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Linear Differential Games with Delayed and Noisy Information

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Abstract. This paper deals with a linear–quadratic–Gaussian zero-sum game in which one player has delayed and noisy information and the other has perfect information. Assuming that the player with perfect information can deduce his opponent's state estimate, the optimal closed-loop control laws are derived. Then, it is shown that the separation theorem is satisfied for the player with imperfect information and his optimal state estimate is given by a delay-differential equation.

Key Words. Differential games, information delays, linear–quadratic games, stochastic games.

1. Introduction

In control and game theories, the information structure plays an important role in determining the optimal decisions. Recently, the interplay of information and decision has been generally discussed in Refs. 1–2. Analytical discussions have been made for linear–quadratic–Gaussian zero-sum games (Refs. 3–6). In Refs. 3–4, it has been shown that the separation theorem is satisfied for the game in which one player's measurement is additively corrupted by white Gaussian noise and the other player has perfect information or no measurement. On the other hand, the game in which both players have noisy measurements (Refs. 5–6) cannot be easily solved but it possesses a certainty-coincidence property.

In dynamic games of large-complex systems, an important type of nonclassical information pattern is the delayed information pattern,

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which does not appear in static games. In physical situations, the information is usually obtained with time delay because of transmitting and processing the data and computing the control. In Ref. 7, a min-max solution to the game with delayed measurement is derived under the condition that one player with delayed information selects a control first and informs his opponent with perfect information of his control.

The game considered here is a linear–quadratic–Gaussian zero-sum game in which Player I has delayed measurement additively corrupted by white Gaussian noise and Player II has perfect information. Assuming that Player II can derive the error of Player I's state estimate, it will be shown that the optimal closed-loop control laws are obtained and the separation theorem for Player I's optimal closed-loop control law is satisfied.

2. Statement of the Problem

We consider a special but interesting problem such that Player I has delayed and noisy measurement and Player II has perfect information. The system is described by the equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t), \qquad t_0 \leqslant t \leqslant t_f, \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

and the observation data available at time t are given by

$$y_I(t) = 0, \qquad t_0 \leqslant t \leqslant t_0 + \theta, \tag{3}$$

$$\dot{y}_I(t) = H_I(t-\theta)x(t-\theta) + W_I(t-\theta)\dot{w}_I(t-\theta), \qquad t_0 + \theta \leqslant t \leqslant t_f, \tag{4}$$

$$y_{II}(t) = x(t), \qquad t_0 \leqslant t \leqslant t_f. \tag{5}$$

Here, the n-vector $x(t)$ is the system state; $u(t) \in R^q$ and $v(t) \in R^r$ are the controls of Player I and Player II; $y_I(t) \in R^m$ and $y_{II}(t) \in R^n$ are the observations of Player I and Player II; $w_I(t) \in R^p$ is a standard Wiener process with

$$E[(w_I(t) - w_I(s))(w_I(t) - w_I(s))'] = |t - s| I$$

and (4) is interpreted as the meaning of

$$y_I(t) = \int_{t_0+\theta}^{t} H_I(\tau - \theta)x(\tau - \theta)\, d\tau + \int_{t_0+\theta}^{t} W_I(\tau - \theta)\, dw_I(\tau - \theta); \tag{6}$$

θ is the information delay; $A(t)$, $B(t)$, and $C(t)$ are $n \times n$, $n \times q$, and $n \times r$ piecewise-continuous matrices; $H_I(t)$ and $W_I(t)$ are $m \times n$ and

$m \times p$ differentiable matrices and $W_I(t)\, W_I'(t)$ is positive definite. It is assumed that Player I knows the initial state to be Gaussian with mean and covariance given by

$$E[x(t_0)] = \bar{x}_0, \qquad \operatorname{cov}[x(t_0), x(t_0)] = M_0;$$

it is also assumed that Player II knows these statistics of the initial state and that $x(t_0)$ is independent of the increment $w_I(t) - w_I(t_0)$, $t_0 < t \leqslant t_f - \theta$. All random processes are defined on a measurable space $(\Omega, \mathscr{B})$. The cost, which Player I wishes to minimize and Player II wishes to maximize, is given by

$$J_{t_0}(u, v) = x'(t_f)P'\, Px(t_f) + \int_{t_0}^{t_f} [u'(\tau)\, Q_I(\tau)\, u(\tau) - v'(\tau)\, Q_{II}(\tau)\, v(\tau)]\, d\tau, \tag{7}$$

where $Q_I(t)$ and $Q_{II}(t)$ are symmetric and positive-definite piecewise-continuous matrices; P is an $s \times n$ matrix.

Let $C_m[t_0, t_f]$ and $C_n[t_0, t_f]$ denote the classes of continuous functions defined on $[t_0, t_f]$ with values in R^m and R^n, respectively; and let the past accumulative data be given by

$$(\pi_t y_I)(s) = \begin{cases} y_I(s), & t_0 \leqslant s \leqslant t, \\ y_I(t), & t \leqslant s \leqslant t_f, \end{cases} \tag{8}$$

$$(\pi_t y_{II})(s) = \begin{cases} y_{II}(s), & t_0 \leqslant s \leqslant t, \\ y_{II}(t), & t \leqslant s \leqslant t_f. \end{cases} \tag{9}$$

Since $y_I(t) \in C_m[t_0, t_f]$ and $y_{II}(t) \in C_n[t_0, t_f]$, it follows that $\pi_t y_I \in C_m[t_0, t_f]$ and $\pi_t y_{II} \in C_n[t_0, t_f]$. We denote by $\mathscr{Y}_{It} \subset \mathscr{B}$ the minimal σ-algebra induced by the observation $\{\pi_t y_I\}$.

