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Optimal Control of a Class of Systems with Continuous Lags: Dynamic Programming Approach and Economic Interpretations[1]

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Abstract. This paper derives a maximum principle for dynamic systems with continuous lags, i.e., systems governed by integrodifferential equations, using dynamic programming. As a result, the adjoint variables can be provided with useful economic interpretations.

Key Words. Dynamic programming, maximum principle, distributed parameter systems, integrodifferential equations, economic applications.

1. Introduction

Dynamic systems with continuous lags occur frequently in modeling of population, economic, and ecological systems; e.g., see Sethi and McGuire (Ref. 1), Arthur and McNicoll (Ref. 2), Banks and Manitius (Ref. 3), Bailey (Ref. 4), and Pauwels (Ref. 5). Optimization of these systems is usually carried out with the maximum principle (see Refs. 6–9). This approach yields adjoint variables, which are difficult to interpret in the framework of the maximum principle.

This paper derives the maximum principle by using the method of dynamic programming. This involves a transformation of the original system to an optimal control problem governed by both lumped-parameter and distributed-parameter equations. Our approach uses a method similar to those of Wang (Ref. 10) and Brogan (Ref. 11). The result of our analysis is a characterization of the Hamiltonian and the adjoint variables in terms of the value function satisfying the Hamilton–Jacobi–Bellman equation of

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dynamic programming. We illustrate our interpretation with respect to a continuous-time version of the example in Burdet and Sethi (Ref. 12).

In the next section, we define the optimal control problem and make necessary transformations. Section 3 derives the Hamilton–Jacobi–Bellman equation and obtains the maximum principle. In Section 4, we provide economic interpretations to the adjoint variable. For a simpler class of problems with continuous lags, an alternative form of the adjoint equation is derived in Section 5. We illustrate our results with an optimal advertising problem in Section 6, and Section 7 concludes the paper.

2. Optimal Control Model with Continuous Lags

Let $x(t) \in R^n$ and $u(t) \in \Omega \subset R^m$ denote the state and control trajectory,[4] respectively, defined on the time interval $(-\infty, T]$. Let $F(x, u, t), f(x, u, t), g(x, u, \tau, t)$ be continuously differentiable functions on $R^n \times \Omega \times [0, T]$ and $R^n \times \Omega \times \{(\tau, t): -\infty < \tau \leq t \leq T\}$. Then, the optimal control model is as follows:

$$\max\left\{J = \int_0^T F(x(t), u(t), t)\, dt\right\}, \tag{1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t) + \int_{-\infty}^{t} g(x(\tau), u(\tau), \tau, t)\, d\tau, \tag{2}$$

with the initial conditions

$$\begin{aligned} x(t) &= \underline{x}(t), \qquad t \in (-\infty, 0], \\ u(t) &= \underline{u}(t), \qquad t \in (-\infty, 0]. \end{aligned} \tag{3}$$

In order to transform the integrodifferential equation (2), we introduce the new state variable $y(t, s)$, defined on $\{(t, s): 0 \leq t \leq T, 0 \leq s \leq T - t\}$ by

$$y(t, s) \triangleq \int_{-\infty}^{t} g(x(\tau), u(\tau), \tau, t+s)\, d\tau. \tag{4}$$

By introducing the distributed-parameter state variable $y(t, \cdot)$, denoting $y(t, s)$ for $s \in [0, T - t]$, the original problem (1)–(3) is easily transformed

[4] In the sequel, x and u are treated as scalars in order to simplify the notation. The generalization to the case of $n > 1$, $m > 1$ is straightforward. Note that, in this paper, $x(t)$ is only a part of the state of the system under consideration. Some writers prefer the term phase variable for $x(t)$.

