

The Structure of Admissible Points with Respect to Cone Dominance^{1,2,3}

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Abstract. We study the set of admissible (Pareto-optimal) points of a closed, convex set X when preferences are described by a convex, but not necessarily closed, cone. Assuming that the preference cone is strictly supported and making mild assumptions about the recession directions of X , we extend a representation theorem of Arrow, Barankin, and Blackwell by showing that all admissible points are either limit points of certain *strictly admissible* alternatives or translations of such limit points by rays in the closure of the preference cone. We also show that the set of strictly admissible points is connected, as is the full set of admissible points.

Relaxing the convexity assumption imposed upon X , we also consider local properties of admissible points in terms of Kuhn-Tucker type characterizations. We specify necessary and sufficient conditions for an element of X to be a Kuhn-Tucker point, conditions which, in addition, provide local characterizations of strictly admissible points.

Key Words. Vector optimization, multiple objective optimization, cone dominance, efficient points, Pareto optimality, optimality conditions.

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1. Introduction

Formal approaches to decision making almost always presume that an underlying preference relation governs choices from available alternatives. Rich theories now go far toward either characterizing or computing solutions that are in some sense *optimal*. Mathematical programming techniques predominate when preferences can be embodied in a real-valued utility function [Debreu (Refs. 1 and 2) discusses appropriate conditions, see also Bowen (Ref. 3) and Arrow and Hahn (Ref. 4, page 106)]. Multi-attribute utility theory (Keeney and Raiffa, Ref. 5) provides one means for considering multi-objective situations which involve several, possibly conflicting, criteria. In summarizing methods for studying multi-objective decision making, MacCrimmon (Ref. 6) has classified approaches as weighting methods, including statistical analysis, sequential elimination techniques, mathematical programming procedures, and spatial proximity methods.

Although these theories have made impressive contributions to decision making, they have yet to resolve a number of issues that are fundamental to both descriptive and prescriptive theory. For arbitrary preference relations, still little is known, and possibly can be said, about such an essential concept as admissible alternatives, also called nondominated, efficient, or Pareto-optimal alternatives. Even when a convex cone \mathbb{P} , the set of points

$$x + \mathbb{P} \equiv \{x + p : p \in \mathbb{P}\}$$

specifying those alternatives preferred to x , describes preferences, admissible points have not been characterized completely. Is the set of admissible points connected? Are there representation theorems which characterize admissible points? What are local characterizations of admissible points? Can the notion of admissible points be exploited within the context of solving mathematical programs?

In this paper, we consider several of these issues. In Section 2, we introduce notation and concepts to be used throughout the paper. The next section considers global characterizations of admissible points, including existence. We show that the sets of admissible and strictly admissible points are both connected when (i) the preference cone \mathbb{P} is convex, (ii) the set

$$\mathbb{P}_s^+ \equiv \{p^+ \in \mathbb{R}^n : p^+ \cdot p > 0 \quad \text{for all nonzero } p \in \mathbb{P}\}$$

is nonempty, (iii) the set of available alternatives X is convex and closed, and (iv) some element of \mathbb{P}_s^+ makes an obtuse angle with every direction of recession of X . In this section, we also present a representation theorem which partially characterizes admissible points. This result says that *all*

admissible points can be expressed as limit points of strictly admissible alternatives or as a translation of such limit points by certain rays in the closure of the preference cone \mathbb{P} . Section 4 considers local characterization of admissible points in terms of linear approximations. These results are related to the usual Kuhn–Tucker characterizations of nonlinear programming. The final section discusses possible extensions and applications.

Our analysis is based upon results of convex analysis and mathematical programming. This approach is an outgrowth of work conducted around 1950 in mathematical statistics [Wald (Ref. 7), Arrow, Barankin, and Blackwell (Ref. 8)], in linear and nonlinear programming [Gale, Kuhn, and Tucker (Ref. 9), Kuhn and Tucker (Ref. 10)], and in economic planning (Koopmans, Ref. 11). These fundamental contributions either proved or suggested many of the properties that we consider here under various restrictions on the problem structure, most notably that the set of available alternatives is polyhedral and/or that preference $x \succ y$, x is preferred to y , is defined by $x \geq y$, $x \neq y$. Later, in a series of papers (Refs. 12–14), Geoffrion studied properties and computational aspects of certain nonlinear vector maximization problems. More recently, Yu (Ref. 15) considered preferences defined by cones as here, including several results related to this paper. In the economic literature, Smale (Refs. 16–18), Rand (Ref. 19), Simon and Titus (Ref. 20), and Wan (Ref. 21) have studied local properties of admissible points from the viewpoint of differential topology. A number of other papers [Charnes and Cooper (Ref. 22), Ecker and Kouada (Refs. 23–24), Evans and Steuer (Ref. 25), Gal (Ref. 26), Geoffrion, Dyer, and Feinberg (Ref. 27), Philip (Ref. 28), Sachtman (Ref. 29), and Yu and Zeleny (Ref. 30)] have treated algorithms for determining and investigating admissible points, primarily for linear problems.

Applications of the concept of cone dominance are varied and include efficiency in economic planning (Refs. 4, 11), mathematical statistics (Ref. 31), maximizing utility vectors in exchange equilibrium (Refs. 4, 32), risk–return trade-offs in portfolio selection (Refs. 33–34), risk sharing and group decisions (Ref. 35), and many others as suggested in Ref. 36 and the collection in Ref. 37.

2. Preliminaries

Throughout our discussion, we let \mathbb{P} be a nonempty and nontrivial, i.e., $\mathbb{P} \neq \{0\}$, cone in \mathbb{R}^n . We say that an n -vector x is *preferred* to an n -vector y with respect to the cone \mathbb{P} , denoted by $x \succ y$, when $x \neq y$ and

$$x \in y + \mathbb{P} \equiv \{y + p : p \in \mathbb{P}\}.$$

We say that a point y is *admissible* for the cone \mathbb{P} over a given set X when $y \in X$ and X contains no points preferred to y , that is,

$$X \cap (y + \mathbb{P}) = \{y\}.$$

Let $A(X)$ denote the set of all admissible points y in X .

Frequently, the cone \mathbb{P} is the nonnegative orthant \mathbb{R}_+^n . Then, $x \succ y$ when $x \geq y$ and $x \neq y$, and admissibility is commonly called Pareto optimality. As a useful variant of this example, the preference cone \mathbb{P} is given by

$$\mathbb{P}_k = \{0\} \cup \{(\lambda_1, \dots, \lambda_k, \gamma_{k+1}, \dots, \gamma_n) \in \mathbb{R}^n : \text{all } \lambda_i \geq 0 \text{ and } (\lambda_1, \dots, \lambda_k) \neq 0\}.$$

In this case, the preference relation compares only the first k components of any vector.

The preference cone \mathbb{P}_k arises from the vector maximization problem of *optimizing* a vector

$$f(z) = (f_1(z), f_2(z), \dots, f_k(z))$$

of k real-valued criteria over a subset Z of \mathbb{R}^{n-k} . A point $\bar{z} \in Z$ is called *efficient* in this problem if there is no point $z \in Z$, satisfying

$$f(z) \geq f(\bar{z}),$$

with the inequality strict in at least one component.

If

$$X = \{x = (y, z) \in \mathbb{R}^n : z \in Z \text{ and } y \leq f(z)\},$$

then a point $\bar{x} \in X$ is admissible with respect to the cone \mathbb{P}_k iff

$$\bar{x} = (\bar{y}, \bar{z}), \quad \bar{y} = f(\bar{z}),$$

and \bar{z} is efficient in the vector maximization problem. By this construction, we have expressed, and can study, the preference order $z \succ \bar{z}$ defined by

$$f(z) \geq f(\bar{z}), \quad f(z) \neq f(\bar{z}),$$

in terms of a cone preference \mathbb{P}_k in an enlarged space.

These examples suggest that both closed and nonclosed preference cones might be studied profitably, since both arise in practice.

Many properties of admissible points depend upon the separation of the sets $y + \mathbb{P}$ and X by a hyperplane. For any admissible point y , such a separation is possible whenever \mathbb{P} and X are convex, since

$$(y + \mathbb{P}) \cap X = \{y\}.$$

That is, there is a nonzero n -vector p^+ such that

$$p^+ \cdot x \leq p^+ \cdot y \leq p^+ \cdot (y + p), \quad \text{for all } x \in X \text{ and } p \in \mathbb{P}. \tag{1}$$

Because the rightmost inequality can be restated as

$$p^+ \cdot p \geq 0, \quad \text{for all } p \in \mathbb{P},$$

we may reexpress (1) by saying that there is a nonzero vector p^+ contained in the *positive polar cone* \mathbb{P}^+ of \mathbb{P} defined by

$$\mathbb{P}^+ \equiv \{p^+ \in \mathbb{R}^n : p^+ \cdot p \geq 0 \text{ for all } p \in \mathbb{P}\},$$

with the property that y solves the optimization problem

$$\max\{p^+ \cdot x : x \in X\}. \tag{2}$$

The positive polar cone is closed and convex without any assumption on \mathbb{P} .

There is a partial converse to this necessary condition for a point y to be admissible, which does not require any convexity assumptions. Let \mathbb{P}_s^+ denote the set of *strict supports* of \mathbb{P} , defined by

$$\mathbb{P}_s^+ \equiv \{p^+ \in \mathbb{R}^n : p^+ \cdot p > 0 \text{ for all nonzero } p \in \mathbb{P}\}.$$

If this set is nonempty, we say that \mathbb{P} is *strictly supported* or that \mathbb{P} is a *strictly supported cone*: any $p^+ \in \mathbb{P}_s^+$ defines a supporting hyperplane

$$\{x \in \mathbb{R}^n : p^+ \cdot x = 0\}$$

that intersects \mathbb{P} only at the origin. In this terminology, the converse states that, if y solves (2) for any strict support $p^+ \in \mathbb{P}_s^+$, then y is admissible. This fact is a simple consequence of observing that $p^+ \in \mathbb{P}_s^+$ and $x \succ -y$ (i.e., $x \neq y$, $x - y \in \mathbb{P}$) implies that $p^+ \cdot (x - y) > 0$ and, therefore, that y does not solve (2). We distinguish these special admissible points in the following definition.

Definition 2.1. An admissible point $y \in A(X)$ is *strictly admissible* if it solves the maximization problem

$$\max\{p^+ \cdot x : x \in X\},$$

for some $p^+ \in \mathbb{P}_s^+$. Any other admissible point is said to be *nonstrict*. We let $A^s(X)$ denote the set of strictly admissible points in X .

Most of our subsequent results require that preference cones be strictly supported. When \mathbb{P} is closed and convex, this condition is equivalent to it being pointed, that is, containing no lines. In Appendix A, we extend this characterization, in order to interpret the strictly supported condition directly in terms of the underlying preference cone whenever \mathbb{P} is convex. The following characterization is a consequence of this development.

Proposition 2.1. Let \mathbb{P} be a convex cone in \mathbb{R}^n . \mathbb{P} is supported strictly iff the relative interior of \mathbb{P}^+ is contained in \mathbb{P}_s^+ .