It is assumed that Player II can deduce Player I's state estimate $\hat{x}(t \mid t)$, that is,

$$\hat{x}(t \mid t) = E(x(t) \mid \mathscr{Y}_{It}). \tag{10}$$

Let $\mathscr{Y}_{IIt} \subset \mathscr{B}$ be the minimal σ-algebra induced by $\{\pi_t y_{II}, \hat{x}(t \mid t)\}$. Note that Player I has no observation in the interval $[t_0, t_0 + \theta)$ and then

$$\mathscr{Y}_{It} = \mathscr{Y}_{It_0}, \qquad t_0 \leqslant t \leqslant t_0 + \theta. \tag{11}$$

The admissible controls are defined as the closed-loop controls

$$u(t) = \varphi(t, \pi_t y_I), \tag{12}$$

$$v(t) = \psi(t, \pi_t y_{II}, \hat{x}(t \mid t)), \tag{13}$$

where the mappings

$$\varphi(\cdot, \cdot) : [t_0, t_f] \times C_m[t_0, t_f] \to R^q,$$

$$\psi(\cdot, \cdot, \cdot) : [t_0, t_f] \times C_n[t_0, t_f] \times R^n \to R^r$$

satisfy the Lipschitz conditions

$$| \varphi(t, \xi_1) - \varphi(t, \xi_2) | \leqslant K_1 \| \xi_1 - \xi_2 \|, \quad \xi_1, \xi_2 \in C_m[t_0, t_f], \quad t_0 \leqslant t \leqslant t_f, \tag{14}$$

$$| \psi(t, \zeta_1) - \psi(t, \zeta_2)| \leqslant K_2 \| \zeta_1 - \zeta_2 \|, \quad \zeta_1, \zeta_2 \in C_n[t_0, t_f] \times R^n, \quad t_0 \leqslant t \leqslant t_f. \tag{15}$$

Under these conditions, there exists a unique solution $(x(t), y_I(t))$ to (1)–(4) (see Ref. 8).

Our problem is to find the admissible controls (u^*, v^*) such that, for any admissible controls u and v,

$$E\{J_{t_0}(u^*, v^*)| \mathscr{Y}_{It_0}\} \leqslant E\{J_{t_0}(u, v^*)| \mathscr{Y}_{It_0}\}, \tag{16}$$

$$E\{J_{t_0}(u^*, v)| \mathscr{Y}_{IIt_0}\} \leqslant E\{J_{t_0}(u^*, v^*)| \mathscr{Y}_{IIt_0}\}. \tag{17}$$

These closed-loop controls u^* and v^* are said to be optimal. From (16) and (17), we get the saddle-point condition

$$E\{J_{t_0}(u^*, v)\} \leqslant E\{J_{t_0}(u^*, v^*)\} \leqslant E\{J_{t_0}(u, v^*)\} \tag{18}$$

for any admissible controls u and v.

3. Estimate of the State and Optimality Criterion

The zero-sum games have the property of equivalence and interchangeability of all the solutions satisfying the saddle-point condition (Ref. 3). This suggests the derivation of the optimal controls as follows. First, guess the admissible controls (u^*, v^*) and solve the optimal control problems such that

$$E\{J_{t_0}(u^{**}, v^*)| \mathscr{Y}_{It_0}\} \leqslant E\{J_{t_0}(u, v^*)| \mathscr{Y}_{It_0}\}, \tag{19}$$

$$E\{J_{t_0}(u^*, v)| \mathscr{Y}_{IIt_0}\} \leqslant E\{J_{t_0}(u^*, v^{**})| \mathscr{Y}_{IIt_0}\} \tag{20}$$

for any admissible controls (u, v). Second, find the condition such that the following equalities are satisfied simultaneously:

$$u^{**} = u^*, \tag{21}$$

$$v^{**} = v^*. \tag{22}$$

Then, u^* and v^* are optimal.

Let Φ and Ψ be the classes of functions

$$\Phi = \{\hat{\varphi} : [t_0, t_f] \times R^n \to R^q\}, \tag{23}$$

$$\Psi = \{\hat{\psi} : [t_0, t_f] \times R^n \times R^n \to R^r\}, \tag{24}$$

such that

$$|\hat{\varphi}(t, \lambda_1) - \hat{\varphi}(t, \lambda_2)| \leqslant K_3 |\lambda_1 - \lambda_2|, \qquad \lambda_1, \lambda_2 \in R^n, \qquad t_0 \leqslant t \leqslant t_f, \tag{25}$$

$$|\hat{\psi}(t, \mu_1) - \hat{\psi}(t, \mu_2)| < K_4 |\mu_1 - \mu_2|, \qquad \mu_1, \mu_2 \in R^{2n}, \qquad t_0 \leqslant t \leqslant t_f. \tag{26}$$

Now, we assume that the optimal controls are given in Φ and Ψ by

$$u^*(t) = W(t)\hat{x}(t \mid t), \tag{27}$$

$$v^*(t) = S(t)x(t) + T(t)\tilde{x}(t \mid t), \qquad t_0 \leqslant t \leqslant t_f, \tag{28}$$

where $\tilde{x}(t \mid t) = x(t) - \hat{x}(t \mid t)$.

When Player II's control is assumed to be given by (28), Player I's optimal state estimate $\hat{x}(t \mid t)$ is easily derived by a delay-differential equation

$$d\hat{x}(t \mid t)/dt = \{A(t) + C(t)\, S(t)\}\, \hat{x}(t \mid t) + B(t)\, \varphi(t, \pi_t y_I)$$
$$+ \kappa(t)\, \Phi(t, t - \theta)\, G(t)\{\dot{y}_I(t) - H_I(t - \theta)\, \hat{x}(t - \theta \mid t)\}, \tag{29}$$

$$\hat{x}(t_0 \mid t_0) = \bar{x}_0, \qquad t_0 \leqslant t \leqslant t_f, \tag{30}$$

$$\hat{x}(t - \theta \mid t) = \Phi(t - \theta, t)\, \hat{x}(t \mid t) - \int_{t-\theta}^{t} \Phi(t - \theta, \tau)\, \{B(\tau)\, \varphi(\tau, \pi_\tau y_I)$$
$$- C(\tau)\, T(\tau)\, \hat{x}(\tau \mid \tau)\}\, d\tau, \qquad t_0 + \theta \leqslant t \leqslant t_f, \tag{31}$$

where

$$\hat{x}(t - \theta \mid t) = E\{x(t - \theta) | \mathscr{Y}_{It}\},$$

$$G(t) = M(t - \theta \mid t)\, H_I'(t - \theta)(W_I(t - \theta)\, W_I'(t - \theta))^{-1},$$

and

$$\kappa(t) = \begin{cases} 0, & t_0 \leqslant t < t_0 + \theta, \\ 1, & t_0 + \theta \leqslant t \leqslant t_f; \end{cases} \tag{32}$$

