in Section 4 we treat the example of Ref. 32 to illustrate the points made in Sections 2 and 3 so that the merits as well as the weaknesses of Salukvadze's solution may be more apparent.

2. Compromise Solutions

Suppose that we have to make a choice from a set of alternatives $X \subset R^n$ and that we can associate each alternative $x \in X$ with a set of criteria $(f_1(x),..., f_l(x))$. Let $f(x) = (f_1(x),..., f_l(x))$. We shall call X the *decision space*, while $Y = \{f(x) \mid x \in X\}$ is the *criteria space*. An element $y \in Y$ is often called the outcome of a decision.

Remark 2.1. Observe that the above definitions of decision and criteria spaces can be extended to dynamic cases; thus, the solution concepts that we describe here in fact are applicable to such cases. For instance, we can denote the set of all admissible strategies or controls μ by X (see Refs. 2, 4, 35 for admissible strategies or controls). Then, if for each μ we have $(f_1(\mu),..., f_l(\mu))$ as the performance indices, the criteria space can be specified by

$$Y = \{(f_1(\mu),..., f_l(\mu)) \mid \text{all admissible strategies (or controls) } \mu\}.$$

On the other hand, since we can convert integral payoffs into terminal payoffs by adding extra variables (see Refs. 2, 35), we may regard X as the set of all attainable terminal states in the enlarged space. Then, Y can be defined accordingly on X. Although conceptually we have no difficulty in defining X and Y in dynamic cases, to actually visualize them is a very difficult task. We shall describe the solution concepts in static cases and leave it to the reader to extend them to dynamic cases. However, in Section 4, we shall supply an example of a dynamic case.

Let

$$y_j^* = \sup\{f_j(x) \mid x \in X\}.$$

Then, $y^* = (y_1^*,..., y_l^*)$ is called the *utopia point* or *ideal point* of our problem with the interpretation that, whenever y^* is feasible, $f^{-1}(y^*)$ simultaneously maximizes each criterion. For simplicity, we shall assume that Y is compact. With some slight modification, the compactness assumption can be relaxed.

Given $y \in Y$, we define a class of *regret functions* by

$$R_p(y) = \| y^* - y \|_p = \Big[\sum_j (y_j^* - y_j)^p\Big]^{1/p}, \qquad p \geqslant 1. \tag{1}$$

Definition 2.1. y^p and $f^{-1}(y^p)$ is the compromise solution with parameter $p \geqslant 1$ iff y^p minimizes $R_p(y)$ over Y.

For convenience, we shall call $R_p(y)$ the group regret of y with respect to p, while $y_j^* - y_j$ is the jth individual regret.

Remark 2.2. The solution concept proposed by Salukvadze is a compromise solution with $p = 2$. The corresponding group regret is associated with the Euclidean norm. The solution resulting from usual goal programming (Ref. 10) or simple majority rule can be regarded as a compromise solution with $p = 1$ (Ref. 28). Compromise solutions with $p = \infty$ correspond to a minimax criterion, because $y^{p=\infty}$ solves

$$\min_y \max_j \{y_j^* - y_j \mid j = 1 ..., l\}.$$

Remark 2.3. In group decision problems, we may use $y_j = f_j(x)$ to denote the jth decision maker's utility. It is reasonable to assume that compromise solutions cannot be acceptable to each decision maker unless each one has nonnegative utility. Thus, in Ref. 24, instead of X and Y, the decision and utility space for compromise solutions are defined respectively by

$$X_0 = \{x \in X \mid f(x) \geqq 0\} \quad \text{and} \quad Y_0 = \{f(x) \mid x \in X_0\}.$$

The utopia point y^* is then the point which has its jth component y_j^* maximizing $f_j(x)$ over X_0. With this modification, compromise solutions enjoy the property of *individual rationality* (i.e., nobody in the group has negative utility). In this note, we do not make such a modification because the individual rationality is an unnecessary nicety for one decision maker with multiple criteria. Observe that, except for individual rationality, all properties of compromise solutions (listed below) hold no matter whether or not we introduce the modification. Since compromise solutions can be applied to group decision problems or multicriteria decision problems with one decision maker, $y_j = f_j(x)$ in this note is called the jth objective, criterion, or utility, interchangeably. Also, since minimizing $f_j(x)$ over X is equivalent to maximizing $-f_j(x)$ over X, if $f_j(x)$ is the level of disutility we may interpret $-f_j(x)$ as the level of utility, and *vice versa*. Thus, without loss of generality, we can focus on the properties of compromise solutions defined in Definition 2.1. The extension to other cases is obvious.

