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Application of Functional Analysis to Models of Efficient Allocation of Economic Resources[1]

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Abstract. The present paper studies existence and characterization of efficient paths in infinite-horizon economic growth models: the method used is based on techniques of nonlinear functional analysis on Hilbert spaces developed earlier by Chichilnisky. Necessary and sufficient conditions are given for the existence of positive competitive price systems in which the efficient programs maximize present value and intertemporal profit. Approximation of these competitive price systems by strictly positive ones with similar properties is studied. A complete characterization is also given of a class of welfare functions (nonlinear operators defined on consumption paths) for continuity in a weighted l_2-norm.

Key Words. Hilbert spaces, existence theorems, functional analysis, applied mathematics.

1. Introduction

We study a recurrent problem in intertemporal economic analysis, the dual characterization of infinite-horizon efficient programs by competitive prices. From an economic viewpoint, if an efficient program x admits a competitive price system p at which x maximizes present value and intertemporal profit, then a centralized notion of efficiency can be translated to

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one of decentralized maximization of value or profit through time. Hence, efficiency can in principle be obtained, under these conditions, by decentralized decision-making. A program is a point of a sequence space, each element of the sequence denoting dated consumption. Hence, a program is a stream of consumption through time.

An efficient program within a producible set Y is one that cannot be strictly dominated, or improved, in the vector order of sequences. From a mathematical viewpoint, an efficient program x in a set of producible programs Y can be described as one with the following property: the set Y and the translation of the positive cone P_x^+ of the sequence space with vertex x only intersect at x. A competitive price p for x is a continuous linear functional which takes its maximum over the set Y at the point x. The existence of such a price can then be translated into the existence of an appropriate closed hyperplane separating Y and P_x^+. A problem arises because Y and P_x^+ are both contained, by their definition, in the positive cone of the space of consumption sequences. In order to apply Hahn–Banach type theorems to prove existence of separating hyperplanes, one needs at least one of the convex sets being separated to contain an interior point or at least an internal point.[4]

The only l_p-space of sequences which has a positive cone with nonempty interior, or with internal points, is l_∞. However, the sup norm is fine enough that its dual l_∞^*, the space of prices, contains elements which are not representable by sequences[5] and do not have an adequate economic interpretation.[6] For this, among other reasons, l_p-spaces with $1 \leq p < \infty$ and especially l_2-spaces seem natural candidates for spaces of consumption paths. However, these spaces have positive cones with an empty interior and no internal points, and this rules out the application of the usual Hahn–Banach type separation theorems which require one of the two disjoint convex sets to have an interior or internal point. Because of this, in Section 2 we prove a generalization of a Hahn–Banach separation theorem which is

[4] The hypothesis that one of the convex sets being separated contains an interior point can be weakened to the assumption that one of the sets has an internal point (see Ref. 1) relative to the least closed vector subspace containing the set; this latter hypothesis, however, cannot be eliminated. For a counterexample, see Dieudonné reference in Ref. 1. Dieudonné also shows that, in a nonreflexive space, such as l_∞, two closed convex bounded sets without a common point may not have any closed separating hyperplane. If the space is reflexive (e.g., l_2), such sets can be separated by a closed hyperplane. In our problem, however, the two closed sets *do* have one point in common, namely, the efficient or optimal path, so this last result also does not apply, and new tools have to be used here.

[5] i.e., purely finitely additive elements, Ref. 1.

[6] This occurs, for instance, when the function part of a price p is given by a purely finitely additive measure on l_∞, and hence its sequence part is identically zero, while p as a functional on l_∞ is not zero.

shown to enable many standard results to be rescued. Furthermore, we give a complete characterization of certain nonlinear operators (welfare functions) for continuity in a weaker weighted l_2 norm, and we prove that if the efficient program x maximizes the value of such a continuous welfare function then the problem can also be overcome. Basically, one shows that, in this case, one of the sets being separated is contained in a convex set which has an interior in a weighted l_2-norm, since it is the inverse image under an l_2 continuous map, and intersects the other convex set at the point x only. Thus, the separating hyperplanes can be chosen so as to be representable by sequences, effectively elements of $l_2^* \approx l_2$. Thus, it is shown that the question of existence of prices is also related to the appropriate continuity of welfare functionals, if one is to work on l_2.