$\Phi(t, \tau)$ is the transition matrix given by

$$\partial\Phi(t, \tau)/\partial t = [A(t) + C(t)\, \{S(t) + T(t)\}]\, \Phi(t, \tau), \tag{33}$$

$$\Phi(\tau, \tau) = I; \tag{34}$$

and the conditional covariance $M(t-\theta \mid t)$ on $[t_0+\theta, t_f]$ is defined by

$$M(t-\theta \mid t) = E\{(x(t-\theta) - \hat{x}(t-\theta \mid t))(x(t-\theta) - \hat{x}(t-\theta \mid t))' \mid \mathscr{Y}_{It}\} \quad (35)$$

and it is the solution to the equation

$$\begin{aligned} dM(t-\theta \mid t)/dt &= [A(t-\theta) + C(t-\theta)\{S(t-\theta) + T(t-\theta)\}]\, M(t-\theta \mid t) \\ &\quad + M(t-\theta \mid t)[A'(t-\theta) + \{S'(t-\theta) + T'(t-\theta)\}\, C'(t-\theta)] \\ &\quad - G(t)\, W_I(t-\theta)\, W_I'(t-\theta)\, G'(t), \end{aligned} \quad (36)$$

$$M(t_0 \mid t_0+\theta) = M_0, \qquad t_0+\theta \leqslant t \leqslant t_f. \quad (37)$$

Since $G(t)$ is differentiable, it is easily shown that the solution $\hat{x}(t \mid t)$ to (29)–(31) satisfies the Lipschitz condition in $\pi_t y_I$. This implies that the assumed controls (27) and (28) are admissible. In (29), the process $\dot{\nu}(t-\theta)$ given by

$$\dot{\nu}(t-\theta) = \dot{y}_I(t) - H_I(t-\theta)\, \hat{x}(t-\theta \mid t), \qquad t_0+\theta \leqslant t \leqslant t_f, \quad (38)$$

is called an innovation process, that is, $\dot{\nu}(t)$ is a white Gaussian process

$$E\{\dot{\nu}(t)\} = 0, \quad \operatorname{cov}\{\dot{\nu}(t), \dot{\nu}(\tau)\} = W_I(t)\, W_I'(t)\, \delta(t-\tau), \quad t_0+\theta \leqslant t \leqslant t_f. \quad (39)$$

From (36) and (37), the covariance matrix $M(t-\theta \mid t)$ is independent of Player I's data $\pi_t y_I$ and his control $u(t)$. These facts imply that $\mathscr{Y}_{It}$ and $\hat{x}(t \mid t)$ are equivalent statistics (Ref. 9). Thus, Player I's information is nested in Player II's information, that is,

$$\mathscr{Y}_{It} \subset \mathscr{Y}_{IIt}, \qquad t_0 \leqslant t \leqslant t_f, \quad (40)$$

so that, for any $\mathscr{Y}_{It}$-measurable function $h(t)$,

$$E\{h(t) \mid \mathscr{Y}_{IIt}\} = h(t); \quad (41)$$

and, from (16) and (17), we have the saddle-point condition

$$E\{J_t(u^*, v) \mid \mathscr{Y}_{It}\} \leqslant E\{J_t(u^*, v^*) \mid \mathscr{Y}_{It}\} \leqslant E\{J_t(u, v^*) \mid \mathscr{Y}_{It}\} \quad (42)$$

for any $t \in [t_0, t_f]$ and any admissible controls (u, v). Then, the closure problem (Ref. 3) is not raised, and Player I can derive $E\{x(t) \mid \mathscr{Y}_{It}\}$ based on only an estimator (29) which depends on the data $y_I(t)$ and the history of the state estimate $\hat{x}(\tau \mid \tau)$, $\tau \in (t-\theta, t)$.

Player I's state estimate $\hat{x}(t \mid t)$ given by (29)–(31) is derived on the assumption that Player II uses the optimal control (28). Suppose that Player II does not use (28); $\hat{x}(t \mid t)$ obtained by (29)–(31) is different from (10). Then, the error of $\hat{x}(t \mid t)$ given by (29)–(31) is derived by a couple of equations, namely,

$$\begin{aligned} d\tilde{x}(t \mid t)/dt &= A(t)\,\tilde{x}(t \mid t) + C(t)\,[\psi(t, \pi_t y_{II}, \hat{x}(t \mid t)) - S(t)\,\{x(t) - \tilde{x}(t \mid t)\}] \\ &\quad - \kappa(t)\Phi(t, t-\theta)G(t)\{H_I(t-\theta)\tilde{x}(t-\theta \mid t) + W_I(t-\theta)\hat{w}_I(t-\theta)\}, \end{aligned} \tag{43}$$

$$\tilde{x}(t_0 \mid t_0) = x_0 - \bar{x}_0, \qquad t_0 \leqslant t \leqslant t_f, \tag{44}$$