Definition 2.2. Let

$$\Lambda^{\leqq} = \{d \in R^l \mid d \leqq 0\}.$$

We say that Y is $\Lambda^{\leqq}$-convex iff $Y + \Lambda^{\leqq}$ is convex.

For more discussion on cone convexity, see Ref. 25.

Compromise solutions enjoy the following properties (See Ref. 24).

(i) *Feasibility*. For each $p \geqq 1$, under the assumption that Y is compact, there is always a compromise solution. Observe that some solution concepts, such as *stable set* or *core*, may have no feasible solution.

(ii) *Least Group Regret*. Since y^p is the closest point over Y to the utopia point, the group regret is minimized in the sense of distance.

(iii) *No Dictatorship*. That is, the group decision is not completely determined by any one criterion $f_j(x)$. In contrast to lexicographical maximization[4] in which some f_j may not be considered in the final decision, each f_j is considered in the compromise solution.

(iv) *Pareto Optimality*. For $1 \leqq p < \infty$, each compromise solution is Pareto optimal. That is, there is no other $y \in Y$ such that $y \geqq y^p$ and $y \neq y^p$. This property comes directly from Definition 2.1.

(v) *Uniqueness*. Suppose that Y is $\Lambda^{\leqq}$-convex. Then, each y^p, $1 < p < \infty$, is unique.

(vi) *Symmetry or Principle of Equity*. If Y is convex and closed with respect to cyclical rotation, then for each p, $1 < p \leqslant \infty$, the $y_j^p, j = 1,..., l$, are identical. For $p = 1$, there is at least one compromise solution y^1 such that the $y_j^1, j = 1,..., l$, are identical. Thus, the principle of equity is implicit in the concept of compromise solutions.

(vii) *Independence of Irrelevant Alternatives*. Suppose that $X \subset X'$ and

$$\max_X f_j(x) = \max_{X'} f_j(x), \qquad j = 1,..., l.$$

If a compromise solution with respect to X' happens to be x^0 in X, then x^0 is also a compromise solution with respect to X. Thus, the irrelevant alternatives may be discarded from the consideration for compromise solution. For instance, for $1 \leqq p < \infty$, we may discard those y of Y which are not Pareto optimal without affecting the final compromise solution.

If we treat p as a parameter of y^p, then $\{y^p \mid p \geqq 1\}$ enjoys the following properties.

(viii) *Continuity*. Suppose that Y is $\Lambda^{\leqq}$-convex. Then, as a function of p, y^p is continuous over $1 < p < \infty$. If y^1 (or y^∞) is unique, then y^p is continuous at $p = 1$ (or $p = \infty$) (see Ref. 36).

(ix) *Monotonicity and Bounds*. If $l = 2$ and Y is $\Lambda^{\leqq}$-convex, under a very mild condition it can be shown that $\{y_j^p\}$, $j = 1, 2$, is bounded by $[y_j^1, y_j^\infty]$; furthermore, it is a monotone function of p (see Ref. 24; for some generalization of this result for $l > 2$, see Ref. 36).

(x) *Monotonicity of the Group Utilities and the Individual Regrets*. Under the same assumptions as in (ix), it can be shown that both $\sum_j y_j^p$ and $\max_j\{y_j^* - y_j^p\}$ are decreasing functions of p. Observe that, if

[4] Let $X^0 = X$. For $k = 1,..., l$, define

$$X^k = \{x^0 \in X^{k-1} \mid f_k(x^0) \geqq f_k(x), x \in X^{k-1}\}.$$

In lexicographical maximization, the final decision is on the set X^l. Observe that, if X^k contains only one point, the final decision is uniquely determined; $f_{k+1} ,..., f_l$ need not be considered.