In Theorem 2.1, necessary and sufficient conditions for a separation of the feasible set Y from the set of programs which are strictly larger in the vector order are given. This separation result is equivalent, in this case, to the existence of nonzero competitive prices for the efficient programs. Such prices are shown to assign strictly larger present and intertemporal profit value to strictly larger programs. They define continuously a bounded present value and intertemporal profit for all programs in the space, which is maximized in Y at the efficient program. A sufficient condition is also given on the feasible set Y for existence of an efficient program in Y. In Proposition 2.1, a complete characterization of continuous utility functions in a weighted l_2-norm is given; these utilities are represented by sums of discounted time-dependent utilities. Theorem 2.2 is an extension of Arrow, Barankin, and Blackwell (Ref. 2) and Radner's (Ref. 3) results. This theorem gives an approximation of a competitive price p for an efficient program x by a sequence of competitive prices p^α which maximize the value at x^α in the set Y, where the x^αs are efficient paths in Y, $x^\alpha \to x$, and $p^\alpha \to p$. This result extends those of Ref. 2, adapting the proof of Ref. 3 for programs and prices in weighted l_2-spaces. The results given in this paper are based on previous work by Chichilnisky (see Ref. 4).

2. Competitive Prices for Efficient Programs

A *production program* is a sequence $\{a_t, b_{t+1}\}$, $t = 1, 2, \ldots$, where $a_t \in R^n$ represents inputs, $b_{t+1} \in R^n$ represents outputs at period t and $t+1$, respectively, $a_t \geq 0$, $b_{t+1} \geq 0$, and b_{t+1} is in $T_t(a_t)$, T_t a correspondence[7] from R^{n+} to R^{n+} representing the production possibilities or technology at date t. For a production program $\{a_t, b_{t+1}\}$, let $\{x_t\}$ denote the sequence $\{b_t - a_t\}$,

[7] i.e., a set-valued function.

$t \geq 2$, and $x_1 = -a_1$, which is called the *net output program*. A feasible set of net output vectors Y is defined as a set of nonnegative net output programs $\{x_t\}$, $t = 1, 2, \ldots,$ where $b_{t+1} \in T_t(a_t)$ for all t. From now on, the word program is used to denote net output programs.

If $x = \{x_t\}$ and $y = \{y_t\}$, $t = 1, 2, \ldots,$ are two infinite sequences of vectors, we denote $x \geqq y$ if $x_t \geqq y_t$ for all t, $x \geq y$ if $x \geqq y$ and $x \neq y$, and $x > y$ if $x_t > y_t$ for all t. A program x is *efficient* or maximal in a feasible set Y if there is no y in Y with $y \geq x$, i.e., x is efficient in Y if $Y \cap P_x^+ = \{x\}$, where P_x^+ is the translation of the positive cone in the sequence space with vertex x,

$$P_x^+ = \{z \mid z \geqq x\}.$$

A system of prices p is called a *competitive price system* for the program x in Y if

$$p(x) = \max_{y \in Y} p(y).$$

If $p = \{p_t\}$ is a sequence of prices at each data t, the *intertemporal profit* of the production program $\{a_t, b_{t+1}\}$ at price $\{p_t\}$ and time $t+1$ is defined by $p_{t+1} \cdot b_{t+1} - p_t \cdot a_t$, where $p_t \cdot a_t$ denotes the inner product of the vectors p_t and a_t. For a review and discussion of these models, see for instance Ref. 5.