As an example, if

$$\mathbb{P} = \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 : (p_1, p_2, p_3) = (0, 0, 0) \text{ or } p_1 \geq 0 \text{ and } p_2 > 0\},$$

then

$$\mathbb{P}^+ = \text{cl } \mathbb{P} \cap \{p \in \mathbb{R}^3 : p_3 = 0\}, \quad \mathbb{P}_s^+ = \mathbb{P}^+ \setminus \{0\},$$

and the relative interior of \mathbb{P}^+ is the set

$$\{p \in \mathbb{R}^3 : p_1 > 0, p_2 > 0 \text{ and } p_3 = 0\},$$

a strict subset of \mathbb{P}_s^+ .

Remark 2.1. Throughout the remainder of this paper, we frequently apply elementary results of convex analysis without reference. We also translate many properties of polar cones usually formulated in terms of the negative polar of any set S , denoted by

$$S^* \equiv \{y \in \mathbb{R}^n : y \cdot x \leq 0 \text{ for all } x \in S\},$$

into statements concerning the positive polar S^+ of S . Standard texts in convex analysis [Rockafellar (Ref. 38), Stoer and Witzgal (Ref. 39)] discuss those results that we use.

In addition to the notation introduced earlier in this section, let $\text{RC}(X)$ denote the recession cone of a convex set; let $\text{cl}(S)$, $\text{ri}(S)$, and

$$-S \equiv \{x : -x \in S\}$$

denote the closure, relative interior, and negative of any set S ; and let $S \setminus T$ denote the set-theoretic difference of two sets S and T . We adopt Rockafellar's terminology (Ref. 38) by including the origin in $\text{RC}(X)$, but by defining the direction of recessions as only the nonzero elements of this cone. Finally, we use the Euclidean norm to define the open and closed unit balls in \mathbb{R}^n .

3. Global Characterization

When the set of available alternatives X is convex and closed, and the preference cone \mathbb{P} is convex and supported strictly, the admissible set $A(X)$ has important global properties: it is connected (see Theorem 3.4) if some strict support to \mathbb{P} makes an obtuse angle with every direction of recession of X , and it can be characterized in terms of strictly admissible points (see Theorem 3.1) if no direction of recession of X belongs to the closure of \mathbb{P} .

Before establishing these properties, we briefly consider the existence of admissible points.

Existence. Yu (Ref. 15) has observed that $A(X)$ is nonempty if X is compact and the interior of \mathbb{P}^+ is nonempty ($\mathbb{P}_s^+ \neq \emptyset$ suffices) or if the problem

$$\max\{p^+ \cdot x : x \in X\}$$

has a unique solution for some $p^+ \in \mathbb{P}^+$. Neither condition requires convexity of \mathbb{P} or X . He also notes that $A(X)$ may be empty when these conditions are not satisfied.

The following propositions characterize the existence of admissible points for closed and convex, but not necessarily bounded, sets of alternatives when preferences are defined by strictly supported closed convex cones.

Proposition 3.1. Let \mathbb{P} be a strictly supported closed convex cone, and let X be a nonempty closed convex set. Then, $A(X) \neq \emptyset$ iff the origin is the only element contained in both \mathbb{P} and the recession cone of X .

Proof. If

$$y \neq 0 \in \mathbb{P} \cap \text{RC}(X),$$

then

$$x + y \in (x + \mathbb{P}) \cap X,$$

for any $x \in X$, and so no point $x \in X$ is admissible.

If

$$\mathbb{P} \cap \text{RC}(X) = \{0\},$$

then $(y + \mathbb{P}) \cap X$ is compact for any $y \in X$ (Ref. 38, Theorem 8.4). Consequently, for any $p_s^+ \in \mathbb{P}_s^+$, there is an optimal solution z to the optimization problem

$$\max\{p_s^+ \cdot x : x \in (y + \mathbb{P}) \cap X\}.$$

The point z is strictly admissible in $(y + \mathbb{P}) \cap X$ with respect to \mathbb{P} . It also must be admissible in X ; for, if

$$p \neq 0 \in \mathbb{P} \quad \text{and} \quad z + p \in X,$$

then

$$z + p \in (y + \mathbb{P}) \cap X$$

is preferred to z . □

To obtain a dual version of this proposition, we note the following theorem of the alternative.

Proposition 3.2. (a) Let \mathbb{P} be a strictly supported closed convex cone and let X be a closed convex set. Then, exactly one of the following alternatives is valid:

- (I) $\mathbb{P} \cap \text{RC}(X) \neq \{0\}$,
- (II) $\mathbb{P}_s^+ \cap \text{RC}(X)^* \neq \emptyset$.

Consequently, the set of admissible points in X is nonempty iff

$$\mathbb{P}_s^+ \cap \text{RC}(X)^* \neq \emptyset.$$

(b) Moreover, if \mathbb{P} is any strictly supported convex cone, not necessarily closed, then

$$A^s(X) \neq \emptyset$$

implies alternative (II), and alternative (II) implies that

$$A^s(X) \neq \emptyset$$

whenever X is a polyhedron.

Proof. Alternatives (I) and (II) cannot both be valid, since any

$$p \neq 0 \in \mathbb{P} \cap \text{RC}(X) \quad \text{and} \quad p_s^+ \in \mathbb{P}_s^+ \cap \text{RC}(X)^*$$

must satisfy⁶

$$p_s^+ \cdot p > 0 \quad \text{and} \quad p_s^+ \cdot p \leq 0.$$

So, suppose that

$$\mathbb{P}_s^+ \cap \text{RC}(X)^* = \emptyset.$$

Since \mathbb{P}_s^+ and $\text{RC}(X)^*$ are nonempty convex sets, they can be separated; that is, there is a nonzero vector $u \in \mathbb{R}^n$ and a real number β satisfying

$$u \cdot y \leq \beta \leq u \cdot p_s^+, \quad \text{for all } y \in \text{RC}(X)^* \text{ and all } p_s^+ \in \mathbb{P}_s^+.$$

Since the origin is a limit point of \mathbb{P}_s^+ and is contained in $\text{RC}(X)^*$, $\beta = 0$. The leftmost inequality implies that

$$u \in \text{RC}(X)^{**} = \text{RC}(X),$$

since the recession cone of a closed convex set is closed. The rightmost inequality shows that

$$u \cdot p_s^+ \geq 0, \quad \text{for all } p_s^+ \in \mathbb{P}_s^+,$$

and thus,⁷

$$u \cdot p^+ \geq 0, \quad \text{for all } p^+ \in \mathbb{P}^+.$$

⁶ The closure of \mathbb{P} is not required for this conclusion.

⁷ Every point of \mathbb{P}^+ is a limit point of \mathbb{P}_s^+ , since, by Proposition 2.1, $\text{ri}(\mathbb{P}^+) \subseteq \mathbb{P}_s^+ \subseteq \mathbb{P}^+$.

Consequently,

$$u \in \mathbb{P}^{++} = \mathbb{P} \quad \text{and} \quad u \in \mathbb{P} \cap \text{RC}(X),$$

so that one of alternatives (I) and (II) is always satisfied.

Since, by the previous proposition, $A(X)$ is nonempty iff alternative (I) is not satisfied, $A(X) \neq \emptyset$ iff alternative (II) is valid.

To prove the final assertions of the proposition, consider the optimization problem

$$\max\{p_s^+ \cdot x : x \in X\}, \tag{3}$$

where $p_s^+ \in \mathbb{P}_s^+$. If $z \in A^s(X)$, we can select p_s^+ so that z solves this problem. Therefore,

$$p_s^+ \cdot y \leq 0, \quad \text{for all } y \in \text{RC}(X),$$

implying that

$$p_s^+ \in \text{RC}(X)^*$$

and that

$$\mathbb{P}_s^+ \cap \text{RC}(X)^* \neq \emptyset.$$

If

$$p_s^+ \in \mathbb{P}_s^+ \cap \text{RC}(X)^*$$

and X is polyhedral, then

$$p_s^+ \cdot y \leq 0, \quad \text{for all } y \in \text{RC}(X),$$

implying, by linear programming theory, that there is a solution to problem (3). This solution is strictly admissible. \square

The following examples show that the hypothesis that \mathbb{P} is closed is required for these propositions and that converses to part (b) of the last proposition are not possible.

Example 3.1. Let

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\},$$

and let

$$\mathbb{P} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0 \text{ and } x_2 \geq 0\} \cup \{(0, 0)\}.$$

Then,

$$p_s^+ = (-1, 0) \in \mathbb{P}_s^+ \cap \text{RC}(X)^* \quad \text{and} \quad A(X) = \emptyset.$$

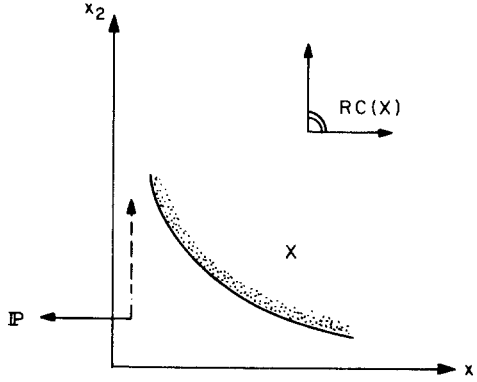


Fig. 1. $\mathbb{P} \cap RC(X) = \{0\}$ and no admissible points.

The preference cone \mathbb{P} is not closed, and neither of the conditions

$$\mathbb{P} \cap RC(X) \neq \{0\} \quad \text{or} \quad \mathbb{P}_s^+ \cap RC(X)^* = \emptyset$$

for $A(X)$ to be empty is valid. In addition, the converse to the first assertion in part (b) of the last proposition is violated. See Fig. 1.

Example 3.2. Let \mathbb{P} be the positive orthant in \mathbb{R}^3 , except for those points in the x_1x_2 -plane not on the x_2 -axis, i.e.,

$$\mathbb{P} = \{(0, \lambda, 0) : \lambda \geq 0\} \cup \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 \geq 0, \lambda_2 \geq 0 \text{ and } \lambda_3 > 0\}.$$

Let

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0, x_1 \geq x_2, \text{ and } x_1 \leq x_2 + 1\}.$$

Then, $RC(X)$ is the line

$$\{\lambda(1, 1, 0) : \lambda \in \mathbb{R}\};$$

$RC(X)^*$ is the subspace

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = 0\};$$

also,

$$\mathbb{P}_s^+ = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 : \lambda_1 \geq 0, \lambda_2 > 0 \text{ and } \lambda_3 > 0\}$$

and

$$A(X) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0 \text{ and } x_1 = x_2\}.$$

In this instance, \mathbb{P} is not closed, and neither alternative (I) nor (II) of Proposition 3.2 is valid. The example also shows that $A(X) \neq 0$ does not

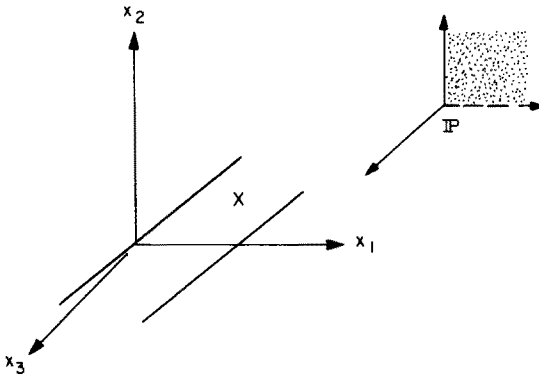


Fig. 2. Proposition 3.2 requires \mathbb{P} to be closed.

necessarily imply alternative II, even when X is polyhedral, unless \mathbb{P} is closed. See Fig. 2.