$y_j = f_j(x)$ is the utility function for the jth decision maker, then $\sum_j y_j{}^p$ is the sum of the utilities and $\max_j\{y_j{}^* - y_j{}^p\}$ is the maximum individual regret. We may consider $\sum_j y_j{}^p$ and $\max_j\{y_j{}^* - y_j{}^p\}$ as group utility and individual regret, respectively, resulting from the compromise solution with parameter p. Our result says that, as p increases, the group utility decreases; however, individual regret reduces too. In this sense, the parameter p has a meaning for balancing group utility and individual regret. As a consequence, we see that simple majority rule (see Remark 2.3) is the rule which maximizes the group utility and most neglects the individual regret in the entire domain of compromise solutions (see Ref. 28).

The concept of compromise solutions can be generalized in several directions. For instance, we may replace $R_p(y)$ by

$$R_p(\alpha, y) = \Big[\sum_j \alpha_j{}^p(y_j{}^* - y_j)^p\Big]^{1/p}.$$

Then, most of the above properties, with a suitable modification, remain the same (see Ref. 24).

Although compromise solutions have merits, they are by no means perfect. Some associated assumptions are discussed in the next section. In addition, compromise solutions implicitly impose an intercomparison among the criteria or utilities through the group regret function (negative side of social welfare function). This imposition is not acceptable on some occasions. One should also observe that compromise solutions are not independent of a positive linear transformation of the $f_j(x)$. In fact, changing the scale of $f_j(x)$ has the same effect as changing the weight α in $R_p(\alpha, y)$. In applying this solution concept, one should be aware of this defect.

3. Domination Structures

In this section, we shall introduce briefly the concept of domination structures so that we can understand how strong an assumption has been imposed in compromise solutions. For more detailed discussion, we refer to Ref. 25; a simple summary can be found in Refs. 16, 27. An attempt to apply the concept to solve some practical problems can be found in Ref. 37.

Given two outcomes, y^1 and y^2, in the criteria space Y, we can write $y^2 = y^1 + d$, with $d = y^2 - y^1$. If y^1 is prefered to y^2, written $y^1 \succ y^2$, we can think of this preference as occurring because of d.

Now, suppose that the nonzero d has the additional property that, if $y = y^1 + \lambda d$ and $\lambda > 0$, then $y^1 \succ y$. Then, d will be called a *domination factor* for y^1. Note that, by definition, given a domination factor for y^1, any positive multiple of it is also a domination factor. It follows that, given $y^1 \succ y^2$, it is not necessarily true that $d = y^2 - y^1$ is a domination factor for y^1.

Let $D(y)$ be the set of all domination factors for y together with the zero vector of R^l. The family $\{D(y) | y \in Y\}$ is the *domination structure* of our decision problem. For simplicity, the structure will be denoted by $D(\cdot)$.

One important class of domination structures is $D(y) = \Lambda$, Λ a convex cone, for all $y \in Y$. In this case, we shall call Λ the *domination cone*.

Definition 3.1. Given Y, $D(\cdot)$, and two points y^1, y^2 of Y, by y^1 *is dominated by* y^2 we mean

$$y^1 \in y^2 + D(y^2) = \{y^2 + d \mid d \in D(y^2)\}.$$

A point $y^0 \in Y$ is a *nondominated solution* iff there is no $y^1 \in Y$, $y^1 \neq y^0$, such that $y^0 \in y^1 + D(y^1)$. That is, y^0 is nondominated iff it is not dominated by any other outcomes. Likewise, in the decision space X, a point $x^0 \in X$ is a nondominated solution iff there is no x^1 in X such that

$$f(x^0) \neq f(x^1) \qquad \text{and} \qquad f(x^0) \in f(x^1) + D(f(x^1)).$$

The set of all nondominated solutions in the decision and criteria space will be denoted by $N_X(D(\cdot))$ and $N_Y(D(\cdot))$, respectively. Because of its geometric significance, a nondominated solution with respect to a domination cone Λ is also called a Λ-extreme point. The set of all Λ-extreme points is denoted by Ext $[Y \mid \Lambda]$.

Example 3.1. Let

$$\Lambda^{\leqq} = \{d \in R^l \mid d \leqq 0\}.$$

We see that y is Pareto optimal iff y is a $\Lambda^{\leqq}$-extreme point. That is, in the concept of Pareto optimality, one uses a constant domination cone $\Lambda^{\leqq}$. Observe that $\Lambda^{\leqq}$ is only $1/2^l$ of the entire space. When $l = 6$, for instance, $\Lambda^{\leqq}$ is only 1/64 of R^6.