The approach that we follow here is to give these spaces of consumption paths a weighted l_2-norm induced by a discount factor. The notion of distance of paths in this space seems quite well fitted for discounted types of models; the results apply to nondiscounted models as well. Some of the difficulties noticed by Majumdar and Radner (Ref. 4), among others who work on l_∞-spaces, seem surmountable in this framework; in particular, a difficulty that their approach runs into now disappears. Every value functional in the dual of a Hilbert space of sequences can be represented as a sequence of prices, and thus the difficulty that the sequence part of a nonzero value functional may be zero is removed. In addition, in these prices, the value is given by an inner product and therefore has a ready interpretation. This brings together the concepts introduced by Malinvaud (Ref. 6), Debreu (Ref. 7), and Radner (Ref. 3) for infinite programs in this space. Further, economic relations between the concepts of efficiency, present value maximization, and intertemporal profit maximization of finite programs are shown to be inherited by these programs.

Let x and y be two bounded programs. Define the inner product:

$$(x, y)\lambda = \sum_{t=1}^{\infty} \lambda^t (x_t \cdot y_t), \tag{1}$$

where $0 < \lambda < 1$. This inner product can be thought of as representing the present value of program x in price system y with discount factor λ. It induces a normed topology on l_∞, with norm $\|\cdot\|_\lambda$ given by

$$\|x\|_\lambda = (x, x)^{1/2}.$$

We consider the completion of l_∞ under this topology. This space is denoted H_λ to call attention to the parameter λ in its definition; in Proposition 2.1, the relationship between the parameter λ and the continuity of discounted additive welfare functionals is shown. The inner product defined in (1) extends to an inner product on H_λ and defines a Hilbert space structure for the space H_λ, which is an l_2-space of sequences with the finite measure induced by the density function λ^t, $t = 1, 2, \ldots$.

A *price p* is a function that assigns to every program in H_λ a *present value*, which is a continuous linear functional on the space of all programs. Thus, the space of prices is isomorphic to the dual space of H_λ, H_λ^*. Since H_λ is a Hilbert space, H_λ^* is isomorphic to H_λ.

Thus, the space of prices H_λ^* is a sequence space; and, if $p = \{p_t\} \in H_\lambda^*$ and $y = \{y_t\}$ is a program in H_λ^+ then the present value of y at price p is equal to the inner product

$$(p, y) = \sum_{t=1}^{\infty} \lambda^t (p_t \cdot y_t).$$

The space of prices l_∞^* (continuous linear functionals on l_∞ with the sup norm) must be strictly larger than the space of prices of l_∞ with the $\|\cdot\|_\lambda$ topology. Intuitively, since $\|\cdot\|_\lambda$ is weaker than $\|\cdot\|_{\sup}$ on l_∞, i.e., $\|\cdot\|_\lambda$ on l_∞ has fewer open sets than $\|\cdot\|_{\sup}$, there exists then fewer continuous linear functions on l_∞ with the $\|\cdot\|_\lambda$ norm than with the $\|\cdot\|_{\sup}$ norm. A problem for the choice of $(l_\infty, \|\cdot\|_{\sup})$ as a space of programs is that l_∞^* contains elements which are not sequences: there are nonzero continuous linear functionals on $(l_\infty, \|\cdot\|_{\sup})$ whose sequence part is zero, the purely finitely additive measures (Ref. 1). By weakening the topology of l_∞, the purely finitely additive measure part of l_∞^* disappears (i.e., looses continuity in the new norm), and we are left only with a sequence space H_λ^*.

The following results show necessary and sufficient conditions for the existence of nonzero prices supporting efficient programs under technological assumptions on the set Y of feasible programs. We first need a lemma; a result related to this, but for sup norms instead of l_2-norms, is stated without proof in Ref. 8, page 52, E.

Lemma 2.1. Let f be a linear functional defined on an l_2-space of real sequences, f nonnegative on l_2^+, the set of nonnegative sequences of l_2. Then, f is also continuous on l_2, i.e., $f \in l_2$.