Representation. Arrow, Barankin, and Blackwell (Ref. 8) have shown that every admissible point is a limit point of strictly admissible points whenever $\mathbb{P} = \mathbb{R}_+^n$ and X is convex and compact. The following example shows that this property is not valid for all preference cones.

Example 3.3. Let X , as illustrated in Fig. 3, be a truncated cone in \mathbb{R}^3 with vertex

$$V = (0, 1, 1)$$

and circular base

$$\{x \in \mathbb{R}^3: x_1^2 + x_2^2 \leq 1 \text{ and } x_3 = 0\}.$$

Let

$$\mathbb{P} = \mathbb{P}_2 = \{0\} \cup \{(\lambda_1, \lambda_2, \lambda_3): \lambda_1 \geq 0, \lambda_2 \geq 0 \text{ and } (\lambda_1, \lambda_2) \neq 0\},$$

which ignores the direction $\pm(0, 0, 1)$. The set of strictly admissible points $A^s(X)$ consists of those points on the interior of the arc KS , the closure of these points is the entire arc, but the admissible points also include the line segment KV .

Observe that the admissible points in this example are those points in X which are translations of the arc KS by the direction $(0, 0, 1)$, a direction which belongs to the closure of the preference cone \mathbb{P}_2 , but not \mathbb{P}_2 itself. Our next result, which shows that this observation characterizes admissible points under very general hypothesis, extends the Arrow, Barankin, and

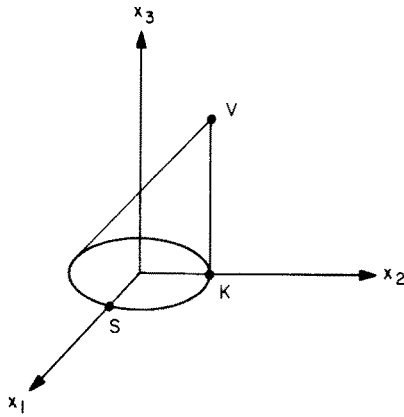


Fig. 3. $A(X) \subseteq \text{cl } A^s(X)$.

Blackwell representation by permitting more general preference cones and by relaxing the compactness of X . We build upon their clever proof techniques, using the following result in place of the von Neumann minimax theorem.

Lemma 3.1. Let C and D be nonempty closed convex sets in \mathbb{R}^n satisfying the conditions

- (i) $C^* \cap \text{RC}(D) = \{0\}$,
- (ii) $\text{RC}(C) \cap D^+ = \{0\}$.

Then, the function $u \cdot v$ has a saddlepoint on $C \times D$, in the sense that, for some $u^0 \in C$ and $v^0 \in D$,

$$u \cdot v^0 \leq u^0 \cdot v^0 \leq u^0 \cdot v, \quad \text{for all } u \in C \text{ and } v \in D.$$

Proof. The lemma is a specialization of a theorem due to Rockafellar (Ref. 38, Theorem 37.6). By this theorem, a saddlepoint exists if the functions

$$f_u(v) \equiv u \cdot v, \quad \text{for } u \in \text{ri}(C),$$

have no common direction of recession over their domain D and the functions

$$g_v(u) \equiv -u \cdot v, \quad \text{for } v \in \text{ri}(D),$$

have no common direction of recession over their domain C . A direction of recession y for $f_u(v)$ is a nonzero element of $RC(D)$ satisfying

$$u \cdot y \leq 0.$$

Therefore, the functions $f_u(v)$ for $u \in ri(C)$ have no common direction of recession if no nonzero $y \in RC(D)$ satisfies

$$u \cdot y \leq 0, \quad \text{for all } u \in C,$$

that is,

$$RC(D) \cap C^* = \{0\}.$$

Similarly, the functions $g_v(u)$ for $v \in ri(D)$ have no common direction of recession if the inequalities

$$-y \cdot v \leq 0, \quad \text{for all } v \in D,$$

are impossible to satisfy simultaneously whenever $y \in RC(C)$ is nonzero, which is condition (ii). □

Theorem 3.1. Let X be a closed convex set, let \mathbb{P} be a strictly supported convex cone, and suppose that

$$c1 \mathbb{P} \cap RC(X) = \{0\}.$$

Then, any point $x \in A(X)$ can be written as

$$x = x^* - \bar{p},$$

where x^* belongs to $c1 A^s(X)$ and either $\bar{p} = 0$ or \bar{p} belongs to $c1 \mathbb{P} \setminus \mathbb{P}$.

Proof. We first establish preliminary results to be used in the proof. Let B be the closed unit ball in \mathbb{R}^n , and let S_j , for $j = 1, 2, \dots$, be nonempty, closed, convex, and increasing cones⁸ (i.e., $S_j \subseteq S_{j+1}$), whose union is $ri(\mathbb{P}^+) \cup \{0\}$. Then,

$$c1 \mathbb{P} = \mathbb{P}^{++} = ri(\mathbb{P}^+)^+ = \bigcap_{j \geq 1} S_j^+.$$

⁸ Several choices are possible. We may, in fact, choose polyhedral cones for the S_j . Let B^0 denote the open unit ball in \mathbb{R}^n ; and, for each $j = 1, 2, \dots$, let F_j be a finite set of points in $Q \equiv [ri(\mathbb{P}^+) \cup \{0\}] \cap B^0$, with the property that $(1/j)$ balls about these points cover Q . The convex hull H_j of $\{0\} \cup F_1 \cup F_2 \cup \dots \cup F_j$ is polyhedral, and so then is the cone S_j that it generates (Ref. 38, Corollary 19.7.1). Any $x \in Q$ belongs to the interior of a simplex in Q , whose dimension equals that of Q . For some j , points in F_j are close enough to the vertices of this simplex, so that $x \in H_j$. Therefore, $\cup \{H_j: j \geq 1\} = Q$, and the union of the S_j 's is $ri(\mathbb{P}^+) \cup \{0\}$.

Note that, for some positive integer J ,

$$S_j^+ \cap \text{RC}(X) = \{0\}, \quad \text{whenever } j \geq J.$$

Otherwise,

$$(B \cap S_j^+) \cap (B \cap \text{RC}(X)) \neq \{0\}, \quad \text{for all } j,$$

which implies, by the finite intersection property [since X is closed, so is $\text{RC}(X)$], that

$$(B \cap c1 \mathbb{P}) \cap (B \cap \text{RC}(X)) \neq \{0\},$$

contrary to hypothesis.

The sets

$$T_j \equiv S_j \cap B, \quad \text{for } j = 1, 2, \dots,$$

are nonempty, convex and compact. In addition,

$$T_j^+ = S_j^+.$$

Now, let x^0 be any admissible point in X , and let

$$Z = \{z \in \mathbb{R}^n : z = x - x^0 \text{ for some } x \in X\}.$$

Then, Z is convex and

$$\text{RC}(Z) = \text{RC}(X).$$

We apply Lemma 3.1 with

$$C = Z, \quad D = T_j, \quad j \geq J,$$

where J is defined as above. Since T_j is compact, conditions (i) and (ii) of the lemma reduce to

$$\text{RC}(Z) \cap T_j^+ = \{0\},$$

which we established previously. Therefore, for each $j \geq J$, there are points $t^j \in T^j$ and $z^j \in Z$ satisfying

$$z \cdot t^j \leq z^j \cdot t^j \leq z^j \cdot t, \quad \text{for all } z \in Z, t \in T^j. \quad (4)$$

Since $x^0 \in X$, $z = 0$ belongs to Z and

$$z^j \cdot t \geq z^j \cdot t^j \geq 0, \quad \text{for all } t \in T^j.$$

The definitions of S^j and T^j imply that the inequalities

$$z^j \cdot s \geq 0, \quad \text{for all } s \in S^j, \quad (5)$$

are valid as well.

Because the union of the increasing sets S_j , for $j = 1, 2, \dots$, is $\text{ri}(\mathbb{P}^+) \cup \{0\}$, Ineqs. (5) imply that, for any $p^+ \in \text{ri}(\mathbb{P}^+)$,

$$p^+ \cdot z^j \geq 0, \quad \text{for all } j \text{ sufficiently large.}$$

Consequently, if $z^* = x^* - x^0$ is a limit point of the sequence $\{z^j\}_{j \geq 1}$, then

$$p^+ \cdot z^* \geq 0, \quad \text{for all } p^+ \in \text{ri}(\mathbb{P}^+),$$

and z^* belongs to

$$\text{ri}(\mathbb{P}^+)^+ = \mathbb{P}^{++} = \text{c1 } \mathbb{P}.$$

If $x^* \neq x^0$, then $(x^* - x^0) \notin \mathbb{P}$, because x^0 was chosen from $A(X)$.

Therefore, any limit point z^* to the sequence $\{z^j\}_{j \geq 1}$ gives $x^0 = x^* - z^*$, with

$$\text{either } z^* = 0 \quad \text{or } z^* \in \text{c1 } \mathbb{P} \setminus \mathbb{P}.$$

This representation satisfies the conclusion of the theorem if

$$x^* \in \text{c1 } A^s(X).$$

But x^* fulfills this condition, since, with $z^j = x^j - x^0$, expression (4) gives

$$t^j \cdot [x^j - x^0] \geq t^j \cdot [x - x^0], \quad \text{for all } x \in X.$$

Consequently, x^j solves

$$\max\{t^j \cdot x : x \in X\},$$

and $x^j \in A^s(X)$, because $t^j \in \mathbb{P}_s^+$. Thus, x^* is a limit point of the strictly admissible points $\{x^j\}_{j \geq 1}$.

To complete the proof, we must show that the sequence $\{z^j\}_{j \geq 1}$ contains a limit point. Suppose, to the contrary, that it does not. Then, the Euclidean norms λ^j of the z^j must grow without bound as j increases. As we have noted previously, for each $p^+ \in \text{ri}(\mathbb{P}^+)$,

$$p^+ \cdot z^j \geq 0,$$

and hence

$$p^+ \cdot (z^j / \lambda_j) \geq 0, \quad \text{for all } j \text{ sufficiently large.}$$

But then, for any limit point y to the sequence $\{z^j / \lambda^j\}_{j \geq 1}$,

$$p^+ \cdot y \geq 0, \quad \text{for all } p^+ \in \text{ri}(\mathbb{P}^+).$$

Therefore,

$$y \in \text{ri}(\mathbb{P}^+)^+ = \mathbb{P}^{++} = \text{c1 } \mathbb{P} \quad \text{and} \quad y \in \text{RC}(X) = \text{RC}(Z)$$

(Ref. 38, Theorem 8.2), contrary to hypothesis, and our assumption that the sequence $\{z^j\}_{j \geq 1}$ contains no limit point is untenable. \square

When the preference cone \mathbb{P} is closed, either

$$\mathbb{P} \cap \text{RC}(X) \neq \{0\} \quad \text{and} \quad A(X) = \emptyset$$

or

$$\mathbb{P} \cap \text{RC}(X) = \{0\},$$

which fulfills the hypothesis of the theorem. In either case, the characterization simplifies as follows.

