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TECHNICAL NOTE

Laguerre Series Direct Method for Variational Problems

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Abstract. A direct method for solving variational problems via Laguerre series is presented. First, an operational matrix for the integration of Laguerre polynomials is introduced. The variational problems are reduced to the solution of algebraic equations. An illustrative example is given.

Key Words. Laguerre polynomials, variational problems, Ritz direct method, optimization.

1. Introduction

The direct methods of Ritz and Galerkin in solving variational problems are well known (Ref. 1). Recently, Chen and Hsiao (Ref. 2) introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solutions obtained are piecewise constant.

Laguerre polynomials are well known in mathematics. In this paper, an operational matrix for the integration of Laguerre polynomials is introduced. This matrix enables one to solve variational problems by Laguerre series. The method consists of the following steps: (i) assuming the candidate functions as Laguerre series with unknown coefficients to be determined; (ii) finding the necessary conditions for extremization; and (iii) solving the algebraic equations obtained to evaluate the Laguerre coefficients. The new method is simple, as illustrated by an example. First, a brief review of the properties of Laguerre polynomials is outlined.

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2. Laguerre Polynomials

The Laguerre polynomials are known as (Ref. 3)

$$
\begin{aligned}
&L_0(t)=1,\\
&L_1(t)=1-t,\\
&L_2(t)=1-2t+\tfrac{1}{2}t^2, \qquad (1)\\
&\dots\dots\dots\dots\\
&L_{i+1}(t)=[(1+2i-1)/(i+1)]L_i(t)-[i/(i+1)]L_{i-1}(t).
\end{aligned}
$$

The polynomials are orthonormal in $t\in[0,\infty)$ with respect to the weighting function $\exp(-t)$. That is,

$$
\begin{aligned}
&\int_0^\infty \exp(-t)L_i^2(t)\,dt=1,\\
&\int_0^\infty \exp(-t)L_i(t)L_j(t)=0, \qquad i\neq j.
\end{aligned} \tag{2}
$$

A function $\tilde{f}(t)$, which is absolutely square integrable in $t\in[0,\infty)$, may be approximated as a sum of Laguerre polynomials,

$$\tilde{f}(t)\cong\sum_{i=0}^{m-1} f_iL_i(t)=f^TL(t), \tag{3}$$

where T means transpose and m is a sufficiently large integer. The Laguerre coefficient vector f and the Laguerre vector $L(t)$ are, respectively,

$$f=[f_0,f_1,\dots,f_{m-1}]^T, \tag{4}$$

$$L(t)=[L_0(t),L_1(t),\dots,L_{m-1}(t)]^T. \tag{5}$$

The Laguerre coefficients f_i are obtained by the minimization of the integral weighted square error ϵ,

$$\epsilon=\int_0^\infty [\tilde{f}(t)-f^TL(t)]^2\exp(-t)\,dt, \tag{6}$$

and are given by

$$f_i=\int_0^\infty \exp(-t)\tilde{f}(t)L_i(t)\,dt. \tag{7}$$

The choice of m in the approximation of $\tilde{f}(t)$ given by Eq. (3) is based on the criterion that

$$\left| \sum_{i=0}^{m-1} f_i L_i(t) - \sum_{i=0}^{m-1} f_i L_i(t) \right| < \epsilon,$$

for every t in the interval $[0, \infty)$. The value of ϵ can be chosen arbitrarily.

3. Operational Matrix

It is also known that

$$\int_0^t L_i(t')\, dt' = L_i(t) - L_{i+1}(t), \qquad i = 0, 1, 2, \ldots \tag{8}$$

For $i = m - 1$, by approximating

$$\int_0^t L_{m-1}(t')\, dt' = L_{m-1}(t)$$

with the omitting of $L_m(t)$, Eq. (8) can be written in matrix form as follows:

$$\int_0^t L(t')\, dt' = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} L_0(t) \\ L_1(t) \\ \cdots \\ L_{m-2}(t) \\ L_{m-1}(t) \end{bmatrix} = PL(t). \tag{9}$$

P is called the operational matrix for integrating Laguerre polynomials. Notice that the $m \times m$ constant matrix P is an upper bidiagonal matrix. The unique form of the operational matrix plays an important role in the direct method for solving variational problems.