Proof. Let $\{\xi^n\}$ be the canonical base of l_2. Consider the sequence of real numbers S_f, defined by $S_f = (f(\xi^1), f(\xi^2), \ldots)$, the sequence part of f. We shall show first that S_f is in l_2. Consider a sequence $\beta = (\beta_1, \beta_2, \ldots)$ in l_2^+. Then,

$$f(\beta) = f\Big(\sum_{i=1}^{\infty} \beta_i \xi^i\Big) = f\Big(\sum_{i=1}^{N} \beta_i \xi^i\Big) + f\Big(\sum_{N+1}^{\infty} \beta_i \xi^i\Big),$$

which by linearity is equal to

$$\sum_{i=1}^{N} \beta_1 f(\xi^i) + f\Big(\sum_{N+1}^{\infty} \beta_i \xi^i\Big).$$

Since $\beta \in l_2^+$ and f is well defined and nonnegative on $\xi^i \in l_2^+$, it is obvious that

$$\infty > f\Big(\sum_{i=1}^{\infty} \beta_i \xi^i\Big) \geq f\Big(\sum_{i=1}^{N} \beta_i \xi^i\Big) = \sum_{i=1}^{N} \beta_i f(\xi^i),$$

so that, for any β in l_2^+,

$$\lim_{N\to\infty} \sum_{i=1}^{N} \beta_i f(\xi^i) = \sum_{i=1}^{\infty} \beta_i f(\xi^i) < \infty.$$

Then, for any β in l_2,

$$\sum_{i=1}^{\infty} \beta_i f(\xi^i) < \infty$$

also. Since l_2 is self-dual, and S_f is nonnegative and it well defines a continuous linear function on l_2, it follows that S_f is in l_2. Now, let $h = f - S_f$, that is, h is the *nonsequence part* of f. We shall show that h is identically zero. First, note that $h(\xi^i) = 0$, for all ξ^i in the base of l_2.

For all α in l_2^+, there exists a β in l_2^+ with

$$\lim_{i\to\infty} (\beta_i/\alpha_i) = \infty;$$

then, given any $N > 0$, if k is large enough,

$$\sum_{i=k+1}^{\infty} (\beta_i - N\alpha_i)\xi^i \qquad \text{is in } l_2^+.$$

Note that, given that, for k large enough,

$$S_f\Big(\sum_{i=k+1}^{\infty} (\beta_i - N\alpha_i)\xi^i\Big)$$

is as close to zero as desired, then since $h = f - S_f$ and f is nonnegative on l_2^+, this implies that, for k large enough,

$$h\Big(\sum_{i=k+1}^{\infty} (\beta_i - N\alpha_i)\xi^i\Big)$$

is a nonnegative number, and so

$$h\left(\sum_{i=k+1}^{\infty} \beta_i \xi^i\right) \geq Nh\left(\sum_{i=k+1}^{\infty} \alpha_i \xi^i\right).$$

Also,

$$h\left(\sum_{i=1}^{k} \beta_i \xi^i\right) = Nh\left(\sum_{i=1}^{k} \alpha_i \xi^i\right) = 0,$$

since by definition $h = f - S_f$. Thus,

$$h(\alpha) \leq (1/N)h(\beta) \qquad \text{for all } N.$$

Since N is arbitrarily chosen, this implies that $h(\alpha) = 0$, which completes the proof.

We need some more definitions. A point x is said to be *internal* to a set Y in a linear space X if, for all z in X, there is an $\epsilon > 0$ such that $x + \lambda z \in Y$ for all λ with $|\lambda| < \epsilon$. Note that an internal point may not be interior. A real-valued function u on H_λ is called *strictly increasing when* $z > y$ implies that $u(z) > u(y)$.

Let Y be a convex set, and $x \in Y$. The *cone with vertex* x *generated by* Y is the smallest cone with vertex x containing the set Y denoted $C(Y, x)$. It is easy to see that

$$C(Y, x) = \{z \mid z = a(y - x) + x,\ y \in Y,\ a \geq 0\}.$$

Let λ be any real number in (0, 1).

Theorem 2.1. If Y is nonempty, norm-bounded, closed, and convex in H_λ, then there exists a maximal element x in Y. For any maximal x, the following conditions (a), (b), (c) are equivalent, and are each necessary and sufficient for the existence of a nonzero continuous supporting hyperplane $p \in H_\lambda^{*+}$ for Y, supported at x; further, for any such hyperplane, if $z \in H_\lambda$, $z \geq x$, then $p(z) \geq p(x)$ and if $z > x$, then $p(z) > p(x)$. This hyperplane p defines a price system with respect to which x is value maximizing and discounted intertemporal profit maximizing; and, in this price system, any program $y \in H_\lambda$ has a finite present value given by

$$(p, x)_\lambda = \sum_{t=1}^{\infty} \lambda^t (p_t \cdot x_t).$$

(a) There exists a vector $w \geq x$ which is at a positive distance from the set $C(Y, x)$.