to the following problem:

$$\max\left\{J=\int_0^T F(x(t), u(t), t)\, dt\right\},$$

subject to

$$\dot{x}(t)=f(x(t), u(t), t)+y(t, 0), \tag{5}$$

$$y_t(t, s)=g(x(t), u(t), t, t+s)+y_s(t, s), \tag{6}$$

with the initial conditions

$$x(0)=x_0 \triangleq \underline{x}(0), \tag{7}$$

$$y(0, s)=h(s) \triangleq \int_{-\infty}^{0} g(\underline{x}(\tau), \underline{u}(\tau), \tau, s)\, d\tau, \tag{8}$$

with subscripts denoting partial derivatives. The additional state equation (6) is easily obtained by computing y_t and y_s from (4) and combining them. It should be noted that the state variable $x(t)$, together with the past history $y(t, \cdot)$, represents the complete *state* of the system described above.

The transformation (4) is in the spirit of Burdet and Sethi (Ref. 12), who treated a discrete-time version of our model. In their model, it is possible to transform the problem into a standard control problem in a higher-dimensional state space.[5]

In our model, Eq. (5) is a lumped-parameter equation, and Eq. (6) is a distributed-parameter equation, i.e., a partial differential equation. Thus, the transformed problem is not a standard distributed-parameter optimal control problem.

In the next section, we will use the dynamic programming approach to obtain necessary optimality conditions and adjoint functions associated with $x(t)$ and $y(t, \cdot)$ for the problem represented by (1) and (5)–(8).

3. Dynamic Programming Formulation and the Maximum Principle

Let us define the *value function*

$$J(\bar{x}, \bar{y}(\cdot), \bar{t}) \triangleq \max\left\{\int_{\bar{t}}^{T} F(x(t), u(t), t)\, dt \colon x(\bar{t})=\bar{x},\ y(\bar{t}, \cdot)=\dot{y}(\cdot);\right.$$
$$\left. u(t)\in\Omega,\ \bar{t}\le t\le T\right\}, \tag{9}$$

[5] Equations (4), (5), (6) correspond, respectively, to Eqs. (12), (4), (11) of Ref. 12; see also Ref. 13.

which is a functional of $\bar{x}, \bar{t}$, and the continuously differentiable function $\bar{y}(\cdot)$, denoting $\bar{y}(s)$ for $s \in [0, T-\bar{t}]$.

Using Bellman's principle of optimality, we have

$$J(x(t), y(t,\cdot), t) = \max_{u(\tau); t \le \tau \le t+\Delta} \left\{ \int_t^{t+\Delta} F(x(\tau), u(\tau), \tau)\, d\tau + J(x(t+\Delta), y(t+\Delta, \cdot), t+\Delta) \right\}, \tag{10}$$

with $x(t)$, $y(t,\cdot)$ denoting the state of the system at time t. We now follow the procedure used in Refs. 10 and 11 in order to derive the Hamilton–Jacobi–Bellman equation. Expanding x and y in Taylor series, we have

$$x(t+\Delta) = x(t) + \Delta[f(x(t), u(t), t) + y(t, 0)] + o(\Delta),$$

$$y(t+\Delta, s) = y(t, s) + \Delta[g(x(t), u(t), t, t+s) + y_s(t, s)] + o(\Delta).$$

Assuming the value function $J(\bar{x}, \bar{y}, \bar{t})$ to be sufficiently smooth with respect to $\bar{x}$ and $\bar{y}(\cdot)$, we obtain the Taylor series expansion of J as

$$\begin{aligned} J(x(t+\Delta), y(t+\Delta,\cdot), t+\Delta) &= J(x(t), y(t,\cdot), t) \\ &\quad + \Delta[\partial J(x(t), y(t,\cdot), t)/\partial x][f(x(t), u(t), t) + y(t,0)] \\ &\quad + \Delta \int_0^{T-t} [\delta J(x(t), y(t,s), t)/\delta y][g(x(t), u(t), t, t+s) + y_s(t,s)]\, ds \\ &\quad + \Delta[\partial J(x(t), y(t,\cdot), t)/\partial t] + o(\Delta). \end{aligned} \tag{11}$$

Here, $\delta J(\bar{x}, \bar{y}(\bar{s}), \bar{t})/\delta y$ denotes the functional or variational partial derivative (see Refs. 14 and 15), which is defined as the variation of the functional $J(\bar{x}, \bar{y}(\cdot), \bar{t})$ with respect to the function $\bar{y}(\cdot)$ at the point $s = \bar{s}$.