4. Laguerre Direct Method

Let us consider the problem of finding the minimum (or the maximum) of the functional

$$J(x) = \int_0^1 F[t, x(t), \dot{x}(t)]\, dt, \tag{10}$$

where $(\dot{\ }) = d(\ )/dt$. The necessary condition to minimize $J(x)$ is that $x(t)$ satisfies the Euler–Lagrange equation

$$\partial F/\partial x - (d/dt)(\partial F/\partial \dot{x}) = 0, \tag{11}$$

with appropriate boundary conditions. The Euler–Lagrange equation can be integrated easily only for simple cases. Therefore, direct methods and numerical methods have been used to solve variational problems. The Ritz and Galerkin methods are well-known direct methods. The Ritz direct method, using Laguerre series, is developed as follows.

First, assume that the rate variable $\dot{x}(t)$ can be expressed as a truncated Laguerre series,

$$\dot{x}(t)=\sum_{i=0}^{m-1} d_i L_i(t)=d^T L(t), \tag{12}$$

where d_i's are the Laguerre coefficients of $\dot{x}(t)$. Using Eq. (9), $x(t)$ can be represented as

$$\begin{aligned} x(t)&=\int_0^t \dot{x}(t')\,dt'+x(0)\\ &=d^T PL(t)+[x(0),0,\ldots,0]L(t). \end{aligned} \tag{13}$$

Notice that

$$L_0(t)=1$$

has been used in the last term of Eq. (13). Expressing t in Laguerre series gives

$$t=[1,-1,0,\ldots,0]L(t)\triangleq h^T L(t). \tag{14}$$

Substituting the Laguerre series of $\dot{x}(t)$, $x(t)$, t into Eq. (10), the functional $J(x)$ becomes a function of d_i. Thus, the necessary condition for extremizing of $J(x)$ is that

$$\partial J/\partial d_i=0, \qquad i=0,1,\ldots,m-1. \tag{15}$$

In fact, Eqs. (15) form a set of algebraic equations which are used to solve d_i. When the d_i's are obtained, the extremal function $x(t)$ is thus determined. The procedure is best illustrated by an example.

5. Illustrative Example

Consider the problem of finding the minimum of

$$J(x)=\int_0^1 [\dot{x}^2(t)+t\dot{x}(t)]\,dt, \tag{16a}$$

with boundary conditions

$$x(0)=0, \tag{16b}$$

$$x(1)=\tfrac{1}{4}. \tag{16c}$$

Inserting Eqs. (12) and (14) into Eq. (16a) gives

$$J=\int_0^1 [d^T L(t) L^T(t) d + d^T L(t) L^T(t) h]\, dt. \tag{17}$$

Letting

$$W=\int_0^1 L(t) L^T(t)\, dt, \tag{18}$$

where W is an $m \times m$ matrix, Eq. (17) becomes

$$J = d^T W d + d^T W h. \tag{19}$$

Recursive formulas for the evaluation of the W-matrix are given in the Appendix. To satisfy the boundary conditions (16b) and (16c), integration of $\dot{x}(t)$ gives

$$x(t)=\int_0^t \dot{x}(t)\, dt' + x(0). \tag{20}$$

Hence,

$$x(1)=d^T \int_0^1 L(t)\, dt = d^T v = \tfrac{1}{4}, \tag{21}$$

where

$$v=\int_0^1 L(t)\, dt. \tag{22}$$

The variational problem becomes to minimize J given Eq. (19), subject to the constraint of Eq. (21). Let α be the Laguerre multiplier; Eq. (19) is rewritten in terms of an augmented functional $\tilde{J}$,