(b) y maximizes a strictly increasing concave $\|\cdot\|_\lambda$ continuous function u defined on a neighborhood of Y,

(c) There exists a convex set $Y_1 \supset P_y^+$, $Y \cap Y_1 = \{x\}$, and Y_1 contains an internal point.

Proof. First, we prove existence of a maximal element in Y. Note that, since H_λ is a Hilbert space for any $\lambda \in (0, 1)$, it is reflexive. Thus, by Alaoglu's theorem (Ref. 1), Y is weakly compact. It follows that Y is compact in the pointwise convergence topology (see Ref. 1). Thus, by Ref. 9, Theorem 2.2 there exists a maximal element x in Y.

We now study the existence of the separating hyperplane for Y and P_x^+ with the above properties.

We first prove sufficiency of (a). Consider the set

$$L = C(Y, x) - P_x^+ = \{z \mid z = y - u, \text{ where } y \in C(Y, x),\ u \in P_x^+\}.$$

L is a convex cone with vertex $\{0\}$, since $C(Y, x)$ and P_x^+ are convex cones and $x \in P_x^+ \cap C(Y, x)$.

Let w be the element of P_x^+ at a positive distance from $C(Y, x)$. Then, the vector $w_1 = w - x$ is in H_λ^+ and it is at a positive distance from L. For, if it is not [i.e., if for all $\epsilon < 0$ there is a u in k with $d(w_1, u) < \epsilon$], then since

$$d(w_1, u) = d(w_1 + x, u + x) < \epsilon, \qquad u + x \in C(Y, x), \qquad w_1 + x = w$$

this would imply that w is not at a positive distance from $C(Y, x)$, a contradiction.

Therefore, by Theorem V.2.12 of Ref. 1, the closure of the cone L, $\bar{L}$, and the point $w_1 \in H_\lambda^+$ can be separated by a nonzero continuous linear functional, say, p. In addition, since 0 is the vertex of the cone L, and $p(0) = 0$ by linearity of p, p can be chosen so that $p(z) \le 0$ for all z in L. This last point can be seen as follows. Since p separates L and w_1, there is a constant c such that

$$p(u) \le c < p(w_1)$$

for all u in L.

If there would exist a z in L with $a = p(z) > p(0) = 0$, then, by linearity of p,

$$p(\gamma z) = \gamma p(z) = \gamma a.$$

Since γ is arbitrary, and γz is in L for all $\gamma > 0$, this would contradict the fact that $p(u) \le c\ \forall u$ in L.

We now complete the proof of sufficiency of (a).

As shown in Ref. 1, the positive cone P_0^+ is supported by a continuous tangent functional at $p = (p_i)$ iff $p_i = 0$ for some $i \ge 0$ (see Ref. 1, page 458, No. 9). Suppose now that $z \in P_x^+$ and $p(z) = p(y)$. Then, by the above result, $z_t = y_t$ for some t. Thus, $p(z) > p(y)$ if $z > y$. This completes the proof of sufficiency of (a). To see the necessity of (a), note that, if p separates $C(Y, y)$ from P_x^+, then

$$C(Y, y) \supset \{z \in H_\lambda \text{ with } p(z) \le p(z)\};$$

thus, $\overline{C(Y, y)}$ is actually contained in a closed half-space, and, by definition of P_x^+, this implies (a).

We now prove (b). If y maximizes a strictly increasing concave continuous function u defined on a neighborhood of Y, then the set

$$S = \{z : z \in H \text{ and } u(z) > u(y)\}$$

is convex, and its interior is not empty. Thus, Y and S can be separated by a nonzero continuous hyperplane p. Note that $p(z) > p(y)$ if $z > y$.