Example 3.2. In the additive weight method, one first finds a suitable weight $\lambda = (\lambda_1, \ldots, \lambda_l)$ and then maximizes

$$\lambda \cdot f(x) = \sum_j \lambda_j f_j(x)$$

over X or $\lambda \cdot y = \sum \lambda_j y_i$ over Y. Given λ, the solution concept implicitly uses the domination cone

$$\{d \in R^l \mid \lambda \cdot d < 0\}.$$

The concept of the additive weight method is closely related to that of *trade-off* in economic analysis. In fact, the latter can be a way of obtaining the weight λ. In order to illustrate this, let us limit ourselves to $l = 2$. The trade-off ratio of $f_2(x)$ with respect to $f_1(x)$ is defined by how many units of $f_2(x)$ we want to sacrifice in order to increase a unit of $f_1(x)$. Thus, the ratio gives the value of $f_2(x)$ in terms of $f_1(x)$. Although the ratio may be nonlinear in reality, in practice one often interprets it as a constant ratio. For the time being, assume that the ratio is constant and given by λ_2/λ_1, with $\lambda_1 > 0$. Clearly, our decision problem becomes one of maximizing $f_1(x) + (\lambda_2/\lambda_1)\, f_2(x)$ over X, which is equivalent to maximizing $\lambda_1 y_1 + \lambda_2 y_2$ over Y. The latter is essentially an additive weight method. To correctly predetermine the ratio λ_2/λ_1 is a very difficult task. In practice, one may first use his experience or judgement to set up its bounds. Once the bounds are set, we have partial information on preference. For instance, suppose that $1 < \lambda_2/\lambda_1 < 3$ is given. Then, we obtain valuable information, because it implies that, by using

$$\Lambda = \{(d_1, d_2) \mid d_1 + d_2 \leqq 0,\ d_1 + 3d_3 \leqq 0\}$$

as our domination cone, the optimal solution is contained in Ext $[Y \mid \Lambda]$ (see Fig. 1 and Ref. 27).

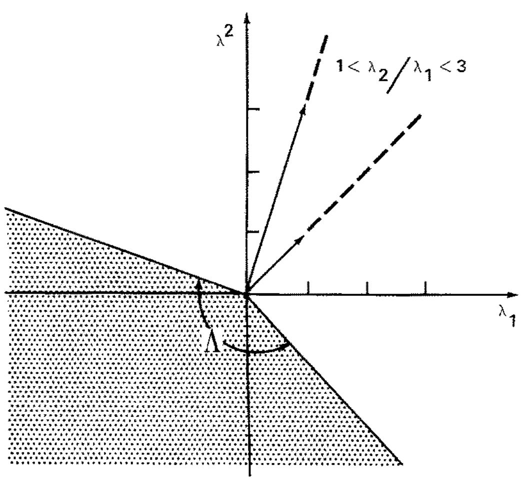

Fig. 1

Example 3.3. For compromise solutions with $1 \leqq p < \infty$, we can define the related domination structures by

$$D(y) = \{d \in R^l \mid \nabla R_p(y) \cdot d > 0\}$$

for $y \in Y$, where $\nabla R_p(y)$ is the gradient of $R_p(y)$, which is given by (1). This is because, in compromise solutions, the smaller $R_p(y)$ is, the more it is preferred. Since $R_p(y)$ is a differentiable convex function, $D(y)$ forms a domination cone for y. When $p = \infty$, we can again specify its domination structure; we shall not do so here in order not to distract from the main ideas. Observe that each $D(y)$ contains a half space, no matter what the parameter p is.