$$\tilde{J} = d^T W d + d^T W h + \alpha(d^T v - \tfrac{1}{4}). \tag{23}$$

The necessary condition to minimize $\tilde{J}$ becomes

$$\partial \tilde{J}/\partial d = 0, \tag{24}$$

or

$$2Wd + Wh + \alpha v = 0, \tag{25}$$

subject to

$$d^T v - \tfrac{1}{4} = 0.$$

Equations (25) and (21) consist of $m+1$ simultaneous linear equations which are used to determine $d_0, d_1, \ldots, d_{m-1}$ and α. For $m=4$, we have

$$d^T = [0, 0.5, 0, 0].$$

Thus, the extremal function is

$$\dot{x} = \tfrac{1}{2}(1-t) \quad \text{and} \quad x(t) = \tfrac{1}{2}t(1-\tfrac{1}{2}t),$$

with

$$\alpha = -1.$$

6. Concluding Remarks

A method for solving variational problems is proposed in this paper. The method reduces a variational problem to the solution of algebraic equations. After finding the operational matrix for the integration of the Laguerre vector, the calculation is recursive and useful in digital computation.

If the function $F(t, x, \dot{x})$ appearing in Eq. (10) is a polynomial function in $t, x, \dot{x}$, the Laguerre series may serve well as a particular case of the Ritz method. However, if F is not a polynomial function in $t, x, \dot{x}$, the Laguerre series method might be impractical.

For more general computing techniques for solving variational problems, see Refs. 4–8.

7. Appendix: Recursive Formulas for the Matrix W

An operator $R_k(t)$ is defined as

$$R_k(t)L(t) = \int_0^t L(t')(t')^k \, dt'. \tag{26}$$

Hence,

$$R_0(t) = P. \tag{27}$$

Integration by parts of Eq. (26) gives

$$R_k(t) = t^k P - kR_{k-1}(t)P. \tag{28}$$

Since (Ref. 3)

$$L_i(t) = \sum_{j=0}^{i} [(-1)^j/j!]\binom{i}{j} t^j, \tag{29}$$

the $(i+1)$th column of the matrix W becomes

$$\int_0^t L(t')L_i(t')\,dt' = \sum_{j=0}^{i} \int_0^t L(t')[(-1)^j/j!]\binom{i}{j}(t')^j\,dt'$$

$$= \sum_{j=0}^{i} [(-1)^j/j!]\binom{i}{j} R_k(t)L(t). \tag{30}$$

References

1. SCHECHTER, R. S., *The Variational Method in Engineering*, McGraw-Hill, New York, New York, 1967.
2. CHEN, C. F., and HSIAO, C. H., *A Walsh Series Direct Method for Solving Variational Problems*, Journal of the Franklin Institute, Vol. 300, No. 4, 1975.
3. SANSONE, G., *Orthogonal Functions*, John Wiley and Sons (Interscience), New York, New York, 1959.
4. KELLEY, H. J., *Gradient Theory of Optimal Flight Paths*, ARS Journal, Vol. 30, No. 10, 1960.
5. BRYSON, A. E., JR., and DENHAM, W. F., *A Steepest-Ascent Method for Solving Optimum Programming Problems*, Journal of Applied Mechanics, Vol. 84, No. 3, 1962.
6. MIELE, A., PRITCHARD, R. E., and DAMOULAKIS, J. N., *Sequential Gradient Restoration Algorithm for Optimal Control Problems*, Journal of Optimization Theory and Applications, Vol. 5, No. 4, 1970.
7. MIELE, A., TIETZE, J. L., and LEVY, A. V., *Summary and Comparison of Gradient-Restoration Algorithms for Optimal Control Problems*, Journal of Optimization Theory and Application, Vol. 10, No. 6, 1972.
8. MIELE, A., *Recent Advances in Gradient Algorithms for Optimal Control Problems*, Journal of Optimization Theory and Application, Vol. 17, Nos. 5/6, 1975.