Inserting Eq. (11) in (10) and taking the limit as $\Delta \to 0$, we arrive at the following partial differential-integral equation:

$$\begin{aligned} &\partial J(x(t), y(t,\cdot), t)/\partial t + \max_{u(t)} \Big\{ F(x(t), u(t), t) \\ &\quad + [\partial J(x(t), y(t,\cdot), t)/\partial x] f(x(t), u(t), t) \\ &\quad + \int_0^{T-t} [\delta J(x(t), y(t,s), t)/\delta y] g(x(t), u(t), t, t+s)\, ds \Big\} \\ &\quad + [\partial J(x(t), y(t,\cdot), t)/\partial x] y(t, 0) \\ &\quad + \int_0^{T-t} [\delta J(x(t), y(t,s), t)/\delta y] y_s(t,s)\, ds = 0, \end{aligned} \tag{12}$$

which is the Hamilton–Jacobi–Bellman equation for our problem.

Let us now define the adjoint functions

$$\lambda(t) \triangleq \partial J(x(t), y(t, \cdot), t)/\partial x, \qquad \mu(t, s) \triangleq \delta J(x(t), y(t, s), t)/\delta y, \tag{13}$$

and the Hamiltonian

$$H(x, u, t, \lambda, \mu(t, \cdot)) = F(x, u, t) + \lambda f(x, u, t) + \int_0^{T-t} \mu(t, s) g(x, u, t, t+s)\, ds, \tag{14}$$

where the argument $\mu(t, \cdot)$ indicates that H depends on $\mu(t, s)$ for $s \in [0, T-t]$.

Integration by parts,

$$\int_0^{T-t} \mu(t, s) y_s(t, s)\, ds = \Big[\mu(t, s) y(t, s)\Big]_{s=0}^{T-t} - \int_0^{T-t} \mu_s(t, s) y(t, s)\, ds,$$

allows Eq. (12) to be written as

$$\partial J(x(t), y(t, \cdot), t)/\partial t + \max_{u(t)} H(x(t), u(t), t, \lambda(t), \mu(t, \cdot)) + [\lambda(t) - \mu(t, 0)] y(t, 0) + \mu(t, T-t) y(t, T-t) - \int_0^{T-t} \mu_s(t, s) y(t, s)\, ds = 0. \tag{15}$$

Note that H as well as the maximized Hamiltonian

$$H^0(x, t, \lambda, \mu(t, \cdot)) = \max_{u \in \Omega} H(x, u, t, \lambda, \mu(t, \cdot))$$

do not depend on $y(t, \cdot)$. Since Eq. (15) must hold for every x and y, we can take appropriate partial and functional derivatives with respect to $y(t, 0)$, $y(t, s)$, $x(t)$, to obtain

$$\lambda(t) - \mu(t, 0) = 0,$$

$$\mu_t(t, s) = (\partial/\partial t)(\delta J/\delta y) = (\delta/\delta y)(\partial J/\partial t) = \mu_s(t, s),$$

$$\dot{\lambda}(t) = (\partial/\partial t)(\partial J/\partial x) = (\partial/\partial x)(\partial J/\partial t) = -\partial H^0(x(t), t, \lambda(t), \mu(t, \cdot))/\partial x.$$