The converse is trivial, since p itself is continuous concave and can be taken to be positive, and thus increasing.

We now prove (c). For the sufficiency of (c), note that, if $Y_1 \cap Y = \{x\}$, $Y_1 \supset P_x^+$, and Y_1 contains an internal point, then by Ref. 1, Theorem V.I.12, there exists a linear function p separating Y_1 and Y, and thus Y and P_x^+. We next note that, by Lemma 2.1 above, if p is positive on P_x^+, p is continuous.

The reciprocal is immediate: if p separates Y and P_x^+, then

$$P_x^+ \subset \{z \text{ in } H_\lambda, p(z) \geq p(y)\}.$$

Remark 2.1. For an example of a maximal program in a convex set which does not satisfy the above conditions, see McFadden (Ref. 10).

Remark 2.2. Note that, in the above results, the separation theorem yields a separation between Y and the set P_x^+; and, if $z > x$, then $p(z) > p(x)$. For some economic purposes, this strong separation is not needed: it may suffice that y maximizes present value and intertemporal profit with respect to a positive price system, without being concerned with the value of programs which are strictly larger than y.

Corollary 2.1. Let Y be a convex subset in H_λ^+. For any maximal x, the following are necessary sufficient conditions for the existence of a nonzero price p in H_λ^{*+} with respect to which x is present value maximizing and discounted intertemporal profit maximizing.

(a) $C(Y, x)$ is not dense in H_λ.

(b) y maximizes a concave function u which is continuous in a neighborhood of Y.

Proof. First, we prove the sufficiency of (a). Assume that there exists w in H_λ with $d(c(Y, x), w) > 0$. By Ref. 1, V.2.12, there exists a continuous linear function h, with

$$h(w) \geq c \geq h(C(Y, x)).$$

We shall see that h is maximized in $C(Y, x)$ at x. Let $h(x) = c_1$. If $z \in C(Y, x)$,

$$h(z) = h(r(y-x)+x)) = rh(y-x)+h(x),$$

so that, if $h(y) > c_1$ for some y in $C(Y, x)$ then

$$rh(y-x)+h(x) > c$$

for some $r \geq 0$. Thus, $h(y) \leq c_1$, for all y in Y, which completes the proof of separation.

On the necessity of (a), note that, if there exists a continuous linear function supporting Y at x, then $\overline{C(Y, x)}$ is contained in a closed half-space.

To see that (b) is necessary and sufficient, note that the proof of (b) in Theorem 2.1 holds:

$$S = \{z: u(z) > u(x)\} \cap Y = \emptyset.$$

Note that S does not necessarily contain P_x^+ here, since u may not be monotone nondecreasing.

Remark 2.3. The condition (a) of Corollary 1 is equivalent to (a) of Theorem 2.1, when there is free disposal, i.e., when if $y \in Y$ and $z \leqq y$, then $z \in Y$.

In the following, in view of the conditions (b) of Theorem 2.1 and Corollary 2.1, we study necessary and sufficient conditions for continuity in H_λ of utility functions of a usual type in economics, given by a discounted sum of time-dependent utility of consumption. The next result gives a complete characterization to the class of such functions that satisfy the continuity condition (b) of Theorem 2.1. First, we need more definitions.

Let H_λ^1 be the Banach space of all sequences x satisfying

$$\sum_{t=1}^{\infty} \lambda^t |x_t| < \infty, \qquad 0 < \lambda < 1,$$

with the norm

$$\|\cdot\|_\lambda^1 = \sum_{t=1}^{\infty} \lambda^t |x_t|.$$

Let $u(c, t)$ be a nonnegative real-valued function of two variables, for $-\infty < c < \infty$, $t = 1, 2, \ldots$. Assume that u is continuous with respect to c for all values of t. Then, u induces a real-valued map W on any real-valued function $c(t)$ on $\{1, 2, \ldots\}$ by

$$W(c) = \sum_{t=1}^{\infty} \lambda u(c(t), t)$$

when this sum exists. $u(c(t), t)$ represents, for instance, a time-dependent utility derived from consumption.