$$\begin{aligned} d\tilde{x}(t-\theta \mid t)/dt &= A(t-\theta)\tilde{x}(t-\theta \mid t) + C(t-\theta)[\psi(t-\theta, \pi_{t-\theta} y_{II}, \hat{x}(t-\theta \mid t-\theta)) \\ &\quad - \{S(t-\theta) + T(t-\theta)\}\{x(t-\theta) - \tilde{x}(t-\theta \mid t)\} + T(t-\theta)\{x(t-\theta) \\ &\quad - \tilde{x}(t-\theta \mid t-\theta)\}] - G(t)\{H_I(t-\theta)\tilde{x}(t-\theta \mid t) + W_I(t-\theta)\hat{w}_I(t-\theta)\}, \end{aligned} \tag{45}$$

$$\tilde{x}(t_0 \mid t_0 + \theta) = x_0 - \bar{x}_0, \qquad t_0 + \theta \leqslant t \leqslant t_f, \tag{46}$$

where

$$\tilde{x}(t-\theta \mid t) = x(t-\theta) - \hat{x}(t-\theta \mid t).$$

Consider two optimal control problems such that Player I and II select their controls satisfying (19) and (20), respectively, where their controls are restricted to be in the classes Φ and Ψ, that is,

$$u(t) = \hat{\varphi}(t, \hat{x}(t \mid t)), \tag{47}$$

$$v(t) = \hat{\psi}(t, x(t), \tilde{x}(t \mid t)). \tag{48}$$

Here, each player chooses his optimal control under the assumption that his opponent is using the optimal control. Hence, the estimate $\hat{x}(t \mid t)$ which the players use in their closed-loop controls (47) and (48) is the solution to (29)–(31). Let the functionals $V_I(t, \hat{x}(t \mid t))$ and $V_{II}(t, x(t), \tilde{x}(t \mid t))$ be defined by

$$\begin{aligned} V_I(t, \hat{x}(t \mid t)) &= \min_{\hat{\varphi} \in \Phi} E[x'(t_f)\,P'Px(t_f) \\ &\quad + \int_t^{t_f} \hat{\varphi}'(\tau, \hat{x}(\tau \mid \tau))\,Q_I(\tau)\,\hat{\varphi}(\tau, \hat{x}(\tau \mid \tau)) - \{S(\tau)\,x(\tau) + T(\tau)\,\tilde{x}(\tau \mid \tau)\}' \\ &\quad \cdot Q_{II}(\tau)\,\{S(\tau)x(\tau) + T(\tau)\tilde{x}(\tau \mid \tau)\}\,d\tau \mid \mathscr{Y}_{It}], \qquad t_0 \leqslant t \leqslant t_f, \end{aligned} \tag{49}$$

$$\begin{aligned} V_{II}(t, x(t), \tilde{x}(t \mid t)) &= \max_{\hat{\psi} \in \Psi} E[x'(t_f)\,P'Px(t_f) \\ &\quad + \int_t^{t_f} \{\hat{x}'(\tau \mid \tau)\,W'(\tau)\,Q_I(\tau)\,W(\tau)\,\hat{x}(\tau \mid \tau) - \hat{\psi}'(\tau, x(\tau), \tilde{x}(\tau \mid \tau)) \\ &\quad \cdot Q_{II}(\tau)\,\hat{\psi}(\tau, x(\tau), \tilde{x}(\tau \mid \tau))\}\,d\tau \mid \mathscr{Y}_{IIt}], \qquad t_0 \leqslant t \leqslant t_f. \end{aligned} \tag{50}$$

By the principle of optimality and Ito–Dynkin's formula (Ref. 10), the functional equations are obtained by

$$\min_{\hat{\phi}\in\Phi} [\hat{\varphi}'(t, \hat{x}(t \mid t)) Q_I(t) \hat{\varphi}(t, \hat{x}(t \mid t)) - \hat{L}(t, \hat{x}(t \mid t)) + V_{It}(t, \hat{x}(t \mid t)) + \mathscr{L}_{\hat{\varphi}} V_I(t, \hat{x}(t \mid t))] = 0, \qquad t_0 \leqslant t \leqslant t_f, \tag{51}$$

$$\max_{\hat{\psi}\in\Psi} [(x(t) - \tilde{x}(t \mid t))' W'(t) Q_I(t) W(t) (x(t) - \tilde{x}(t \mid t)) - \hat{\psi}'(t, x(t), \tilde{x}(t \mid t)) Q_{II}(t) \hat{\psi}(t, x(t), \tilde{x}(t \mid t)) + V_{IIt}(t, x(t), \tilde{x}(t \mid t)) + \mathscr{L}_{\hat{\psi}} V_{II}(t, x(t), \tilde{x}(t \mid t))] = 0, \qquad t_0 \leqslant t \leqslant t_f. \tag{52}$$

Here, V_{it}, $i = I, II$, denotes $\partial V_i/\partial t$; also,

$$\hat{L}(t, \hat{x}(t \mid t)) = \hat{x}'(t \mid t) S'(t) Q_{II}(t) S(t) \hat{x}(t \mid t) + \operatorname{tr}[M(t \mid t)\{S(t) + T(t)\}' Q_{II}(t)\{S(t) + T(t)\}], \tag{53}$$

$$\mathscr{L}_{\hat{\varphi}} V_I(t, \hat{x}(t \mid t)) = \{A(t) \hat{x}(t \mid t) + B(t) \hat{\varphi}(t, \hat{x}(t \mid t)) + C(t) S(t) \hat{x}(t \mid t)\}' \cdot [\partial V_I(t, \hat{x}(t \mid t))/\partial \hat{x}(t \mid t)] + \tfrac{1}{2} \kappa(t) \operatorname{tr}\{[\partial^2 V_I(t, \hat{x}(t \mid t))/\partial \hat{x}^2(t \mid t)] \Phi(t, t - \theta) \cdot G(t) W_I(t - \theta) W_I'(t - \theta) G'(t) \Phi'(t, t - \theta)\}, \tag{54}$$