The second equation above implies that

$$\mu(t, s) = \mu(t+s, 0);$$

thus,

$$\mu(t, s) = \lambda(t+s). \tag{16}$$

Note that (16) is the continuous-time counterpart of Eq. (21) in the discrete model of Ref. 12. The adjoint equation can be written as

$$\dot{\lambda}(t) = -\partial H^0/\partial x = -\partial H/\partial x, \tag{17}$$

where the second equality is known as the *envelope theorem*. A general proof of this result is given in Derzko, Sethi, and Thompson [Ref. 16, Eq. (24)); see also Ref. 7, Eq. (13)]. Finally, from

$$J(\bar{x}, \bar{y}(\cdot), T) = 0,$$

we obtain the transversality condition

$$\lambda(T) = [\partial J/\partial x]_{t=T} = 0. \tag{18}$$

Making use of the identity

$$\int_0^{T-t} \mu(t, s) g(x, u, t, t+s)\, ds = \int_t^T \lambda(\tau) g(x, u, t, \tau)\, d\tau,$$

we can express the results (15)–(18) derived so far as the following theorem.

Theorem 3.1. *Maximum Principle.* Let $u(t)$ be an optimal control of the problem (1)–(3) with associated state trajectory $x(t)$. Then, there exists a continuous adjoint function $\lambda(t)$ on $[0, T]$ such that, with the Hamiltonian defined as

$$H[x, u, t, \lambda(\cdot)] = F(x, u, t) + \lambda(t) f(x, u, t) + \int_t^T \lambda(\tau) g(x, u, t, \tau)\, d\tau, \tag{19}$$

the following conditions are satisfied:

$$H[x(t), u(t), t, \lambda(\cdot)] \geq H[x(t), u, t, \lambda(\cdot)], \qquad \forall u \in \Omega, \tag{20}$$

$$\dot{\lambda}(t) = -\partial H[x(t), u(t), t, \lambda(\cdot)]/\partial x, \tag{21}$$

$$\lambda(T) = 0. \tag{22}$$

This theorem corresponds to the necessary optimality conditions obtained by Bate (Ref. 6; see also the appendix in Ref. 17). Sethi (Ref. 7) has shown, that (20)–(22) are also sufficient for $u(t)$ to be optimal if the maximized Hamiltonian

$$H^0 = \max_u H$$

is concave in x.

The maximum principle also holds if there is a salvage function $S(x(T))$, so that the objective function in (1) becomes

$$\max\left\{J=\int_0^T F(x(t),u(t),t)\,dt+S(x(T))\right\}. \tag{23}$$

In this case, the transversality condition (22) changes to

$$\lambda(T)=\partial S(x(T))/\partial x. \tag{24}$$

Moreover, if $S(x)$ is a concave function, then the maximum principle can easily be shown to be sufficient as in Ref. 7.

It is also possible to easily develop the current value version of the maximum principle when $F(x,u,t)$ and $S(x(T))$ defined in (23) have the form $\exp(-rt)\bar{F}(x,u,t)$ and $\exp(-rT)\bar{S}(x(T))$ respectively; see Refs. 19 and 21.

4. Economic Interpretation

In optimal control problems in economics and management science, it is useful to provide the adjoint functions with economic interpretations. This allows additional insights into the structure of the problem.

For standard optimal control problems, these interpretations are in terms of the shadow prices and are given by Dorfman (Ref. 18), Arrow and Kurz (Ref. 19, Chapter 2), Peterson (Ref. 20), and Sethi and Thompson (Ref. 21, Chapter 2). With our derivation based on dynamic programming, it is possible to provide similar interpretations for the adjoint functions of optimal control problems with continuous lags. Our discussions will follow very closely the discussion in Ref. 21. For the purpose of discussion, it will be convenient to consider $x(t)$ as the stock of capital at time t and $u(t)$ as the rate of investments in the capital stock. By (2), these investments lead to both instantaneous and delayed increase in the stock of capital (via f and g, respectively). The objective function (23) then represents the aggregate value of the net profit stream F plus the salvage value S at the terminal time T.