Proposition 2.1. The real-valued function

$$W(c)=\sum_{t=1}^{\infty}\lambda^{t}u(c(t),t)$$

is $\|\cdot\|_\lambda$ continuous iff

$$u(x,t)\le b(t)+\alpha|c|^2,$$

where α is a positive number and $b\in H_\lambda^{1+}$.

Proof. Note that

$$c^n\xrightarrow{\|\cdot\|_\lambda}c$$

iff

$$\lambda^{t/2}c^n\to\lambda^{t/2}c$$

in l_2. Also, $c\to W(c)$ is $\|\cdot\|_\lambda$ continuous iff (a) $d\to\lambda^{-t}u(\lambda^{-t/2}d,t)$ is continuous from l_2 to l_1, with $d(t)=\lambda^{t/2}c(t)$. By Ref. 11, Theorems 2.1 and 2.3, pp. 23–28 and remarks on page 28, a necessary and sufficient condition for (a) to be continuous is that

$$\lambda^{t}u(\lambda^{-t/2}d,t)\le a(t)+\alpha|d|^2,$$

where $a(t)\in l_1^+$ and α is a positive constant. Or, equivalently,

$$u(c,t)\le b(t)+\alpha|c|^2\qquad\text{for } b(t)=\lambda^{t}a(t)\in H_\lambda^{1+}.$$

This completes the proof.

Remark 2.4. Let $0\le\rho\le\lambda$. Then,

$$f^\alpha\xrightarrow{\|\cdot\|_\rho}f\Rightarrow f^\alpha\xrightarrow{\|\cdot\|_\lambda}f;$$

also, $H_\lambda\supset H_\rho$. Therefore, if $W: H_\lambda\to R$ is $\|\cdot\|_\lambda$ continuous, when $W|_{H_\rho}: H_\rho\to R$ is also $\|\cdot\|_\rho$ continuous. Therefore, for all $0\le\rho\le\lambda$, the function

$$W(c)=\sum_{t=1}^{\infty}\lambda^{t}u(c(t),t)$$

is $\|\cdot\|_\rho$ continuous; or, equivalently, for all $\rho \geq \lambda$, the function

$$W(c) = \sum_{t=1}^{\infty} \rho^t u(c(t), t)$$

is H_λ-continuous.

Examples of functions which are l_∞-continuous and H_λ-discontinuous can be constructed by considering functions which are essentially given by the $\|\cdot\|_\infty$ norm, which is strictly stronger than $\|\cdot\|_\lambda$. For instance,

$$F(c) = \sup_t (c_t).$$

We now extend Arrow, Barankin, and Blackwell (Ref. 2) and Radner's results (Ref. 3) on approximation of nonnegative continuous competitive prices for efficient programs by strictly positive ones in H_λ. The next result extends a theorem of Ref. 3, page 352, which is valid only for *strongly compact* convex feasible sets Y; here, we prove the result for $\|\cdot\|_\infty$ bounded and closed convex feasible sets Y, which is a strictly weaker condition than that of strong compactness of Ref. 3.

Theorem 2.2. Let x be a maximal point in a convex closed $\|\cdot\|_\infty$ norm bounded set Y in H_λ^+. Assume that Y satisfies one of the conditions (a) or (b) of Theorem 2.1. Then, a price p such as that of Theorem 2.1 can be constructed so that $\|p\| = 1$, $p \geq 0$, and (x, p) is the limit of a net (x^α, p^α) in $Y \times H_\lambda^{*+}$ with the weak convergence on H_λ^*, such that, for all α, x^α is maximal in Y, and it maximizes the value of p^α on Y, and $p^\alpha \geq 0$.