Remark 3.1. The domination cone induced by Pareto optimality (that is, $\Lambda^{\leqq}$) (see Example 3.1) is sometimes not large enough to encompass all information possessed by the decision makers. For instance, in Example 3.2, Λ is larger than $\Lambda^{\leqq}$; $\Lambda - \Lambda^{\leqq}$ is the valuable information missed by Pareto optimality. On the other hand, the domination structure induced by compromise solutions (Example 3.3) is such that each $D(y)$ contains a half space. To make this possible, too strong an assumption may have been imposed. For instance, in Example 3.2, although Λ is much larger than $\Lambda^{\leqq}$, it is much less than a half space. The difference between Λ and the half space of $D(y)$ induced by compromise solutions (when they are comparable) depends on how strong the assumptions are or how much information we have on the preference of the outcomes in Y. From the computational point of view, one has no difficulty in computing a compromise solution because the problem reduces to one of mathematical programming or to a control problem. However, because of the strong assumptions imposed or the information required, the compromise solutions may not be acceptable. A researcher may be able to find the compromise solutions, but these solutions may not be actually desirable for the decision maker (see the next section for further discussion). In daily decision problems, we are usually faced with problems of partial information. Both Pareto and compromise solutions cannot suitably explain the decision situation. The former does not adequately employ the partial information, while the latter may impose too strong an assumption, which is not conformable with the partial information. We believe that one should try to make suitable shoes (mathematical models) for the feet (the decision problems), rather than to cut off the feet in order to wear a given pair of shoes (one can use domination structures to attack some partial information decision problems, see Refs. 16, 27, 37).

4. Example

In order to illustrate the discussion of the previous section, consider a problem treated by Salukvadze (Ref. 32):

(i) *Playing Space*: $\{(z_1, z_2) \mid |z_1| \leqq 3\}$;
(ii) *Dynamic System*: $\dot{z}_1 = z_2$, $\dot{z}_2 = u$, $t \in [0, T]$;
(iii) *Control Set*: $u \in [-1, 1]$;
(iv) *Initial Condition*: $z_1(0) = 1$, $z_2(0) = 0$;
(v) *Terminal Condition*: $z_1(T) = 0$, T not specified;
(vi) *Criteria*: (A) minimize $f_1(u(\cdot)) = T$,
(B) maximize $f_2(u(\cdot)) = z_2(T)$.

As mentioned in Remark 2.1, the decision space X and criteria space Y may be very difficult to visualize. However, in this example, letting X be the set of all admissible controls, the set of all Pareto-optimal solutions can be found. Note that (A) requires minimization; to permit use of the results in the earlier sections, we can consider maximizing $-f_1(u(\cdot))$. Rather than doing so, we shall consider the problem as is; the simple modifications required in the results are obvious.

It is readily established that the set of terminal points reachable from (1, 0) along a solution path that remains in the playing space is given by

$$\{(z_1, z_2) \mid z_1 = 0, -\sqrt{6} \leqq z_2 \leqq \sqrt{6}\}$$

(see Fig. 2).

Let us first consider terminal points on the subset

$$\{(z_1, z_2) \mid z_1 = 0, -\sqrt{2} \leqq z_2 \leqq \sqrt{6}\}$$

of the set of reachable terminal points. The control

$$u^*(t) = \begin{cases} -1 & \text{for } t \in [0, s], \\ 1 & \text{for } t \in (s, T] \end{cases}$$

renders a minimum of the transfer time T from (1, 0) to a given terminal point $(0, z_2)$, $-\sqrt{2} \leqq z_2 \leqq \sqrt{6}$, where the switching time s is a function of z_2. Conversely, for a given switching time s, the control $u^*(\cdot)$ results in a transfer time T and a corresponding terminal point $(0, z_2(T))$.