Corollary 3.1. Let X be a closed convex set, and let \mathbb{P} be a strictly supported closed convex cone. Then, every admissible point is a limit point of strictly admissible points.

A slight modification to Example 3.3 shows that the representation of Theorem 3.1 might not be valid when

$$c1 \mathbb{P} \cap \text{RC}(X) \neq \{0\}.$$

Example 3.4. Let Y be the cone generated by $X - V$ in Example 3.3. Then, if $\mathbb{P} = \mathbb{P}_2$, the admissible set of $V + Y$ is the half line from V passing through K . The set of strictly admissible points is empty, however.

We should emphasize that the previous results do not characterize admissible points completely. Arrow, Barankin, and Blackwell show by an example that a limit point of strictly admissible points need not be admissible. The following example shows that points expressed by $x = x^* - \bar{p}$, as in Theorem 3.4, need not be admissible, even when $x^* \in c1 A^s(X)$ is admissible.

Example 3.5. As in Example 3.2, let \mathbb{P} be defined by

$$\mathbb{P} = \{(0, \lambda, 0) : \lambda \geq 0\} \cup \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 \geq 0, \lambda_2 \geq 0 \text{ and } \lambda_3 > 0\}.$$

Let X be the triangle in \mathbb{R}^3 given by

$$X = \{(x_1, x_2, 0) : x_2 \geq 0, x_1 \geq x_2 \text{ and } x_1 + x_2 \leq 1\}.$$

Every point in X on the line segment l satisfying

$$x_1 + x_2 = 1$$

is strictly admissible, and every point x in X can be written as

$$x = x^* - \bar{p},$$

for some $x^* \in l$ and $\bar{p} \in c1 \mathbb{P} \setminus \{0\}$; but not every point is admissible. The admissible points are the points in X on the lines

$$x_1 = x_2 \quad \text{and} \quad x_1 + x_2 = 1.$$

Connectedness. When X is a polyhedron and \mathbb{P} is a polyhedral cone, parametric analysis in linear programming shows that $A(X)$ is connected [see, for example, Koopmans (Ref. 11) or Yu and Zeleny (Ref. 30) for related results]. To study the connectedness of $A(X)$ in a more general setting, we also use a parametrization of a mathematical programming objective function.

We first set some additional notation. Let $p(0)$ and $p(1)$ be any points in \mathbb{R}^n , and, for each $0 \leq \theta \leq 1$, let

$$p(\theta) = \theta p(1) + (1 - \theta)p(0).$$

Let $X(\theta)$ denote the set of optimal solutions to the optimization problem

$$v(\theta) = \sup\{p(\theta) \cdot x : x \in X\}, \tag{6}$$

for a given set X , not necessarily convex; $x(\theta)$ denotes a generic element of $X(\theta)$.

Berge (Ref. 40) discusses several properties of parametric optimization problems which encompass results for this problem. See also Hildenbrand and Kirman (Ref. 4), whose introductory description of parametric analysis is formulated to include the following result.

Lemma 3.2. Let X be an arbitrary closed subset of \mathbb{R}^n , and assume that the solution set $X(\theta)$ to problem (6) is nonempty and compact for all $0 \leq \theta \leq 1$. Then, the point-to-set mapping $\theta \rightarrow X(\theta)$ is upper semicontinuous; that is, if $0 \leq \theta \leq 1$ and G is an open set in \mathbb{R}^n containing $X(\theta)$, then there is a real number $\delta > 0$ such that $X(\theta') \subseteq G$ whenever

$$0 \leq \theta' \leq 1 \quad \text{and} \quad |\theta - \theta'| \leq \delta.$$

Using this lemma, we first consider the set of strictly admissible points.

Theorem 3.2. Let \mathbb{P} be a strictly supported convex cone, let X be a given subset of \mathbb{R}^n , and suppose that the solution set to the optimization problem

$$\max\{p_s^+ \cdot x : x \in X\}$$

is nonempty, compact, and connected for each $p_s^+ \in \mathbb{P}_s^+$. Then, $A^s(X)$ is connected.

Proof. Suppose, to the contrary, that $A^s(X)$ is not connected; that is, there are disjoint open sets G and H with

$$G \cap A^s(X) \neq \emptyset, \quad H \cap A^s(X) \neq \emptyset, \quad A^s(X) \subseteq G \cup H.$$

Let

$$x(0) \in G \cap A^s(X) \quad \text{and} \quad x(1) \in H \cap A^s(X).$$

The definition of strictly admissible points implies that there are vectors

$$p(0) \in \mathbb{P}_s^+ \quad \text{and} \quad p(1) \in \mathbb{P}_s^+,$$

so that, for $\theta = 0$ and 1 , $x(\theta)$ solves

$$\max\{p(\theta) \cdot x : x \in X\}, \tag{7}$$

where $p(\theta) \in \mathbb{P}_s^+$ is defined (as above) as

$$p(\theta) = \theta p(1) + (1 - \theta)p(0).$$

Since the set of optimal solutions $X(\theta)$ to problem (7) is nonempty and connected, each $X(\theta)$ is either contained in G or contained in H . In particular,

$$X(0) \subseteq G \quad \text{and} \quad X(1) \subseteq H.$$

Let

$$\theta' = \sup\{\bar{\theta} : 0 \leq \bar{\theta} \leq 1 \quad \text{and} \quad X(\theta) \subseteq G \quad \text{for all} \quad 0 \leq \theta \leq \bar{\theta}\}.$$

By the previous lemma, $\theta' > 0$. Now, $X(\theta')$ must be contained in either G or H . But either assumption leads to a contradiction. For, if $X(\theta') \subseteq G$, then $X(\theta) \subseteq G$ for every $\theta \leq \theta' + \delta$ and some $\delta > 0$, by Lemma 3.2, contrary to the definition of θ' ; and, if $X(\theta') \subseteq H$, then by Lemma 3.2 again, $X(\theta) \subseteq H$ for every $\theta \geq \theta' - \delta$ and some $\delta > 0$, contrary to the definition of θ' .

Therefore, our assumption that $A^s(X)$ is not connected is untenable, and the theorem has been proven. \square

The following version of this theorem is probably more useful in applications.

Theorem 3.3. Let \mathbb{P} be a strictly supported convex cone, let X be a closed convex set, and suppose that

$$-\mathbb{P}_s^+ \cap \text{RC}(X)_s^+ \neq \emptyset.$$

Then, $A^s(X)$ is connected.

Note that the condition on $-\mathbb{P}_s^+ \cap \text{RC}(X)_s^+$ states that, for some $p_s^+ \in \mathbb{P}_s^+$, $p_s^+ \cdot y < 0$ for every element y in the recession cone of X . That is, the solution set to the optimization problem

$$\sup\{p_s^+ \cdot x : x \in X\}$$

must be bounded for some, but not necessarily all, strict supports p_s^+ to \mathbb{P} . This weakening of the hypothesis of Theorem 3.2. is offset by the stronger assumption that X is convex.

With slight modifications, the proof of Theorem 3.2 proves Theorem 3.3 as well. Without loss of generality, we may select $x(1)$ in the proof as the solution to

$$\max\{p(1) \cdot x : x \in X\},$$

where $-p(1) \in \text{RC}(X)_s^+$. Then, we invoke the following result, instead of Lemma 3.2. Observe that each solution set $X(\theta)$, $0 \leq \theta \leq 1$, to problem (7) is convex, and hence connected.

Lemma 3.3. Let X be a closed convex set in \mathbb{R}^n . Assume that the solution set $X(\theta)$ to the problem

$$v(\theta) = \sup\{[\theta p(1) + (1 - \theta)p(0)] \cdot x : x \in X\}$$

is nonempty for $\theta = 0$ and nonempty and compact for $\theta = 1$. Then, $X(\theta)$ is nonempty and compact for all $0 < \theta \leq 1$, and the point-to-set mapping $\theta \rightarrow X(\theta)$ is upper semicontinuous.

This lemma is proved in Appendix B (Section 7).

When X is compact,

$$\text{RC}(X) = \{0\},$$

and every point in \mathbb{R}^n is a strict support to $\text{RC}(X)$; consequently, the hypothesis

$$-\mathbb{P}_s^+ \cap \text{RC}(X)_s^+ \neq \emptyset$$

of this theorem is valid whenever \mathbb{P} is strictly supported. Therefore, we have the following corollary.

Corollary 3.2. Let \mathbb{P} be a strictly supported convex cone, and let X be convex and compact. Then, $A^s(X)$ is connected. If, in addition, \mathbb{P} is closed or $\mathbb{P} \setminus \{0\}$ is open, then $A(X)$ is connected.

Proof. As we have just noted, Theorem 3.3 shows that $A^s(X)$ is connected. Whenever $\mathbb{P} \setminus \{0\}$ is open, $\mathbb{P}^+ = \mathbb{P}_s^+$, and so $A(X) = A^s(X)$ is connected. By Corollary 3.1, whenever \mathbb{P} is closed, $A(X)$ is connected, since it is contained in the closure of the connected set $A^s(X)$ and contains $A^s(X)$. □

The next two examples show that the *strictly supported condition* imposed upon \mathbb{P} in this corollary is indispensable, as is the condition

$$-\mathbb{P}_s^+ \cap \text{RC}(X)_s^+ \neq \emptyset$$

of Theorem 3.3.

Example 3.6. Let X be the closed unit ball in \mathbb{R}^2 , and let

$$\mathbb{P} = \{(0, \lambda) \in \mathbb{R}^2 : \lambda \in \mathbb{R}\}.$$

Then, \mathbb{P} is closed and convex, but $\mathbb{P}_s^+ = \emptyset$. Since the only admissible points are $(-1, 0)$ and $(1, 0)$, $A(X)$ is not connected.

Example 3.7. Let

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0, x_3 \leq 0, \text{ and } \exp(x_3) - 1 \leq x_2 \leq 0\},$$

and let T be the half line

$$\{(1, 0, \lambda) \in \mathbb{R}^3 : \lambda \leq 0\}.$$

Define X , which is closed (Ref. 38, Corollary 9.8.1), as the convex hull of S and T ; and let \mathbb{P}_s^+ be generated by nontrivial nonnegative combinations of

$$p(0) = (-1, 0, 0) \quad \text{and} \quad p(1) = (1, -1, 0).$$

Since the third component of every element of \mathbb{P}_s^+ is zero, a point in X is strictly admissible iff it is a translation by negative values of x_3 at some point y that is strictly admissible in the set obtained by projecting X onto the x_1x_2 -axis.