Proof. We first show that, if $Y \subset H_\lambda$ is a closed and $\|\cdot\|_\infty$ bounded set, then Y is H_λ-compact. Since Y is $\|\cdot\|_\infty$ bounded and closed, Y is weak* compact as a subset of $l_\infty \subset H_\lambda$. Let $\{x^n\}$ be a sequence in Y. Then, there exists a subsequence $\{x^m\}$ such that $x^m \to z$ weak* (Ref. 1) for some $z \in Y$. Thus, $x_t^m \to z_t$ for each t. Also, for all t and m

$$|x_t^m - z_t| \leq 2N,$$

where N is the bound for Y in $\|\cdot\|_\infty$. Since

$$\lim_{T\to\infty} \sum_{t>T} \lambda^t = 0,$$

there exists a T_ϵ such that

$$\sum_{t>T_\epsilon} \lambda^t |x_t^m - z_t|^2 \leq 4N^2 \sum_{t>T_\epsilon} \lambda^t > \epsilon.$$

Choose M such that, for $m > M$ and all $t \leq T_\epsilon$,

$$|x_t^m - z_t|^2 < \epsilon/2T_\epsilon.$$

Then

$$\sum_{t<T_\epsilon} \lambda^t |x_t^m - z_t|^2 < \epsilon/2;$$

and thus, for any $\epsilon > 0$, there exists an M with

$$\sum_t \lambda^t |x_t^m - zt|^2 < \epsilon, \qquad \text{for } m > M,$$

i.e.,

$$x^m \xrightarrow{\|\cdot\|_\lambda} z.$$

Let

$$S = \{p \in H_\lambda^*, \|p\|_\lambda = 1 \text{ and } p \geq 0\}.$$

By the construction in Theorem 2.1, the competitive price corresponding to the efficient program x can be assumed to be an element of the set S. Note that the results of Lemma 1, 2, 3 of Ref. 3 hold also in our case. The evaluation map $\phi: H_\lambda \times S \to R$, $\phi(y, p) = p(y)$ is continuous, when S is given the weak topology, and $H_\lambda \times S$ the corresponding product topology.

The set

$$S_q = \{p: p \in S, p \geq q\}$$

for some $q \gg 0$ in H_λ^* is a closed subset of S. Since S is closed, it is compact in the weak topology by Alaoglu's theorem, and therefore so is S_q. The proof of Lemma 3 in Ref. 3 holds also in H_λ, so that the rest of the proof of Ref. 3 is valid here. This completes the proof.

References

1. DUNFORD, N., and SCHWARTZ, J. T., *Linear Operators, Part 1*, John Wiley and Sons (Interscience Publishers), New York, New York, 1957.
2. ARROW, K. J., BARANKIN, E. W., and BLACKWELL, D., *Admissible Points in Convex Sets*, Contributions to the Theory of Games, Edited by H. W. Kuhn and A. W. Tucker, Princeton University Press, Princeton, New Jersey, 1953.
3. RADNER, R., *On Maximal Points in Convex Sets*, Proceedings of the Fifth Berkeley Symposium on Probability and Statistics, University of California Press, Berkeley, California, 1965.
4. CHICHILNISKY, G., *Nonlinear Functional Analysis and Optimal Economic Growth*, Journal of Mathematical Analysis and Applications, Vol. 61, pp. 504–520, 1977.
5. MAJUMDAR, M., and RADNER, R., *Shadow Prices for Infinite Growth Programs: The Functional Analysis Approach*, Techniques of Optimization, Edited by A. V. Balakrishnan, Academic Press, New York, New York, 1972.

6. MALINVAUD, E., *Capital Accumulation and Efficient Allocation of Resources*, Econometrica, Vol. 21, pp. 253–276, 1953.
7. DEBREU, G., *Valuation Equilibrium and Pareto Optimum*, Proceedings of the National Academy of Sciences, USA, Vol. 40, pp. 588–592, 1954.
8. KELLEY, J., and NAMIOKA, I., *Linear Topological Spaces*, D. Van Nostrand Company, New York, New York, 1963.
9. PELEG, B., *On Competitive Prices for Optimal Consumption Plans*, SIAM Journal on Applied Mathematics, Vol. 26, pp. 239–253, 1974.
10. MCFADDEN, D., *An Example of the Non-Existence of Malinvaud Prices in a Tight Economy*, Journal of Mathematical Economics, Vol. 2, pp. 17–19, 1975.
11. KRASNOSEL'SKII, M. A., *Topological Methods in the Theory of Nonlinear Integral Equations*, The Macmillan Company, New York, New York, 1964.