The solution path generated by $u^*(\cdot)$ is given by

$$z_1(t) = -\tfrac{1}{2}t^2 + 1, \qquad z_2(t) = -t, \qquad t \in [0, s],$$

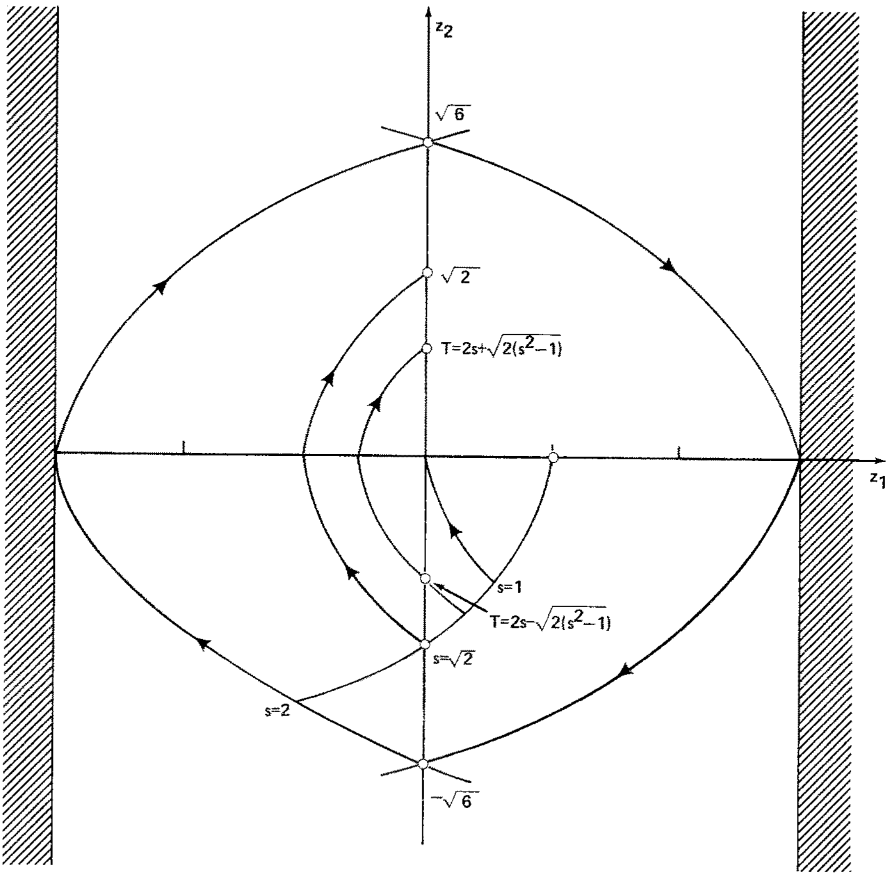

Fig. 2

and

$$z_1(\tau) = \tfrac{1}{2}\tau^2 - s\tau + 1 - s^2, \qquad z_2(\tau) = \tau - s, \qquad \tau \geqq 0, \tag{2}$$

where $\tau = t - s$. At termination, $z_1 = 0$ so that $\tau = s \pm \sqrt{[2(s^2 - 1)]}$, whence

$$T = 2s \pm \sqrt{[2(s^2 - 1)]}. \tag{3}$$

In order that T be defined, i.e., termination takes place, $s \geqq 1$.

Since, by (2), $dz_1/d\tau = 0$ for $\tau = s$, it follows that the minimum value of $z_1(\tau)$ is given by $1 - s^2$. But $z_1(\tau) \geqq -3$, so that $s \geqq 2$. Thus, $s \in [1, 2]$.

Since $\tau \geqq 0$,

$$\tau = s - \sqrt{[2(s^2 - 1)]} \geqq 0,$$

which is met iff $s \leqq \sqrt{2}$. Thus, for $s \in [1, \sqrt{2}]$, the portion of the solution path defined by (2) crosses the z_2-axis twice, the first time at

$$\tau = 2s - \sqrt{[2(s^2 - 1)]}$$

and the second time at

$$\tau = 2s + \sqrt{[2(s^2 - 1)]}.$$

Thus,

$$f_1 = 2s - \sqrt{[2(s^2 - 1)]}, \qquad f_2 = -\sqrt{[2(s^2 - 1)]}, \tag{4}$$

$$f_1 = 2s + \sqrt{[2(s^2 - 1)]}, \qquad f_2 = \sqrt{[2(s^2 - 1)]}. \tag{5}$$

For $s \in [\sqrt{2}, 2]$, the portion of the solution path given by (2) intercepts the z_2-axis once. Thus,

$$f_1 = 2s + \sqrt{[2(s^2 - 1)]}, \qquad f_2 = \sqrt{[2(s^2 - 1)]}. \tag{6}$$

These results are summarized in Table 1.

We shall now consider Curves 1 and 2 (see Fig. 3). Upon use of

$$df_2/df_1 = (df_2/ds)/(df_1/ds) \quad \text{and} \quad d^2f_2/df_1^2 = (d/ds)(df_2/df_1)(ds/df_1), \tag{7}$$

it is readily established that

$$df_2/df_1 > 0, \qquad d^2f_2/df_1^2 < 0 \tag{8}$$

for all points on Curves 1 and 2. Note also that Curve 1 joins Curve 2 smoothly at $(f_1, f_2) = (2, 0)$ corresponding to $s = 1$.