The solutions to problem (7) are (see Fig. 4)

$$X(\theta) = \begin{cases} S, & \text{if } \theta = 0, \\ \emptyset, & \text{if } 0 < \theta < 1, \\ T, & \text{if } \theta = 1. \end{cases}$$

In this case,

$$RC(X)_s^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\},$$

$$-\mathbb{P}_s^+ \cap RC(X)_s^+ = \emptyset,$$

and

$$A^s(X) = S \cup T$$

is not connected.

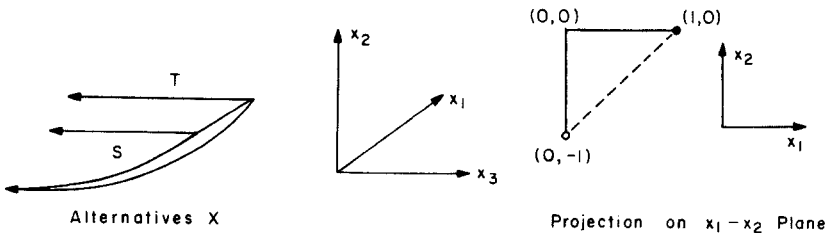


Fig. 4. Set of strictly admissible points that is not connected.

To show that $A(X)$ is connected, without assuming that the underlying preference cone \mathbb{P} is closed or that $\mathbb{P} \setminus \{0\}$ is open, requires additional argument. For any vector $y \in \mathbb{R}^n$, let $\mathcal{L}(y)$ and $\mathcal{L}(y)^\perp$ denote the subspace generated by y , and its orthogonal subspace.

Lemma 3.4. Let \mathbb{P} and X be a cone and arbitrary set in \mathbb{R}^n . Suppose that p^+ belongs to \mathbb{P}^+ and that y solves

$$\max\{p^+ \cdot x : x \in X\}.$$

Then, the following conditions are equivalent:

- (i) $y \in A(X)$;
- (ii) $y \in A(\hat{X})$, where $\hat{X} = X \cap (y + \mathcal{L}(p^+)^\perp)$;
- (iii) y is admissible in $X \cap (y + \mathcal{L}(p^+)^\perp)$ with respect to $\mathbb{P} \cap \mathcal{L}(p^+)^\perp$.

Proof. Suppose that

$$z = y + p \in X, \quad \text{for some } p \in \mathbb{P}.$$

By definition of y ,

$$p^+ \cdot z = p^+ \cdot y + p^+ \cdot p \leq p^+ \cdot y.$$

But, since $p^+ \cdot p \geq 0$, then $p^+ \cdot p = 0$; that is,

$$p \in \mathcal{L}(p^+)^\perp \quad \text{and} \quad z \in (y + \mathcal{L}(p^+)^\perp) \cap X.$$

Consequently, y is not admissible with respect to \mathbb{P} , that is, p can be chosen to be nonzero, iff condition (ii) is violated and iff condition (iii) is violated. □

Corollary 3.3. Let $p^+ \in \mathbb{P}^+ \setminus \mathbb{P}_s^+$, and let y solve the problem

$$\max\{p^+ \cdot x : x \in X\}.$$

Then, either $y \in A(X)$ or $y + p \in X$ for some p contained in the boundary of \mathbb{P} .

Proof. Apply the previous lemma, noting that any point $p \in \mathbb{P} \cap \mathcal{L}(p^+)^\perp$ belongs to the boundary of \mathbb{P} . □

Corollary 3.4. Let $p^+ \in \mathbb{P}^+$, let

$$\hat{X} = \{x \in X : p^+ \cdot x \geq p^+ \cdot z \text{ for all } z \in X\},$$

and let $p_s^+ \in \mathbb{P}_s^+$. Then, any solution y to the problem

$$\max\{p_s^+ \cdot x : x \in \hat{X}\}$$

belongs to $A(X)$.

Proof. Since y belongs to $A^s(\hat{X}) \subseteq A(\hat{X})$ and, by Lemma 3.3, $A(\hat{X}) = A(X) \cap \hat{X}$, then y belongs to $A(X)$. □

These results and the representation Theorem 3.1 provide ingredients for proving that the set of admissible points is connected, without requiring \mathbb{P} to be closed or $\mathbb{P} \setminus \{0\}$ to be open.

Theorem 3.4. Let \mathbb{P} be a strictly supported convex cone, and let X be a closed convex set satisfying

$$-\mathbb{P}_s^+ \cap \text{RC}(X)_s^+ \neq \emptyset.$$

Then, $A(X)$ is connected.

Proof. We use induction on the dimension k of X . Whenever $k = 1$, $A(X)$ is an interval, and hence connected; so, assume that the theorem is valid for all closed convex sets with dimension less than k and that X has dimension k .

Note, first, that no generality is lost by assuming that X has full dimension. For, suppose (by translation, if necessary) that the origin belongs to X . Then, X is connected in \mathbb{R}^n iff it is connected in L , the smallest linear subspace containing X . The definition of admissibility implies that x belongs to $A(X)$ iff it is admissible in X with respect to $\mathbb{P} \cap L$, which we assume is nonempty; otherwise, $A(X) = X$ is connected. Moreover, expressing any

$$p_s^+ \in \mathbb{P}_s^+, \quad -p_s^+ \in \text{RC}(X)_s^+$$

as

$$p_s^+ = p_L^+ + p_\perp^+,$$

where

$$p_L^+ \in L \quad \text{and} \quad p_\perp^+ \in L^\perp,$$

L^\perp denoting the orthogonal subspace of L , shows that

$$Q \equiv (\mathbb{P} \cap L)_s^+ \cap L$$

is nonempty, as is

$$-Q \cap [\text{RC}(X)_s^+ \cap L];$$

i.e., if

$$p \in \mathbb{P} \cap L, \quad p \neq 0, \quad y \in \text{RC}(X),$$

then

$$p_s^+ \cdot p = p_L^+ \cdot p > 0 \quad \text{and} \quad p_s^+ \cdot y = p_L^+ \cdot y < 0.$$

Thus,

$$p_L^+ \in Q \quad \text{and} \quad -p_L^+ \in \text{RC}(X)_s^+ \cap L.$$

Consequently, the hypothesis of the theorem is valid in L , and we may assume that X is an element of \mathbb{R}^k .

Let y be any element of $A(X)$. Then, y solves

$$\max\{p^+ \cdot x : x \in X\}$$

for some nonzero $p^+ \in \mathbb{P}^+$. Let \hat{X} denote the set of optimal solutions to this problem. If (i) $A(X) \cap \hat{X}$ is connected and (ii) $\text{cl } A^s(X) \cap A(X) \cap \hat{X} \neq \emptyset$, then by Theorem 3.3 the set of strictly admissible points in X , together with the admissible points in \hat{X} , is connected. Since $y \in A(X)$ was chosen arbitrarily, the set $A(X)$ will be connected as well.

Consequently, we will establish the theorem by verifying conditions (i) and (ii). Since $\hat{X} \subseteq X$,

$$\text{RC}(\hat{X}) \subseteq \text{RC}(X) \quad \text{and} \quad \text{RC}(X)_s^+ \subseteq \text{RC}(\hat{X})_s^+.$$

Therefore

$$-\mathbb{P}_s^+ \cap \text{RC}(\hat{X})_s^+ \neq \emptyset;$$

and, since \hat{X} has dimension less than k , the inductive hypothesis implies that $A(\hat{X})$ is connected. By Lemma 3.4,

$$A(X) \cap \hat{X} = A(\hat{X}),$$

and condition (i) is satisfied.

To establish condition (ii), let x^0 solve

$$\max\{p_s^+ \cdot x : x \in \hat{X}\}$$

for some $p_s^+ \in \mathbb{P}_s^+$. By Corollary 3.4, $x^0 \in A(X)$. Since

$$-\mathbb{P}_s^+ \cap \text{RC}(X)_s^+ \neq \emptyset,$$

then

$$\text{cl } \mathbb{P} \cap \text{RC}(X) = \{0\},$$

since any vector q and nonzero vector p , with

$$q \in -\mathbb{P}_s^+ \cap \text{RC}(X)_s^+ \quad \text{and} \quad p \in \text{cl } \mathbb{P} \cap \text{RC}(X),$$

would satisfy

$$q \cdot p \leq 0 \quad \text{and} \quad q \cdot p > 0.$$

Thus, the representation Theorem 3.1 applies and x^0 can be represented as $x^0 = x^* - \bar{p}$, where $x^* \in \text{cl } A^s(X)$ and either

$$\bar{p} = 0 \quad \text{or} \quad \bar{p} \in \text{cl } \mathbb{P} \setminus \mathbb{P}.$$

Therefore,

$$p^+ \cdot x^0 = p^+ \cdot x^* - p^+ \cdot \bar{p} \leq p^+ \cdot x^*$$

and

$$p_s^+ \cdot x^0 = p_s^+ \cdot x^* - p_s^+ \cdot \bar{p} \leq p_s^+ \cdot x^*,$$

and so $x^* \in \hat{X}$ and x^* solves

$$\max\{p_s^+ \cdot x : x \in \hat{X}\}.$$

As a result,

$$x^* \in A(X) \cap \text{cl } A^s(X),$$

and condition (ii) is satisfied. □

4. Local Characterization

Studying properties of an underlying set by applying convex analysis to approximations of the set has been a recurring and fruitful theme in optimization. In this section, we adopt this viewpoint, assuming that the set of alternatives X is defined as the intersection of a convex set C with a set D , not necessarily convex. By approximating D at a given point x^0 to form an approximation to X , we investigate admissibility in X via the approximation. We show, with appropriate hypotheses, that strict admissibility in X is equivalent to admissibility in the approximation. We also establish a Kuhn–Tucker theory in the setting of cone dominance, which (when specialized) becomes the Kuhn–Tucker theory of nonlinear programming.

General Setting. Let us call a set $L(x^0)$ a *conical approximation* to D at x^0 if

$$L(x^0) - x^0 \equiv \{(x - x^0) : x \in L(x^0)\}$$

is a closed cone. If, in addition, $D \subseteq L(x^0)$, we say that $L(x^0)$ is a *support* to D at x^0 . A support to D at x^0 is said to be *finite* if it is a polyhedron. In this case, $L(x^0)$ is the intersection of a finite number of half spaces, each supporting D at x^0 .

In many applications, the set D is defined by a system of nonlinear inequalities,

$$D = \{x : h_i(x) \geq 0, i = 1, \dots, m\}.$$

In this case, two conical approximations to D predominate in the optimization literature. When each function $h_i(x)$ is differentiable at x^0 with

gradient $\nabla h_i(x^0)$, then

$$L(x^0) = \{x \in \mathbb{R}^n : \nabla h_i(x^0)(x - x^0) \geq 0 \text{ for all } i \text{ with } h_i(x^0) = 0\}; \quad (8)$$

and, when each function $h_i(x)$ is concave with supergradient s_i at x^0 , i.e., s_i satisfies the *supergradient inequality*

$$h_i(x) \leq h_i(x^0) + s_i(x - x^0), \quad \text{for all } x \in \mathbb{R}^n,$$

then

$$L(x^0) = \{x \in \mathbb{R}^n : s_i(x - x^0) \geq 0 \text{ for all } i \text{ with } h_i(x^0) = 0\}. \quad (9)$$

When the functions $h_i(x)$ are both differentiable and concave, the finite supports (8) and (9) coincide.