$$\begin{aligned}\mathscr{L}_{\hat{\psi}} V_{II}(t, x(t), \tilde{x}(t \mid t)) &= \{A(t) x(t) + B(t) W(t)(x(t) - \tilde{x}(t \mid t)) \\ &+ C(t) \hat{\psi}(t, x(t), \tilde{x}(t \mid t))\}' [\partial V_{II}(t, x(t), \tilde{x}(t \mid t))/\partial x(t)] \\ &+ \{A(t) \tilde{x}(t \mid t) + C(t) \hat{\psi}(t, x(t), \tilde{x}(t \mid t)) - C(t) S(t) (x(t) \\ &- \tilde{x}(t \mid t)) - \kappa(t) \Phi(t, t - \theta) G(t) H_I(t - \theta) \tilde{x}(t - \theta \mid t)\}' \\ &\cdot [\partial V_{II}(t, x(t), \tilde{x}(t \mid t))/\partial \tilde{x}(t \mid t)] + \tfrac{1}{2} \kappa(t) \operatorname{tr}\{[\partial^2 V_{II}(t, x(t), \tilde{x}(t \mid t))/\partial \tilde{x}^2(t \mid t)] \\ &\cdot \Phi(t, t - \theta) G(t) W_I(t - \theta) W_I'(t - \theta) G'(t) \Phi'(t, t - \theta)\}.\end{aligned} \tag{55}$$

For these optimal control problems, we get the following sufficient conditions for optimality.

Lemma 3.1. Let $V_I(t, \hat{x}(t \mid t))$ and $V_{II}(t, x(t), \tilde{x}(t \mid t))$ be functions such that $V_I : [t_0, t_f] \times R^n \to R^1$ and $V_{II} : [t_0, t_f] \times R^n \times R^n \to R^1$; V_I, V_{It}, $V_{I\hat{x}}$, $V_{I\hat{x}\hat{x}}$ and V_{II}, V_{IIt}, V_{IIx}, $V_{II\tilde{x}}$, $V_{II\tilde{x}\tilde{x}}$ are continuous; and, for some K_5 and K_6,

$$|V_I| + |V_{It}| + |\hat{x}|\,|V_{I\hat{x}}| + |V_{I\hat{x}\hat{x}}| \leqslant K_5(1 + |\hat{x}|^2), \tag{56}$$

$$|V_{II}| + |V_{IIt}| + (|x| + |\tilde{x}|)(|V_{IIx}| + |V_{II\tilde{x}}|) + |V_{II\tilde{x}\tilde{x}}| \leqslant K_6(1 + (|x| + |\tilde{x}|)^2). \tag{57}$$

Suppose there exist closed-loop control laws $\hat{\varphi}^{**} \in \Phi$ and $\hat{\psi}^{**} \in \Psi$ such that, for any $(t, \hat{x}, x, \tilde{x}) \in [t_0, t_f] \times R^n \times R^n \times R^n$ and any admissible controls φ and ψ,

$$\begin{aligned} 0 &= V_{It}(t, \hat{x}(t \mid t)) + \mathscr{L}_{\hat{\varphi}^{**}} V_I(t, \hat{x}(t \mid t)) \\ &\quad + \hat{\varphi}^{**\prime}(t, \hat{x}(t \mid t)) Q_I(t) \hat{\varphi}^{**}(t, \hat{x}(t \mid t)) - \hat{L}(t, \hat{x}(t \mid t)) \\ &\leqslant V_{It}(t, \hat{x}(t \mid t)) + \mathscr{L}_{\varphi} V_I(t, \hat{x}(t \mid t)) \\ &\quad + \varphi'(t, \pi_t y_I) Q_I(t) \varphi(t, \pi_t y_I) - \hat{L}(t, \hat{x}(t \mid t)), \end{aligned} \tag{58}$$

$$V_I(t_f, \hat{x}(t_f \mid t_f)) = \hat{x}'(t_f \mid t_f) P' P \hat{x}(t_f \mid t_f) + \operatorname{tr}\{M(t_f \mid t_f) P' P\}, \tag{59}$$

$$\begin{aligned} 0 &= V_{IIt}(t, x(t), \tilde{x}(t \mid t)) + \mathscr{L}_{\hat{\psi}^{**}} V_{II}(t, x(t), \tilde{x}(t \mid t)) \\ &\quad + (x(t) - \tilde{x}(t \mid t))' W'(t) Q_I(t) W(t) (x(t) - \tilde{x}(t \mid t)) \\ &\quad - \hat{\psi}^{**\prime}(t, x(t), \tilde{x}(t \mid t)) Q_{II}(t) \hat{\psi}^{**}(t, x(t), \tilde{x}(t \mid t)) \\ &\geqslant V_{IIt}(t, x(t), \tilde{x}(t \mid t)) + \mathscr{L}_{\psi} V_{II}(t, x(t), \tilde{x}(t \mid t)) \\ &\quad + (x(t) - \tilde{x}(t \mid t))' W'(t) Q_I(t) W(t) (x(t) - \tilde{x}(t \mid t)) \\ &\quad - \psi'(t, \pi_t y_{II}, \hat{x}(t \mid t)) Q_{II}(t) \psi(t, \pi_t y_{II}, \hat{x}(t \mid t)), \end{aligned} \tag{60}$$

$$V_{II}(t_f, x(t_f), \tilde{x}(t_f \mid t_f)) = x'(t_f) P' P x(t_f). \tag{61}$$

Then, the controls $\hat{\varphi}^{**}$ and $\hat{\psi}^{**}$ are optimal for any admissible controls.

This lemma is proved by the method analogous to Ref. 11. Suppose that the conditions (21) and (22) are satisfied simultaneously; it follows from this lemma that Player I's and Player II's optimal closed-loop control laws can be constructed by the state estimate $\hat{x}(t \mid t)$ and by the state $x(t)$ and the estimation error $\tilde{x}(t \mid t)$, respectively.