The domination cone for our problem is

$$\Lambda_1 = \{(d_1, d_2) \mid d_1 \geqq 0, d_2 \leqq 0\}.$$

It follows directly from Corollary 4.3 of Ref. 25 that a point $(f_1^0, f_2^0) \in \text{Ext}[Y \mid \Lambda_1]$ (i.e., is noninferior) iff f_1^0 is the unique minimum of f_1 for all $(f_1, f_2) \in Y$ and $f_2 \geqq f_2^0$.

Table 1

	f_1	f_2	s
Curve 1	$2s - \sqrt{[2(s^2 - 1)]}$	$-\sqrt{[2(s^2 - 1)]}$	$1 \leqq s \leqq \sqrt{2}$
Curve 2	$2s + \sqrt{[2(s^2 - 1)]}$	$\sqrt{[2(s^2 - 1)]}$	$1 \leqq s \leqq 2$

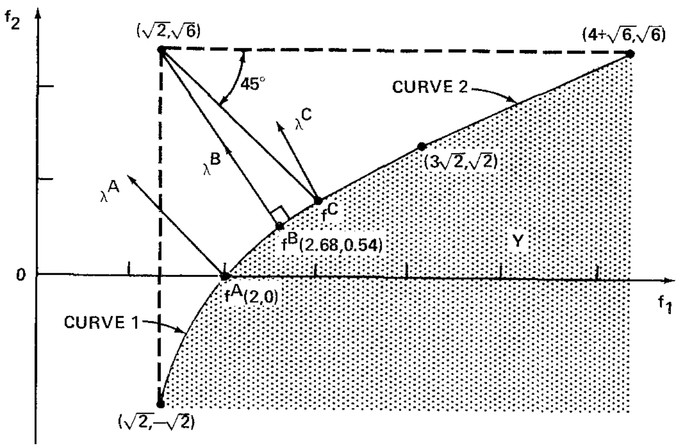

Fig. 3

Since $u^*(\cdot)$ renders the unique minimum of $f_1 = T$ for a given $f_2 = z_2(T) \in [-\sqrt{2}, \sqrt{6}]$, and $df_1/ds > 0$ on Curves 1 and 2, the points on these curves, and only these points, are noninferior among points (f_1, f_2) with $f_2 \in [-\sqrt{2}, \sqrt{6}]$.

On the other hand, consider $z_2(T) \in [-\sqrt{6}, -\sqrt{2})$. Among all (f_1, f_2) with $f_2 \geqslant z_2(T)$, the unique minimum of f_1 is rendered by the control $u^*(t) \equiv -1$. Thus, the point $(\sqrt{2}, -\sqrt{2})$ dominates all $(f_1, f_2) \in Y$ with $f_2 < -\sqrt{2}$. Consequently, the point on Curves 1 and 2 are the only noninferior points for all $(f_1, f_2) \in Y$.

Finally, we observe that, since $d^2f_2/df_1^2 < 0$ along Curves 1 and 2, f_2 is a strictly concave function of f_1. Since these curves represent $\text{Ext}[Y \mid \Lambda_1]$, Y is Λ_1-convex. In order to see this point, we shall show that $Y + \Lambda_1$ is a convex set.

Let $y^1, y^2 \in Y + \Lambda_1$. Given α, $0 < \alpha < 1$, we show that

$$\alpha y^1 + (1 - \alpha) y^2 \in Y + \Lambda_1 .$$

By hypothesis, we can write

$$y^1 = f^1 + h_1, \qquad y^2 = f^2 + h_2,$$

where $f^1, f^2 \in \text{Ext}[Y \mid \Lambda_1]$ and $h_1, h_2 \in \Lambda_1$. Then,

$$\alpha y^1 + (1 - \alpha) y^2 = \alpha f^1 + (1 - \alpha) f^2 + \alpha h_1 + (1 - \alpha) h_2.$$