As a preview of the results to be presented in this section, we begin with the following example. Let

$$\begin{aligned} C &= \mathbb{R}^n, \\ X = D &= \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, x_1 \geq 0, x_2 \geq 0\}, \\ \mathbb{P} &= \mathbb{R}_+^2. \end{aligned}$$

Then,

$$x^0 = (0, 1)$$

is admissible for the cone \mathbb{P} over X , but not over the linear approximation (8) to X at x^0 , while

$$x^1 = (1/\sqrt{2}, 1/\sqrt{2})$$

is strictly admissible over X as well as over its linear approximation (8) at x^1 . In the remainder of this section, we will formalize these observations.

Our first result relates admissibility in $L(x^0)$ to strict admissibility in D . We will apply some simple, but useful, observations concerning admissible points in cones.

Lemma 4.1. Let \mathbb{P} and Y be closed convex cones in \mathbb{R}^n , and suppose that \mathbb{P} is supported strictly. Then, for any $x^0 \in \mathbb{R}^n$,

- (i) either $x^0 \in A(x^0 + Y)$ or $A(x^0 + Y) = \emptyset$;
- (ii) $x^0 \in A^s(x^0 + Y)$ iff $\mathbb{P}_s^+ \cap Y^* \neq \emptyset$;
- (iii) $x^0 \in A^s(x^0 + Y)$ whenever $x^0 \in A(x^0 + Y)$.

Proof. Conclusions (i) and (ii) are immediate consequences of the definitions. If $x^0 \notin A^s(x^0 + Y)$, then $\mathbb{P}_s^+ \cap Y^* = \emptyset$ by (ii), and $A(x^0 + Y) = \emptyset$ by Proposition 3.2. This observation, coupled with part (i), establishes conclusion (iii). □

In the next two propositions, we assume that $C = \mathbb{R}^n$ in the definition $X = C \cap D$ of X .

Proposition 4.1. Let \mathbb{P} be a strictly supported closed convex cone, and let $x^0 \in X \subseteq \mathbb{R}^n$. Then, $x^0 \in A^s(X)$ iff x^0 is admissible in some support $L(x^0)$ to X at x^0 .

Proof. Whenever x^0 is strictly admissible, it solves the problem

$$\max\{p_s^+ \cdot x : x \in X\}$$

for some $p_s^+ \in \mathbb{P}_s^+$. The set

$$L(x^0) = \{x \in \mathbb{R}^n : p_s^+ \cdot x \leq p_s^+ \cdot x^0\}$$

supports X at x^0 , and x^0 is admissible in $L(x^0)$.

If x^0 is admissible in some support $L(x^0)$ to X at x^0 , then by the previous lemma, x^0 is strictly admissible in $L(x^0)$. But, since $x^0 \in X \subseteq L(x^0)$, the definition of strict admissibility implies that $x^0 \in A^s(X)$. \square

Examples 3.3 and 3.4 show that the closedness of \mathbb{P} is necessary in the previous lemma and proposition. Let

$$x^0 = V = (0, 1, 1),$$

and let X and Y be defined as in these examples. Then, x^0 is not strictly admissible in either $x^0 + Y$ or X , even though it is admissible in both of these sets and $L(x^0) = Y$ is a support to X at x^0 .

Certain features of Proposition 4.1 are worth noting. First, the conclusion does not state that, whenever x^0 is strictly admissible, it is admissible in every support $L(x^0)$ to X at this point. For example, let X be the unit square in \mathbb{R}^2 and $\mathbb{P} = \mathbb{R}_+^2$. Although

$$L(x^0) = \{x = (x_1, x_2) : x_1 \leq 1\}$$

is a support to X at the strictly admissible point $x^0 = (1, 1)$, x^0 is not admissible in this support. In fact, the support

$$L(x^0) = \{x \in \mathbb{R}^n : p_s^+ \cdot x \leq p_s^+ \cdot x^0\}$$

to X chosen in the proof of the proposition depends upon knowledge of a strict support $p_s^+ \in \mathbb{P}_s^+$ for which x^0 maximizes $p_s^+ \cdot x$ over X . More useful would be a support that depends only upon local information at x^0 , such as the supports specified in expressions (8) and (9). Our next results delineate a wide class of problems where such supports are possible.

Proposition 4.2. Let \mathbb{P} be a strictly supported closed convex cone and let X be a polyhedron. Then, every admissible point $x^0 \in X$ is strictly

admissible. Moreover, any admissible point $x^0 \in X$ is strictly admissible in the support $L(x^0)$ defined by (8) with $D = X$.

Proof. Let

$$h_i(x) = a^i x - b_i, \quad \text{for } i = 1, 2, \dots, m,$$

denote linear-affine functions defining X , and suppose that $x^0 \in A(X)$ and that $L(x^0)$ is defined by (8). We first note that x^0 is admissible in $L(x^0)$, for otherwise some $z \neq x^0$ belongs to $L(x^0) \cap (x^0 + \mathbb{P})$. But then,

$$a^i \cdot z \geq b_i \quad \text{and} \quad y \equiv x^0 + \theta(z - x^0),$$

where $\theta > 0$, satisfies

$$a^i \cdot y \geq b_i,$$

for all indices i with

$$a^i \cdot x^0 = b_i.$$

Choosing θ small enough,

$$a^i \cdot y \geq b_i,$$

for every i with

$$a^i \cdot x^0 > b_i$$

as well. Therefore,

$$y \in X, \quad y \neq x^0, \quad y \in x^0 + \mathbb{P},$$

contradicting $x^0 \in A(X)$.

Since x^0 is admissible in $L(x^0)$ and $L(x^0) - x^0$ is a closed convex cone, $x^0 \in A^s(L(x^0))$, by Lemma 4.1, and consequently $x^0 \in A^s(X)$. □

Previously, Evans and Steuer (Ref. 25) have shown that

$$A(X) = A^s(X)$$

when \mathbb{P} , as well as X , is polyhedral.

We next consider instances when X is nonpolyhedral. If x^0 is strictly admissible in $X = C \cap D$, then it solves the optimization problem

$$\max_{x \in C \cap D} p_s^+ \cdot (x - x^0), \tag{10}$$

for some $p_s^+ \in \mathbb{P}_s^+$. Replacing D by $L(x^0)$, a support at x^0 , and dualizing with respect to $u \in [L(x^0) - x^0]^+$ removes $L(x^0)$ from the constraints and gives

$$\sup_{x \in C} (p_s^+ + u) \cdot (x - x^0). \tag{11}$$

Note that, when $L(x^0)$ is defined by (8),

$$L(x^0) - x^0 = \{y \in \mathbb{R}^n : \nabla h_i(x^0)y \geq 0 \text{ for all } i \text{ with } h_i(x^0) = 0\};$$

and, by Farkas' lemma,

$$[L(x^0) - x^0]^+$$

$$= \{u \in \mathbb{R}^n : u = \lambda \cdot \nabla h(x^0) \text{ for some vector } \lambda \geq 0 \text{ with } \lambda \cdot h(x^0) = 0\}.$$

In this case, the objective function in (11) is a linear approximation to the Lagrangian function of (10).

Recalling the usual terminology of nonlinear programming for this example, we call x^0 a *Kuhn-Tucker point* in $X = C \cap D$ with respect to the cone \mathbb{P} and support $L(x^0)$ to D at x^0 , if $x^0 \in D$ and

$$\max_{x \in C} (p_s^+ + u)(x - x^0) = 0,$$

for some *Kuhn-Tucker multipliers*

$$p_s^+ \in \mathbb{P}_s^+ \quad \text{and} \quad u \in [L(x^0) - x^0]^+.$$

The following proposition characterizes such Kuhn-Tucker points.

Proposition 4.3. x^0 is a Kuhn-Tucker point in X with respect to the cone \mathbb{P} and support $L(x^0)$ to D at x^0 iff the following conditions are satisfied:

(i) x^0 is strictly admissible in $C \cap L(x^0)$, that is, x^0 solves

$$v = \max\{p_s^+ \cdot (x - x^0) : x \in C \cap L(x^0)\},$$

for some $p_s^+ \in \mathbb{P}_s^+$;

(ii) for this p_s^+ , we have

$$v = \min\{v(u) : u \in [L(x^0) - x^0]^+\},$$

where

$$v(u) = \sup\{(p_s^+ + u)(x - x^0) : x \in C\}.$$

Proof. The validity of conditions (i) and (ii) implies that

$$0 = v = \max\{(p_s^+ + u)(x - x^0) : x \in C\},$$

for some $u \in [L(x^0) - x^0]^+$, so x^0 is a Kuhn-Tucker point. Conversely, if x^0 is a Kuhn-Tucker point with associated Kuhn-Tucker multipliers p_s^+ and u , then $v(u) = 0$. Since $u \in [L(x^0) - x^0]^+$,

$$u \cdot (x - x^0) \geq 0,$$

and consequently,

$$p_s^+ \cdot (x - x^0) \leq (p_s^+ + u)(x - x^0) \leq v(u)$$

whenever $x \in C \cap L(x^0)$. These inequalities imply that

$$v \leq v(u) = 0.$$

Since $x^0 \in C \cap L(x^0)$,

$$v \geq 0.$$

Therefore,

$$v = v(u) = 0$$

satisfies conditions (i) and (ii). □

Whenever $L(x^0)$ is polyhedral, as in (8) or (9), and

$$\text{ri}(C) \cap L(x^0) \neq \emptyset,$$

the duality condition (ii) is fulfilled (Ref. 38, Corollary 28.2.2). In particular, if $C = \mathbb{R}^n$, then condition (ii) becomes superfluous, and we may sharpen Proposition 4.1 by specifying necessary conditions for strict admissibility in terms of easily computed supports.

Corollary 4.1. Let \mathbb{P} be a strictly supported convex cone, let

$$X = D = \{x \in \mathbb{R}^n : h_i(x) \geq 0, i = 1, 2, \dots, m\},$$

and assume that each constraint function h_i is differentiable at a strictly admissible point x^0 solving

$$\max\{p_s^+ \cdot x : x \in X\}, \tag{12}$$

where $p_s^+ \in \mathbb{P}_s^+$. If problem (12) satisfies any constraint qualification,⁹ then x^0 is strictly admissible in the support $L(x^0)$ defined by (8).

Corollary 4.2. Let \mathbb{P} be a strictly supported convex cone, and let $X = C \cap D$, where C is a convex set and

$$D = \{x \in \mathbb{R}^n : h_i(x) \geq 0, i = 1, 2, \dots, m\}$$

is defined by concave functions. Then, if X satisfies the Slater condition

$$h_i(x^*) > 0, \quad \text{for } i = 1, 2, \dots, m, \quad \text{for some } x^* \in C,$$

x^0 is strictly admissible in the support defined by (9).