From (13), it is obvious that the adjoint function $\lambda(t)$ represents the rate of change of the value function with respect to changes in capital stock $x(t)$. Thus, we can interpret $\lambda(t)$ as the *marginal value per unit of capital* or, in other words, the *shadow price of capital.* It should be emphasized that $\lambda(t)\Delta x(t)$ evaluates the value of the change $\Delta x(t)$ at time t and all the future effects of this change along the new optimal path from t to T.

It is also possible at time t to make changes in the past history $x(s), 0\le s\le t$. For example, if old machines of different vintages are traded for new machines, then the past history changes. This will affect the future dynamics from time t on. Moreover, it is possible to make changes in future technology of capital formation, i.e., changes in the form of function g in its fourth argument from time t on. It is easily seen that both kinds of change can be specified by a variation $\Delta y(t, \tau-t)$ in the function

$$y(t, \tau-t)=\int_{-\infty}^{t} g(x(s), u(s), s, \tau)\, ds,$$

for $\tau \geq t$, because the state equation (2) can be expressed as

$$\dot{x}(\tau)=f(x(\tau), u(\tau), \tau)+y(t, \tau-t)+\int_{t}^{\tau} g(x(s), u(s), s, \tau)\, ds. \tag{25}$$

The shadow price for $y(t, \tau-t)$ is

$$\mu(t, \tau-t) \triangleq \delta J(x(t), y(t, \tau-t), t)/\delta y=\lambda(\tau), \tag{26}$$

where the second equality is a restatement of (16). The equality (26) between the shadow price $\mu(t, \tau-t)$ and the shadow price $\lambda(\tau)$ is no longer a surprise, because both these quantities evaluate the value of a change in $x(\tau)$. To see this informally, let us suppose that the change in $y(t, \tau-t)$ is concentrated at one given time $\hat{\tau} \geq t$. This can be accomplished by letting the change $\Delta y(t, \tau-t)$ in $y(t, \tau-t)$ be a δ-function; thus,

$$\Delta y(t, \tau-t)=\delta(\tau-\hat{\tau}), \qquad \tau \geq t. \tag{27}$$

Integrating (25) from t to τ, we can obtain the change $\Delta x(\tau)$, $\tau \geq t$, in $x(\tau)$ as follows:

$$\Delta x(\tau)=\int_{t}^{\tau} \delta(s-\hat{\tau})\, ds=\begin{cases}0, & \tau<\hat{\tau}, \\ 1, & \tau \geq \hat{\tau}.\end{cases} \tag{28}$$

It can now be seen that

$$\lambda(\hat{\tau})=\int_{t}^{T} \mu(t, s-t) \delta(s-\hat{\tau})\, ds=\mu(t, \hat{\tau}-t), \tag{29}$$

where the left-hand side represents the value of the change $\Delta x(\hat{\tau})$ and the right-hand side represents the value of the change

$$\Delta y(t, \tau-t)=\delta(\tau-\hat{\tau}),$$

defined in (27), that results in the change $\Delta x(\hat{\tau})$.

We are now prepared to obtain the interpretation of the Hamiltonian function (19), which we write as

$$\begin{aligned} H &= F(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t) \\ &\quad + \int_t^T \lambda(\tau) g(x(t), u(t), t, \tau)\, d\tau \\ &= F(x(t), u(t), t) \\ &\quad + \int_t^T \lambda(\tau)[f(x(t), u(t), t)\delta(\tau - t) + g(x(t), u(t), t, \tau)]\, d\tau. \end{aligned} \tag{30}$$

The first term represents the marginal *direct contribution* to the profit functional (profit rate) if we are in state $x(t)$ and we apply control $u(t)$. The second term in the second expression in (30) represents the total *indirect contribution* to J. This consists of the usual contribution representing the value λf of the instantaneous marginal change f in $x(t)$ and the additional contributions $\int \lambda(\tau) g\, d\tau$ representing the value of the *marginal change densities* g in $x(\tau)$, $\tau \geq t$.