4. Game with Delayed and Noisy Information

Lemma 3.1 shows that the optimal closed-loop control laws are derived from (51) and (52). Then, we get

$$u^{**}(t) = -\tfrac{1}{2} Q_I^{-1}(t) B'(t) \left[\partial V_I(t, \hat{x}(t \mid t)) / \partial \hat{x}(t \mid t)\right], \tag{62}$$

$$\begin{aligned} v^{**}(t) = \tfrac{1}{2} Q_{II}^{-1}(t) C'(t) \{&\partial V_{II}(t, x(t), \tilde{x}(t \mid t)) / \partial x(t) \\ &+ \partial V_{II}(t, x(t), \tilde{x}(t \mid t)) / \partial \tilde{x}(t \mid t)\}. \end{aligned} \tag{63}$$

Substituting (62), (63) into (51), (52), Bellman's equations are obtained. Suppose that the solution to this Bellman's equation for Player II is given by

$$\begin{aligned} V_{II}(t, x(t), \tilde{x}(t \mid t)) &= x'(t) R(t) x(t) + x'(t) L(t) \tilde{x}(t \mid t) \\ &\quad + \tilde{x}'(t \mid t) N(t) \tilde{x}(t \mid t) + r(t). \end{aligned} \tag{64}$$

Since $\mathscr{Y}_{It} \subset \mathscr{Y}_{IIt}$, we get

$$V_I(t, \hat{x}(t \mid t)) = E\{V_{II}(t, x(t, x(t), \tilde{x}(t \mid t)) \mid \mathscr{Y}_{It}\}$$
$$= \hat{x}'(t \mid t)\, R(t)\, \hat{x}(t \mid t) + \mathrm{tr}[M(t \mid t)\{R(t) + L(t) + N(t)\}] + r(t). \tag{65}$$

Then, we get the following theorem (Appendix A).

Theorem 4.1. The optimal controls which satisfy (16) and (17) subject to (1)–(5) and the assumption that Player II can deduce the value of $\hat{x}(t \mid t)$ at time $t \in [t_0, t_f]$, are given by

$$u^*(t) = -Q_I^{-1}(t)\, B'(t)\, R(t)\, \hat{x}(t \mid t), \tag{66}$$

$$v^*(t) = Q_{II}^{-1}(t)\, C'(t)\{R(t)\, x(t) + N(t)\, \tilde{x}(t \mid t)\}, \qquad t_0 \leqslant t \leqslant t_f; \tag{67}$$

and the optimal cost from t_0 to t_f is given by

$$J_{t_0}(u^*, v^*) = x'(t_0)\, R(t_0)\, x(t_0) + \tilde{x}'(t_0 \mid t_0)\, N(t_0)\, \tilde{x}(t_0 \mid t_0) + r(t_0), \tag{68}$$

where $\hat{x}(t \mid t)$, $\tilde{x}(t \mid t)$, and $M(t - \theta \mid t)$ are, respectively, the solutions to (29)–(31), (43)–(46), and (36)–(37) into which (66) and (67) are substituted; the symmetric matrices $R(t)$ and $N(t)$ are the solutions of

$$dR(t)/dt + A'(t)\, R(t) + R(t)\, A(t) - R(t)\, B(t)\, Q_I^{-1}(t)\, B'(t)\, R(t)$$
$$+ R(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)\, R(t) = 0, \tag{69}$$

$$R(t_f) = P'P, \tag{70}$$

$$dN(t)/dt + A'(t)\, N(t) + N(t)\, A(t) + N(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)\, R(t)$$
$$+ R(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)\, N(t) + N(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)\, N(t)$$
$$+ R(t)\, B(t)\, Q_I^{-1}(t)\, B'(t)\, R(t)$$
$$- \kappa(t)\, N(t)\, \Phi(t, t - \theta)\, G(t)\, H_I(t - \theta)\, \Phi(t - \theta, t)$$
$$- \kappa(t)\, \Phi'(t - \theta, t)\, H_I'(t - \theta)\, G'(t)\, \Phi'(t, t - \theta)\, N(t) = 0, \tag{71}$$

$$N(t_f) = 0; \tag{72}$$

the scalar $r(t)$ satisfies the relations

$$dr(t)/dt + \kappa(t)\, \mathrm{tr}[N(t)\, \Phi(t, t - \theta)\, G(t)\, W_I(t - \theta)\, W_I'(t - \theta)$$
$$\cdot\, G'(t)\, \Phi'(t, t - \theta)] = 0, \tag{73}$$

$$r(t_f) = 0. \tag{74}$$

It is easily shown that, from (43)–(46), (66), and (67), we get

$$\tilde{x}(t \mid t) = \Phi(t, t-\theta)\, \tilde{x}(t-\theta \mid t), \tag{75}$$

$$M(t \mid t) = \Phi(t, t-\theta)\, M(t-\theta \mid t)\, \Phi'(t, t-\theta), \qquad t_0 + \theta \leqslant t \leqslant t_f . \tag{76}$$

Player I's optimal control law (66) shows the separation theorem to be satisfied, in the sense that the state in the optimal control law for the deterministic game (Ref. 12) is replaced by the optimal estimate $\hat{x}(t \mid t)$. In view of (68), the optimal cost from t_0 to t_f consists of three terms. The first term is the optimal cost from t_0 to t_f corresponding to the deterministic game with the initial state x_0 (Ref. 12). The second term depends on the initial error $\tilde{x}(t_0 \mid t_0)$ in Player I's state estimate. The third term is due to the noise in Player I's measurement from $t_0 + \theta$ to t_f. In the interval $[t_0, t_0 + \theta)$, Player I has no observation, so that $\hat{x}(t \mid t) = \hat{x}(t \mid t_0)$ and his control coincides with the open-loop control. Suppose that $\theta > t_f - t_0$, this game is reduced to the one in which one player has perfect information and the other has no observation. In the case where the information delay is reduced to zero, the above results are easily shown to coincide with the results obtained from the game of imperfect information without delay (Refs. 3–4).