Since

$$\alpha h_1 + (1 - \alpha) h_2 \in \Lambda_1,$$

it suffices to show that

$$\alpha f^1 + (1 - \alpha) f^2 \in Y + \Lambda_1 .$$

Let

$$f^1 = (f_1^1, f_2(f_1^1)) \quad \text{and} \quad f^2 = (f_1^2, f_2(f_1^2)),$$

where $f_2(f_1)$ is specified as in Table 1. Then,

$$\begin{aligned}\alpha f^1 + (1-\alpha) f^2 &= (\alpha f_1^1 + (1-\alpha) f_1^2, \alpha f_2(f_1^1) + (1-\alpha) f_2(f_1^2)) \\ &= (\alpha f_1^1 + (1-\alpha) f_1^2, f_2(\alpha f_1^1 + (1-\alpha) f_1^2) - \beta) \\ &= (\alpha f_1^1 + (1-\alpha) f_1^2, f_2(\alpha f_1^1 + (1-\alpha) f_1^2)) + (0, -\beta) \\ &\in Y + \Lambda_1,\end{aligned}$$

where $\beta \geqslant 0$ and where the second equality is due to the fact that f_2 is a concave function of f_1.

Note that the Λ_1-convexity of Y is very desirable because all the properties listed in Section 2 are applicable. It also allows us to visualize the shape of Y as depicted in Fig. 3.

Referring to Fig. 3, let λ^A, λ^B, and λ^C be the normal vectors to ∂Y at the point f^A, f^B, and f^C. Observe that, at point f^A, the slope of the curve which represents $\text{Ext}[Y \mid \Lambda_1]$ is one. Thus, f^A is a compromise solution with $p = 1$. At f^B, the line from the utopia point to f^B is orthogonal to $\text{Ext}[Y \mid \Lambda_1]$. Thus, f^B is a compromise solution with $p = 2$. At f^C, both criteria suffer an equal regret, because $f_1^C - \sqrt{2} = \sqrt{6} - f_2^C$. In fact, f^C is the compromise solution with $p = \infty$.

The following may be worth noting.

(i) The solution suggested by Salukvadze is the point f^B which is the compromise solution with parameter $p = 2$ (see Remark 2.2).

(ii) All compromise solutions as functions of p vary continuously and monotonically from f^A to f^C. There is no special reason to pick f^B [see Properties (viii) and (ix) of Section 2].

(iii) The compromise solutions have the property that, when p is increased, the sum of the *resulting utility* [i.e., $(-f_1) + f_2$] decreases and the maximum of the individual regrets (i.e., $f_1 - \sqrt{2}$ and $\sqrt{6} - f_2$) decreases, too. How to select a p so that the group utility (the sum of the individual utilities) and the individual regret are best balanced remains to be answered [see Property (x)].

(iv) Point f^B also corresponds to the unique maximum of $\lambda^B \cdot f = \lambda_1^B f_1 + \lambda_2^B f_2$. For any weight vector $\lambda \neq \lambda^B$, f^B cannot correspond to the maximum of $\lambda \cdot f$, because f_2 is a strictly concave function of f_1 on $\text{Ext}[Y \mid \Lambda_1]$. By selecting $p = 2$, we determine implicitly the weight vector λ^B. Why should one not use another λ? Is the resulting solution desirable?

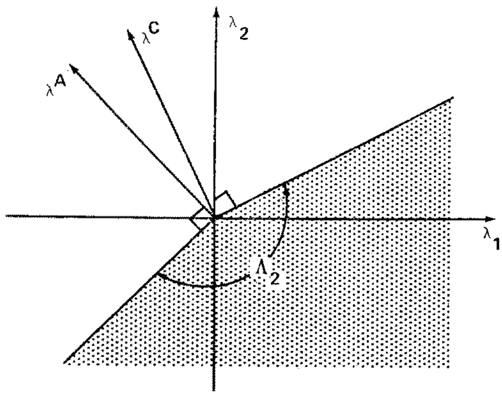

Fig. 4

(v) Let

$$\Lambda_2 = \{d \in R^2 \mid \lambda^A \cdot d \leqq 0, \lambda^C \cdot d \leqq 0\}$$

(see Fig. 4). One can show that

$$\text{Ext}[Y \mid \Lambda_2] = \text{curve}[f^A, f^C].$$

Thus, by restricting ourselves to compromise solutions, we assume implicitly that each domination cone $D(y)$, $y \in Y$, contains Λ_2. Is this assumption too strong?

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