⁹ There are conditions, like linear independence of the vectors $\nabla h_i(x^0)$ for indices i with $h_i(x^0) = 0$, that ensure that x^0 satisfies the Kuhn–Tucker conditions of nonlinear programming for problem (12).

Vector Maximization Problem. To illustrate the previous results in a somewhat more concrete setting, let us consider the vector optimization problem introduced in Section 2 with a criterion function

$$f(z) = (f_1(z), \dots, f_k(z))$$

and a set of alternatives $Z \subseteq \mathbb{R}^{n-k}$. Let

$$X = \{(y, z) \in \mathbb{R}^n : z \in Z \text{ and } y \leq f(z)\},$$

and suppose that Z is defined by $Z = C \cap D$, where

$$D = \{z \in \mathbb{R}^{n-k} : h_i(z) \geq 0, i = 1, 2, \dots, m\}.$$

Let

$$\mathcal{C} = \{(y, z) : y \in \mathbb{R}^k, z \in C\}.$$

For any $z \in D$, let $h_*(z)$ denote the subvector of

$$h(z) = (h_1(z), h_2(z), \dots, h_n(z))$$

restricted to those components with $h_i(z) = 0$. As we noted in Section 2, z^0 is efficient in the vector maximization problem iff

$$x^0 = (f(z^0), z^0)$$

is admissible in X with respect to the preference cone $\mathbb{P} = \mathbb{P}_k$. Note, in this case, that

$$\mathbb{P}_s^+ = \{(\lambda, \gamma) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \lambda > 0 \text{ and } \gamma = 0\}.$$

If each of the functions $f_i(z)$ and $h_i(z)$ is differentiable, then the linear approximation to X at x^0 corresponding to (8) becomes

$$L(x^0) - x^0 = \{(y, z) \in \mathbb{R}^n : y \leq \nabla f(z^0) \cdot z \text{ and } \nabla h_*(z^0) \cdot z \geq 0\}.$$

Since any admissible point (y^*, z^*) in $L(x^0) \cap \mathcal{C}$ with respect to \mathbb{P}_k must satisfy

$$y^* = \nabla f(z^0) \cdot z^*,$$

$x^0 = (y^0, z^0)$ is strictly admissible in $L(x^0) \cap \mathcal{C}$ whenever it solves

$$v = \max\{\lambda \nabla f(z^0) \cdot (z - z^0) : z \in C \text{ and } \nabla h_*(z^0) \cdot (z - z^0) \geq 0\}, \quad (13)$$

for some positive k -vector λ .

As we have noted previously, Farkas' lemma implies that the polar to the polyhedral cone $L(x^0) - x^0$ is given by

$$[L(x^0) - x^0]^+ = \{(u^1, u^2) \in \mathbb{R}^n : u^1 = -\lambda$$

and

$$u^2 = \lambda \nabla f(z^0) + \mu \nabla h_*(z^0) \text{ for some } \lambda \geq 0 \text{ and } \mu \geq 0\}.$$

Therefore, (y^0, z^0) is a Kuhn–Tucker point if it solves

$$\max_{\substack{y \in \mathbb{R}^k \\ z \in C}} \{[\lambda \nabla f(z^0) + \mu \nabla h_*(z^0)] \cdot (z - z^0) + (\sigma - \lambda) \cdot (y - y^0)\},$$

for some positive k -vector σ . The value of this optimization problem is $+\infty$, unless $\sigma = \lambda$. Thus, $(f(z^0), z^0)$ is a Kuhn–Tucker point, or z^0 is a Kuhn–Tucker point in the vector maximization problem, whenever

$$\max_{z \in C} \{[\lambda \nabla f(z^0) + \mu \nabla h_*(z^0)] \cdot (z - z^0)\} = 0, \tag{14}$$

for some positive k -vector λ .

Note that, for z^0 to be a Kuhn–Tucker point to the vector maximization problem, a choice of positive weights λ is required for the vector criterion function $f(z)$, so that z^0 is a Kuhn–Tucker point to the nonlinear program

$$\max\{\lambda f(z) : z \in C \text{ and } h(z) \geq 0\}. \tag{15}$$

Proposition 4.3 shows that necessary and sufficient conditions for z^0 to be a Kuhn–Tucker point are that: (i) z^0 solves the first-order linear approximation (13) to (15) at z^0 ; and (ii) the inequality constraints of (13) can be incorporated within the objective function by an appropriate choice of weights μ , so that the optimal value to the problem remains unaltered. That is, Kuhn–Tucker points are associated with (i) a *regularity condition* guaranteeing that a linear approximation inherits certain solutions from a nonlinear program, as well as (ii) a *duality condition* guaranteeing dualization of the linear approximation problem.

When $k = 1$, the vector maximization problem becomes a nonlinear program, and condition (14), with the positive scalar λ normalized to value 1, reduces to the usual Kuhn–Tucker conditions. Consequently, the regularity and duality conditions for characterizing Kuhn–Tucker points subsume all of the numerous constraint qualification conditions of nonlinear programming (see, for example, Mangasarian, Ref. 42). Fiacco and McCormick (Ref. 43) seem to have first stated this fact when $C = \mathbb{R}^n$ and the duality condition is not required. In a section of an unpublished report, Magnanti (Ref. 44) introduced the duality condition in the context of nonlinear programming. Halkin (Ref. 45) presents related results in the context of nonlinear programming. More recently, Robinson (Ref. 46) derived first-order necessary conditions for a general vector optimization program in infinite dimensions by studying preference orderings in the image of certain multi-valued maps.

When specialized to the vector maximization problem, Corollary 4.1 shows that, if $C = \mathbb{R}^n$ and the regularity condition is fulfilled, then a necessary

condition for z^0 to be strictly admissible [i.e., z^0 solves problem (15) or, equivalently, $(f(z^0), z^0)$ is strictly admissible in X] is that

$$x^0 = (f(z^0), z^0)$$

is admissible in $L(x^0)$. The last condition is equivalent to z^0 being efficient in

$$Z^L \equiv \{z \in \mathbb{R}^{n-k} : \nabla h_*(z^0) \cdot (z - z^0) \geq 0\},$$

with respect to the vector criterion $\nabla f(z^0) \cdot z$. Therefore, efficiency in the linear approximation to the vector optimization problem is necessary for strict admissibility in the problem itself.

Remark 4.1. These results are related to the notion of proper efficiency introduced by Geoffrion (Ref. 12); see also Kuhn and Tucker, Ref. 10. By definition, z^0 is a *proper efficient point* in the vector maximization problem if there is a scalar $M > 0$ with the property that, for every $z \in Z$ and each index i satisfying

$$f_i(z) > f_i(z^0),$$

the inequality

$$[f_i(z) - f_i(z^0)] / [f_j(z^0) - f_j(z)] \leq M$$

is valid for some index j such that

$$f_j(z) < f_j(z^0).$$

As Geoffrion shows, whenever Z is a convex set and each function $f_j(z)$, for $j = 1, 2, \dots, k$, is concave, proper efficiency of z^0 is equivalent to z^0 solving problem (15) for some $\lambda > 0$. As we have noted, this last condition is equivalent to $(f(z^0), z^0)$ being strictly admissible in X . If $C = \mathbb{R}^n$, then we may restate our comment made just prior to this remark as follows: if z^0 is strictly admissible in the vector maximization problem and the regularity condition is fulfilled, then

$$\begin{aligned} z^0 \in Z \text{ is a proper efficient point in } Z^L, \text{ with} \\ \text{respect to the vector criterion } \nabla f(z^0) \cdot z. \end{aligned} \tag{16}$$

We might also note that, if $f(\cdot)$ and $h(\cdot)$ are concave, then condition (16) implies that z^0 is a proper efficient point in the vector maximization problem. To establish this fact, we note that condition (16) implies that z^0 solves problem (13) for some positive k -vector λ . Therefore, z^0 satisfies the Kuhn-Tucker conditions (14) with $C \in \mathbb{R}^n$ and, because of our concavity hypothesis, z^0 solves the Lagrangian maximization

$$\max_{z \in \mathbb{R}^{n-k}} \{\lambda f(z) + \mu h_*(z)\}.$$

Since, by definition,

$$\mu h_*(z^0) = 0,$$

the optimal value to this problem equals $\lambda f(z^0)$; and, since $z^0 \in Z$, it must solve¹⁰ the optimization problem

$$\max\{\lambda f(z) : h(z) \geq 0\};$$

hence, it must be a proper efficient point in the vector maximization problem.

We should point out, though, that a proper efficient point z^0 need not satisfy condition (16). As an example, let

$$z^0 = (0, 0)$$

in the vector maximization problem with criterion

$$f_1(z) = z_1, \quad f_2(z) = z_2,$$

and constraints

$$h_1(z) = z_2 - z_1^2 \geq 0, \quad h_2(z) = -z_2 - z_1^2 \geq 0.$$

In this instance, the regularity condition fails, since the origin is not admissible in

$$Z^L = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = 0\}.$$

In a recent paper, Borwein (Ref. 47) presents, for more general topological spaces, several results intimately related to our local characterizations of efficient points. He defines a point $z^0 \in Z$ to be *proper efficient* with respect to a closed convex cone S , if z^0 is efficient and the tangent cone to the set

$$\{y : y = f(z), z \in Z\} - S$$

at $y^0 = f(z^0)$ intersects S only at the origin. In the absence of convexity, strict efficiency implies this notion of proper efficiency (Ref. 47, Theorem 1), while under convexity assumptions these definitions are equivalent (Ref. 47, Theorem 2). This last result shows that, with convexity, equating $L(x^0) - x^0$ with the tangent cone fulfills the support condition guaranteed by Proposition 4.1.

5. Discussion

In the previous sections, we have studied structural properties of admissible points with respect to a convex cone. Our results provide global

¹⁰ Here, we use the standard weak duality arguments of nonlinear programming.

characterizations of admissible points in terms of strictly admissible points and local characterizations in terms of linear approximations. We have also shown, with appropriate hypotheses imposed upon the problem structure, that the sets of admissible and strictly admissible points are both connected. In this section, we discuss briefly a few potential extensions and applications.

First, we might comment on the frequently evoked assumption that the underlying preference cone is strictly supported. According to Proposition 6.2, this assumption rules out *grass is greener* preferences, in which each of two alternatives is preferred to the other. More generally, it does not permit situations in which $x \succ y$ and $y^j \succ x$, for $j = 1, 2, \dots$, for some points y^j converging to y . As an example, lexicographic orderings define preference cones that are not strictly supported.