Theorem 4.1 is derived on the assumption that Player II can deduce Player I's state estimate $\hat{x}(t \mid t)$. Some of the conditions needed to satisfy this assumption for the game of imperfect information without delay have been discussed variously (Refs. 3–4). These conditions are applicable to this game with delayed information. In the time-invariant case, we get a less restrictive condition as follows.

Lemma 4.1. Consider the time-invariant system (1)–(5), (7). Suppose that Player I uses his optimal control on $[t_0, t_f]$. Player II can deduce $\hat{x}(t \mid t)$ on $[t_0, t_f]$ if it holds that

$$\operatorname{rank}[F_0'(t), F_1'(t), \ldots, F_i'(t)] = n, \qquad t_0 \leqslant t < t_0 + \theta, \tag{77}$$

$$\begin{aligned}
&\mathscr{R}(R(t)\, BQ_I^{-1}B' + N(t)\, CQ_{II}^{-1}C') + [\{R(t)\, A + \kappa(t)\, \Phi'(t-\theta, t)\, H_I' G'(t) \\
&\quad \cdot \Phi'(t, t-\theta)\, R(t)\}\, BQ_I^{-1}B' + \{N(t)\, A + N(t)\, CQ_{II}^{-1}C' R(t) \\
&\quad + R(t)\, BQ_I^{-1}B'\, N(t) + N(t)\, CQ_{II}^{-1}C'\, N(t) + R(t)\, BQ_I^{-1}B'\, R(t) \\
&\quad - \kappa(t)\, N(t)\, \Phi(t, t-\theta)\, G(t)\, H_I\, \Phi(t-\theta, t)\}\, CQ_{II}^{-1}C'] \\
&\quad \cdot \mathscr{N}\{W_I' G'(t)\, \Phi'(t, t-\theta)\, (R(t)\, BQ_I^{-1}B' + N(t)\, CQ_{II}^{-1}C')\} = R^n, \\
&\qquad\qquad\qquad\qquad\qquad\qquad t_0 + \theta \leqslant t \leqslant t_f ,
\end{aligned} \tag{78}$$

where i is some finite integer;

$$F_{i+1}(t) = (d/dt)F_i(t) + F_i(t)\{A - BQ_I^{-1}B' R(t) + CQ_{II}^{-1}C' R(t)\},$$
$$F_0(t) = BQ_I^{-1}B' R(t) + CQ_{II}^{-1}C' N(t);$$

$\mathscr{R}(Y)$ and $\mathscr{N}(Y)$ are the range and the null space of Y, respectively.

This lemma is proved in Appendix B. Player I has no observation in the time interval $[t_0, t_0 + \theta)$, so that $\hat{x}(t \mid t)$ on $[t_0, t_0 + \theta)$ obeys the ordinary differential equation (29). On the other hand, $\hat{x}(t \mid t)$ on $[t_0 + \theta, t_f]$ is the solution to the stochastic differential equation (29) and (31). Hence, the differing types of conditions [(77) and (78)] obtain for $t \in [t_0, t_0 + \theta)$ and $[t_0 + \theta, t_f]$.

5. Conclusions

In this paper, we have formulated a linear-quadratic–Gaussian zero-sum game in which Player I has delayed and noisy information and Player II has perfect information. Here, it is assumed that Player II can derive Player I's state estimate, which means that Player I's information is nested in Player II's information. Then, it is shown that the solution to this problem requires an estimator given by a delay-differential equation and Player I's optimal closed-loop control law satisfies the separation theorem.

6. Appendix A: Proof of Theorem 4.1

Substituting (64)–(65) into (58)–(61), we can get the optimal controls

$$u^*(t) = -Q_I^{-1}(t)\, B'(t)\, R(t)\, \hat{x}(t \mid t), \tag{79}$$

$$v^*(t) = Q_{II}^{-1}(t)\, C'(t)\{(R(t) + \tfrac{1}{2}L'(t))\, x(t) + (N(t) + \tfrac{1}{2}L(t))\, \hat{x}(t \mid t)\}, \tag{80}$$

where

$$dR(t)/dt + A'(t)\, R(t) + R(t)\, A(t) - R(t)\, B(t)\, Q_I^{-1}(t)\, B'(t)\, R(t)$$
$$+ R(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)\, R(t) - \tfrac{1}{4}L(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)L'(t) = 0, \tag{81}$$

$$R(t_f) = P'P, \tag{82}$$

$$dL(t)/dt + A'(t)L(t) + L(t)\, A(t) - R(t)\, B(t)\, Q_I^{-1}(t)\, B'(t)L(t)$$
$$+ R(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)L(t) + L(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)\, R(t)$$
$$+ \tfrac{1}{2}L(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)L(t) + \tfrac{1}{2}L(t)\, C(t)\, Q_{II}^{-1}(t)\, C'(t)L'(t)$$
$$- \kappa(t)L(t)\, \Phi(t, t - \theta)\, G(t)\, H_I(t - \theta)\, \Phi(t - \theta, t) = 0, \tag{83}$$

$$L(t_f) = 0, \tag{84}$$

$$
\begin{aligned}
&dN(t)/dt + A'(t)\,N(t) + N(t)\,A(t) + N(t)\,C(t)\,Q_{II}^{-1}(t)\,C'(t)\,R(t)\\
&\quad + R(t)\,C(t)\,Q_{II}^{-1}(t)\,C'(t)\,N(t) + N(t)\,C(t)\,Q_{II}^{-1}(t)C'(t)\,N(t)\\
&\quad + R(t)\,B(t)\,Q_I^{-1}(t)\,B'(t)\,R(t) - \kappa(t)\,N(t)\,\Phi(t, t-\theta)\,G(t)\\
&\quad\ \cdot H_I(t-\theta)\,\Phi(t-\theta, t) - \kappa(t)\,\Phi'(t-\theta, t)\,H_I'(t-\theta)\,G'(t)\\
&\quad\ \cdot \Phi'(t, t-\theta)\,N(t) + \tfrac{1}{2}(L(t) + L'(t))\,C(t)\,Q_{II}^{-1}(t)\,C'(t)\,N(t)\\
&\quad + \tfrac{1}{2}\,N(t)\,C(t)\,Q_{II}^{-1}(t)\,C'(t)\,(L(t) + L'(t))\\
&\quad + \tfrac{1}{2}\,R(t)\,B(t)\,Q_I^{-1}(t)\,B'(t)L(t) + \tfrac{1}{2}\,L'(t)\,B(t)\,Q_I^{-1}(t)\,B'(t)\,R(t)\\
&\quad + \tfrac{1}{4}L'(t)\,C(t)\,Q_{II}^{-1}(t)\,C'(t)\,L(t) = 0,
\end{aligned}
\tag{85}
$$