With this interpretation of the Hamiltonian representing the surrogate profit rate, it is easy to see why the Hamiltonian must be maximized at each time t. This interpretation of the maximum principle, therefore, becomes the same as in the case of the standard optimal control problem.

Now that we have interpreted the Hamiltonian as the surrogate profit rate at time t to be maximized, the interpretation of the adjoint equation (21), which can be rewritten as

$$-d\lambda = H_x\, dt,$$

is the same as in the standard case. That is, the decrease in the price of capital $-d\lambda$, which can be considered as the *marginal cost of holding the capital*, equals the *marginal revenue of investing the capital*, given by $H_x\, dt$. It is also possible, by integrating (21), to derive

$$\begin{aligned} \lambda(t) &= S_x(x(T)) + \int_t^T F_x(x(s), u(s), s)\, ds \\ &\quad + \int_t^T \lambda(s)\left\{ f_x(x(s), u(s), s) + \int_t^s g_x(x(\tau), u(\tau), \tau, s)\, d\tau \right\} ds. \end{aligned}$$

This equation enables us to explicitly see the marginal value interpretation for the adjoint variable $\lambda(t)$.

In concluding this section, we should remark that the importance of these interpretations can be more clearly seen in the advertising example treated in Section 6.

5. Problems Governed by Integral Equations

This section deals with the following problem:

$$\max\left\{J=\int_0^T F(x(t), u(t), t)\,dt+S(x(T))\right\}, \tag{31}$$

subject to

$$x(t)=\int_{-\infty}^{t} h(x(s), u(s), s, t)\,ds \tag{32}$$

and the initial conditions

$$x(t)=\underline{x}(t),\ u(t)=\underline{u}(t) \text{ given for } t\leq 0. \tag{33}$$

Equation (32) is known as Volterra's integral equation. Different sets of necessary conditions for this class of problems have been given by various authors, e.g., Vinokurov (Refs. 8 and 9), and Bakke (Ref. 22). In what follows, we shall derive these conditions from Theorem 3.1 and establish relations between the adjoint functions of the different approaches and their interpretations in a unified framework.

We differentiate (32) with respect to time t to obtain

$$\dot{x}(t)=h(x(t), u(t), t, t)+\int_{-\infty}^{t} [\partial h(x(s), u(s), s, t)/\partial t]\,ds. \tag{34}$$

Note that this equation can be related to Eq. (2). Define the Hamiltonian for the problem (31)–(33) as

$$\begin{aligned} H=&F(x(t), u, t)+\lambda(t)h(x(t), u, t, t)\\ &+\int_t^T \lambda(\tau)[\partial h(x(t), u, t, \tau)/\partial\tau]\,d\tau. \end{aligned} \tag{35}$$

It is easily seen that the application of Theorem 3.1 to this problem gives (20), (21), (24) as necessary optimality conditions. These conditions correspond to those obtained by Bakke (Ref. 22).

It is now possible to derive an alternative form of the necessary conditions. Define

$$\psi(t)\triangleq-\dot{\lambda}(t). \tag{36}$$

Then, we can prove the following theorem, which corresponds to Theorem 2.1 of Vinokurov (Ref. 8; see also Ref. 9).

Theorem 5.1. Let $u(t)$, with associated state trajectory $x(t)$, be an optimal control of the following problem: max(31), s.t. (32), (33). Then, there exists an adjoint function ψ (not necessarily continuous) such that the Hamiltonian

$$H^{\psi}[x(t), u, t, \psi(\cdot)] = F(x(t), u, t) + [\partial S(x(T))/\partial x]h(x(t), u, t, T) + \int_t^T \psi(\tau)h(x(t), u, t, \tau)\, d\tau \tag{37}$$

is maximized by $u(t)$ and

$$\psi(t) = \partial H^{\psi}[x(t), u(t), t, \psi(\cdot)]/\partial x. \tag{38}$$

Proof. Integrate the last term in (35) by parts to obtain

$$\int_t^T \lambda(\tau)[\partial h(x, u, t, \tau)/\partial \tau]\, d\tau = [\lambda(\tau)h(x, u, t, \tau)]_{\tau=t}^{T} - \int_t^T \dot{\lambda}(\tau)h(x, u, t, \tau)\, d\tau.$$