$$
N(t_f) = 0, \tag{86}
$$

$$
\begin{aligned}
&dr(t)/dt + \kappa(t)\,\mathrm{tr}[N(t)\,\Phi(t, t-\theta)\,G(t)\,W_I(t-\theta)\,W_I'(t-\theta)\\
&\qquad \cdot G'(t)\,\Phi'(t, t-\theta)] = 0,
\end{aligned}
\tag{87}
$$

$$
r(t_f) = 0. \tag{88}
$$

From (83)–(84), we get

$$
L(t) = 0, \qquad t_0 \leqslant t \leqslant t_f\,. \tag{89}
$$

This completes the proof.

7. Appendix B: Proof of Lemma 4.1

Since Player II knows $x(t)$ and $dx(t)/dt$, he can utilize the observation q_0 defined by

$$
\begin{aligned}
q_0(t) &= -dx(t)/dt + \{A + CQ_{II}^{-1}C'(R(t) + N(t))\}\,x(t)\\
&= \{BQ_I^{-1}B'R(t) + CQ_{II}^{-1}C'\,N(t)\}\,\hat{x}(t \mid t) \equiv F_0(t)\,\hat{x}(t \mid t).
\end{aligned}
\tag{90}
$$

Thus, we can recognize the projection of $\hat{x}(t \mid t)$ on the subspace

$$
\mathscr{N}(BQ_I^{-1}B'\,R(t) + CQ_{II}^{-1}C'\,N(t))^{\perp} = \mathscr{R}(R(t)\,BQ_I^{-1}B' + N(t)\,CQ_{II}^{-1}C'). \tag{91}
$$

More knowledge of $\hat{x}(t \mid t)$ can be obtained by taking the derivative of (90) and substituting (29), (69), (71), and (76), as follows:

$$
\begin{aligned}
&[-BQ_I^{-1}B'\,A'R(t) - CQ_{II}^{-1}C'\{A'N(t) + R(t)\,CQ_{II}^{-1}C'\,N(t) + N(t)\,BQ_I^{-1}B'\,R(t)\\
&\quad + N(t)\,CQ_{II}^{-1}C'\,N(t) + R(t)\,BQ_I^{-1}B'\,R(t)\\
&\quad - \kappa(t)\,\Phi'(t-\theta, t)\,H_I'G'(t)\,\Phi'(t, t-\theta)\,N(t)\}\\
&\quad - \kappa(t)\,BQ_I^{-1}B'\,R(t)\,\Phi(t, t-\theta)\,G(t)\,H_I\,\Phi(t-\theta, t)]\,\hat{x}(t \mid t)\\
&\quad = \dot{q}_0(t) - \kappa(t)\,\{BQ_I^{-1}B'\,R(t) + CQ_{II}^{-1}C'\,N(t)\}\,\Phi(t, t-\theta)\,G(t)\\
&\qquad \cdot \{H_I\,\Phi(t-\theta, t)\,x(t) + W_I\dot{w}_I(t-\theta)\} \equiv F_1(t)\,\hat{x}(t \mid t).
\end{aligned}
\tag{92}
$$

For each t on $[t_0 + \theta, t_f]$, let us define the projection $P(t)$ on the subspace

$$\mathscr{R}\{(BQ_I^{-1}B' R(t) + CQ_{II}^{-1}C' N(t)) \Phi(t, t-\theta) G(t) W_I\}^{\perp} = \mathscr{N}\{W_I' G'(t) \Phi'(t, t-\theta) (R(t) BQ_I^{-1}B' + N(t) CQ_{II}^{-1}C')\}. \quad (93)$$

Here, we define the process $q_2(t)$ as follows:

$$\begin{aligned} q_2(t) &= \int_{t_0}^{t} P(s) F_1(s) \hat{x}(s \mid s)\, ds \\ &= \int_{t_0}^{t} P(s) [\dot{q}_0(s) - \kappa(s) \{BQ_I^{-1}B' R(s) + CQ_{II}^{-1}C' N(s)\} \\ &\quad \cdot \Phi(s, s-\theta) G(s) H_I \Phi(s-\theta, s) x(s)]\, ds. \end{aligned} \quad (94)$$

Since $q_2(t)$ is equal to the middle term of (94), it must be differentiable, and so Player II can get the observation $q_1(t)$ by differentiating $q_2(t)$. Thus, he can recognize the projection of $\hat{x}(t \mid t)$ on the subspace

$$\mathscr{N}(P(t) F_1(t))^{\perp} = F_1'(t) \mathscr{R}(P'(t)). \quad (95)$$

From the definition of $P(t)$, the right-hand side of (95) is reduced to

$$F_1'(t) \mathscr{N}\{W_I' G'(t) \Phi'(t, t-\theta) (R(t) BQ_I^{-1}B' + N(t) CQ_{II}^{-1}C')\}. \quad (96)$$

Then, from (91) and (96), we can get the condition (78) for $t \in [t_0 + \theta, t_f]$. In the interval $[t_0, t_0 + \theta)$, $\kappa(t) = 0$, and so $P(t) = I$. Then, iterating the above procedure, we see that the condition (77) is obtained.

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