This allows Eq. (35) to be written as

$$H = F(x(t), u, t) + \lambda(T)h(x(t), u, t, T) + \int_t^T \psi(\tau)h(x(t), u(t), t, \tau)\, d\tau. \tag{39}$$

The application of the transversality condition

$$\lambda(T) = \partial S(x(T))/\partial x$$

implies that

$$H[x, u, t, \lambda(\cdot)] = H^{\psi}[x, u, t, \psi(\cdot)].$$

Thus,

$$\psi(t) = -\dot{\lambda}(t) = \partial H^{\psi}/\partial x,$$

which completes the proof. □

In the next section, we formulate an advertising model and obtain its solution.

6. Application to an Advertising Problem

Let $x(t)$ denote the stock of goodwill, and let $u(t)$ denote the rate of advertising at time t. Then, the system dynamics, with carryover effects of advertising, can be stated as

$$\dot{x}(t) = -\delta x(t) + \rho \int_{-\infty}^{t} f(x(\tau), u(\tau), \tau, t)\, d\tau, \tag{40}$$

where δ and ρ are decay and response constants, respectively. Note that this is the continuous-time version of the system dynamics obtained in Burdet and Sethi (Ref. 12). The function f represents the effects of advertising expenditures at time τ on the sales at time t. Some important special cases reported in the literature are: $f = u \exp(\tau - t)$ (Connors and Teichroew, Ref. 23); $f = u(1-x) \exp[a(\tau - t)]$ (Ireland and Jones, Ref. 24); and $f = g(u) w(t-\tau)$ (Pauwels, Ref. 5), where $g(u)$ is a production function of goodwill and $w(s)$ denotes the density function of the distribution of the time lag between the advertising expenditures u and the increase in x.

As an illustration of the maximum principle, we will consider the following optimal control model:

$$\max\left\{J = \int_0^T \exp(-rt)[\pi x(t) - u(t)]\, dt\right\}, \tag{41}$$

subject to

$$\dot{x}(t) = -\delta x(t) + \rho \int_{-\infty}^{t} g(x(\tau), u(\tau)) w(t-\tau)\, d\tau \tag{42}$$

and

$$x(t) = \underline{x}(t),\ u(t) = \underline{u}(t) \text{ given for } t \leq 0.$$

Here, r denotes the constant discount rate, π is a profit parameter relating goodwill to sales, and $g(x, u)$ is a production function of goodwill with advertising as an input.

From (19) and (21), we have

$$H = (\pi x - u) \exp(-rt) - \lambda \delta x + \rho g(x(t), u(t)) \int_t^T \lambda(\tau) w(\tau - t)\, d\tau,$$

$$\begin{aligned} \dot{\lambda}(t) = -\partial H/\partial x = -\pi \exp(-rt) + \lambda \delta \\ + \rho[\partial g(x(t), u(t))/\partial x] \int_t^T \lambda(\tau) w(\tau - t)\, d\tau. \end{aligned}$$

The transversality condition is

$$\lambda(T) = 0.$$

In order to obtain monotonicity results for u, we make the simplifying assumption

$$g(x, u) = g(u),$$

such that the adjoint equation becomes

$$\dot{\lambda}(t) = \delta\lambda - \pi \exp(-rt), \qquad \lambda(T) = 0,$$

whose solution is

$$\lambda(t)=[\pi/(r+\delta)]\exp(-rt)\{1-\exp[-(r+\delta)(T-t)]\}. \tag{43}$$

Let us first consider the case where the advertising expenditures u are subject to *decreasing returns to scale*:

$$g(0)=0, \qquad g'(0)=\infty,$$

$$g'(u)>0, \quad g''(u)<0, \qquad \text{for } u>0.$$

Then, the maximization of the Hamiltonian yields

$$\partial H/\partial u=-\exp(-rt)+\rho g'(u)\int_t^T \lambda(\tau)w(\tau-t)\,d\tau=0. \tag{44}$$

Making use of (43), we can compute

$$\begin{aligned}
&(d/dt)\left\{\exp(-rt)\int_t^T \lambda(\tau)w(\tau-t)\,d\tau\right\}\\
&\quad=[\pi/(r+\delta)](d/dt)\\
&\qquad\times\left\{\int_0^{T-t}\exp(-rs)\{1-\exp[-(r+\delta)(T-t-s)]\}w(s)\,ds\right\}\\
&\quad=[\pi/(r+\delta)]\left\{-\exp[-r(T-t)][1-e^0]w(T-t)-(r+\delta)\right.\\
&\qquad\left.\times\int_0^{T-t}\exp(-rs)\exp[-(r+\delta)(T-t-s)]w(s)\,ds\right\}\\
&\quad=-\pi\int_t^T\exp[-r(\tau-t)]\exp[-(r+\delta)(T-\tau)]w(\tau-t)\,dt<0
\end{aligned} \tag{45}$$

which, in turn, implies that

$$\dot{u}(t)=-\frac{g'(u)(d/dt)\{\exp(-rt)\int_t^T\lambda(\tau)w(\tau-t)\,d\tau\}}{g''(u)\{\exp(-rt)\int_t^T\lambda(\tau)w(\tau-t)\,d\tau\}}<0$$

when we apply the implicit function theorem to (44). Taking the limit as $t\to T$ in (44), we obtain

$$g'(u(T))=\infty,$$

which is equivalent to

$$u(T)=0.$$

Thus, the optimal advertising policy is one where the advertising expenditures are concentrated at the beginning of the time interval. These are monotonically decreasing and reach zero level at the terminal time T.

Let us now address ourselves to the case of *linear effectiveness* of advertising expenditures $g(u) = u$ with the constraints $0 \leq u \leq \bar{u}$ for the control variable u. The maximization of the Hamiltonian yields [see (44)]

$$u(t) = \begin{cases} 0, & \text{if } \partial H/\partial u < 0, \\ \bar{u}, & \text{if } \partial H/\partial u > 0, \end{cases}$$

that is,

$$u(t) = \begin{cases} 0, & \text{if } \exp(rt) \displaystyle\int_t^T \lambda(\tau) w(\tau - t)\, d\tau < 1/\rho, \\ \bar{u}, & \text{if } \exp(rt) \displaystyle\int_t^T \lambda(\tau) w(\tau - t)\, d\tau > 1/\rho. \end{cases}$$

Using relation (45), we can conclude that there is an interval of time θ such that

$$u(t) = \begin{cases} \bar{u}, & \text{for } t \leq T - \theta, \\ 0, & \text{for } t > T - \theta. \end{cases}$$

In other words, it takes at least θ units of time for any dollar invested in advertising to pay off.

Note that, for concave as well as linear effectiveness functions $g(u)$, the adjoint function $\lambda(t)$ is monotonically decreasing. This is consistent with the interpretation of $\lambda(t)$ as the shadow price of goodwill, since the benefits of an additional unit of goodwill at time t can be reaped only over the interval $[t, T]$, which decreases as t increases.

7. Concluding Remarks

In this paper, we have presented an alternative derivation of the maximum principle for systems with continuous lags. Using the method of dynamic programming, we were able to provide the adjoint functions and the optimality conditions with useful economic interpretations. This allows additional insight into the nature of optimal control models with lags that are encountered frequently in economics and management science.

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