Whenever underlying preferences are described by a closed convex indifference cone \mathbb{P}^I , i.e., $y \in x + \mathbb{P}^I$ iff $y \not\prec x$, the set

$$\mathbb{P} = \{x \in \mathbb{P}^I : 0 \not\geq x\}$$

describes a strictly supported preference cone. This cone is strictly supported; for, if $p \in \mathbb{P}$ belongs to the lineality space of $\text{cl } \mathbb{P}$, i.e.,

$$-p \in \text{cl } \mathbb{P} \in \text{cl } \mathbb{P}^I = \mathbb{P}^I,$$

then

$$0 = p - p \in p + \mathbb{P}^I \quad \text{or} \quad 0 \geq p,$$

a contradiction. We should emphasize, however, that, even though this construction provides strictly supported cones, our development does not presume the existence of any *weak* preference relation \geq .

There are several ways in which our results might be extended. Replacing the preference cone \mathbb{P} by a convex set C or, more generally, by a family of convex sets C_x , C_x denoting the set of points preferred to x , would add possibilities for broader applications. Another line of investigation would be to retain our hypothesis and to see what additional assumptions might lead to stronger conclusions. For example, Arrow and Hahn (Ref. 4) show that, if

$$\mathbb{P} = \{x \in \mathbb{R}^n : x = 0 \text{ or all } x_i > 0\},$$

then the following restrictions on the convex set of alternatives X :

- (i) 0 belongs to the interior of X ,
- (ii) $X \cap \mathbb{R}_+^n$ is compact,
- (iii) free disposal, i.e., $x - y \in X$ for any $y \in \mathbb{R}_+^n$ whenever $x \in X$,

imply that $A(X) \cap \mathbb{R}_+^n$ is homeomorphic to an $(n - 1)$ -dimensional simplex. This substantial strengthening of connectedness is possible, with similar hypothesis, for other convex cones as well. In general, it may be that the set $A(X)$ is homeomorphic to a union of simplices with some special structure.

In a paper not yet available to us, Naccache (Ref. 48) has initiated investigations of another nature. He studies stability of the set of admissible points with respect to perturbations in X and \mathbb{P} . He has also studied connectedness of $A(X)$, but with assumptions that may be related to free disposal.

There are a number of ways in which the structural properties discussed in this paper and these extensions aid decision making. Consider, for example, the vector optimization problem. In practice, it is convenient to generate strictly admissible points by solving

$$\max \left\{ \sum_{j=1}^k \lambda_j f_j(z) : z \in Z \right\}, \tag{17}$$

with positive weights λ_j associated with the criterion functions. By varying the weights, decision makers can generate all strictly admissible points; or, by choosing a sequence of positive weights appropriately, they might move toward some admissible point that is *best* with respect to some auxiliary criterion (see, for example, Ref. 27). Considering, as before, the vector maximization problem in terms of the set

$$X = \{(y, z) \in \mathbb{P}^n : y \leq f(z) \text{ and } z \in Z\}$$

and the preference cone $\mathbb{P} = \mathbb{P}_k$, we see from the representation theorem that every admissible point $(y^0 = f(z^0), z^0)$ is a limit point of solutions $(y^* = f(z^*), z^*)$ to (17) or a translation of such limit points by vectors $(0, z) \in c1 \mathbb{P}_k \setminus \mathbb{P}_k$. That is,¹¹ *in the space \mathbb{R}^k , the image $f(\cdot)$ of efficient points is contained in the closure of the image of the proper efficient points* (see Geoffrion, Ref. 12). In this context, the representation Theorem 3.1 shows that the solutions to (17) delineate all potential values of the criterion function when evaluated at efficient points; connectedness of the admissible points shows that, in order to move from any (proper) efficient point to another, we can restrict ourselves to local movements among (proper) efficient points only, such as local changes in the coefficients λ_j of (17).

We would expect similar benefits from the structural properties of admissible points in general, especially when solving for strictly admissible points is attractive computationally.

One application of admissibility that might be explored profitably concerns the optimization of monotonic functions $h(x)$, where, say, $h(x)$ is strictly increasing in each component x_j of the vector x . In this instance, any

¹¹ Some assumption, such as the hypothesis $c1 \mathbb{P} \cap RC(X) = \{0\}$ of Theorem 3.1, is required for this statement. For example, let V and Y be defined as in Example 3.4; let $Z = V + Y$; and let $f_1(z) = z_1$ and $f_2(z) = z_2$. The efficient set is the half line from V passing through the point K (see Fig. 1). Every efficient point is nonproper, though, so that the statement is not valid in this instance.

optimal solution to the problem

$$\max\{h(x): x \in X\}$$

is admissible for X with respect to $\mathbb{P} = \mathbb{R}_+^n$. This observation suggests that optimization algorithms might restrict their search to the admissible points $A(X)$, especially once any algorithm first identifies a point in this set.

Do mathematical programming algorithms have this property? The answer, at least in terms of the simplex method is negative. In the following example:

$$\begin{aligned} \max \quad & 2x_1 + x_2, \\ \text{subject to} \quad & 4x_1 + 7x_2 \geq 18, \\ & x_1 + 2x_2 \leq 5, \\ & x_1 \geq 0, \quad x_2 \geq 0, \end{aligned}$$

the admissible set is the line segment joining the points $a = (1, 2)$ and the optimal solution $c = (5, 0)$. Starting from the extreme point a , the simplex method moves off the admissible set to the point $b = (4.5, 0)$. In a number of experiments conducted on larger problems (Ref. 36), we have never observed this same phenomenon. In these examples, once the simplex method first encountered an admissible point, it always generated an admissible point at each successive iteration. Our understanding of the simplex method might be enhanced if we could explain this behavior.

6. Appendix A: Characterization of Strictly Supported Cones

The following rather intuitive propositions characterize the strictly supported condition for a convex cone. Recall that the lineality space L of any cone C is the set of lines contained in C , i.e., $L = C \cap (-C)$.

Proposition 6.1. Let L be the lineality space for the closure of the convex cone \mathbb{P} . Then, \mathbb{P} is supported strictly iff $\mathbb{P} \cap L = \{0\}$.

Proof. Let L^\perp denote the orthogonally complementary subspace to L . Then,

$$c1 \mathbb{P} = L \oplus (c1 \mathbb{P} \cap L^\perp)$$

is a direct sum representation, and

$$(c1 \mathbb{P})^+ = L^\perp \cap (c1 \mathbb{P} \cap L^\perp)^+.$$

Now, $p_s^+ \in L^\perp$ whenever

$$p_s^+ \in \mathbb{P}_s^+ \subseteq (c1 \mathbb{P})^+$$

and

$$p_s^+ \cdot p = 0, \quad \text{for all } p \in \mathbb{P} \cap L.$$

Consequently,

$$\mathbb{P} \cap L = \{0\}.$$

To establish the converse, note that, since $c1 \mathbb{P} \cap L^\perp$ is convex, closed, and pointed, its positive polar $(c1 \mathbb{P} \cap L^\perp)^+$ has full dimension (Ref. 38, Corollary 14.6.1). Any point y belonging to the interior of $(c1 \mathbb{P} \cap L^\perp)^+$ must satisfy $y \cdot p > 0$ for all nonzero $p \in (c1 \mathbb{P} \cap L^\perp)$. Expressing y as

$$y = y_L + y^\perp,$$

with $y_L \in L$ and $y^\perp \in L^\perp$, we note that

$$(y_L + y^\perp) \cdot p = y^\perp \cdot p > 0, \quad \text{for all nonzero } p \in c1 \mathbb{P} \cap L^\perp.$$

If $p \in \mathbb{P}$ and $\mathbb{P} \cap L = \{0\}$, then

$$p = p_L + p^\perp$$

for some $p_L \in L$ and some nonzero $p^\perp \in c1 \mathbb{P} \cap L^\perp$, and

$$y^\perp \cdot p = y^\perp \cdot p^\perp > 0.$$

Therefore, $\mathbb{P} \cap L = \{0\}$ implies that $y^\perp \cdot p > 0$ for all $p \in \mathbb{P}$; that is, \mathbb{P} is supported strictly. \square

The proof of this proposition shows that, whenever $\mathbb{P} \cap L = \{0\}$, any point contained in both L^\perp and the interior of $(c1 \mathbb{P} \cap L^\perp)^+$ is a strict support to \mathbb{P} . We next establish the converse to this statement.

Proposition 6.2. Let L be the lineality space for the closure of the convex cone \mathbb{P} . Then,

$$ri(\mathbb{P}^+) \subseteq \mathbb{P}_s^+$$

iff $L \cap \mathbb{P} = \{0\}$. Moreover, if $c1 \mathbb{P} \cap L^\perp = \mathbb{P} \cap L^\perp$, then

$$\mathbb{P}_s^+ \subseteq ri(\mathbb{P}^+);$$

consequently,

$$ri(\mathbb{P}^+) = \mathbb{P}_s^+$$

iff $L \cap \mathbb{P} = \{0\}$.

Proof. Let

$$c1 \mathbb{P} = L \oplus (c1 \mathbb{P} \cap L^\perp)$$

be a direct sum representation. Then,

$$\mathbb{P}^+ = (c1 \mathbb{P})^+ = L^\perp \cap (c1 \mathbb{P} \cap L^\perp)^+;$$

and, since L^\perp is a subspace and $(c1 \mathbb{P} \cap L^\perp)^+$ has a nonempty interior I which intersects L^\perp ,

$$ri(\mathbb{P}^+) = L^\perp \cap I.$$

By the remark preceding the proposition,

$$\mathbb{P} \cap L = \{0\}$$

implies that

$$ri(\mathbb{P}^+) \subseteq \mathbb{P}_s^+.$$

Conversely, if

$$ri(\mathbb{P}^+) \subseteq \mathbb{P}_s^+,$$

then

$$\mathbb{P}_s^+ \neq \emptyset \quad \text{and} \quad L \cap \mathbb{P} = \{0\}$$

by the previous proposition.

Finally, suppose that

$$c1 \mathbb{P} \cap L^\perp = \mathbb{P} \cap L^\perp,$$

and let y belong to the relative boundary of \mathbb{P}^+ . Then, there are vectors $y^j \in L^\perp$ converging to y satisfying $y^j \cdot p^j < 0$ for some $p^j \in \mathbb{P} \cap L^\perp$, which we scale to unit norm. Any limit point p of the sequence $\{p^j\}_{j \geq 1}$ belongs to

$$c1 \mathbb{P} \cap L^\perp = \mathbb{P} \cap L^\perp \subseteq \mathbb{P}$$

and satisfies $y \cdot p \leq 0$. Consequently,

$$y \notin \mathbb{P}_s^+ \quad \text{and} \quad \mathbb{P}_s^+ \subseteq ri(\mathbb{P}^+). \quad \square$$

When combined, these propositions establish Proposition 2.1 of the text. Note that the example following Proposition 2.1 shows that

$$ri(\mathbb{P}^+) \neq \mathbb{P}_s^+$$

is possible.

7. Appendix B: Proof of Lemma 3.3

We prove the continuity lemma required for Theorem 3.3.

Proof of Lemma 3.3. Let y be any recession direction of X . Since $X(0)$ is nonempty and $X(1)$ is nonempty and compact,

$$p(0) \cdot y \leq 0 \quad \text{and} \quad p(1) \cdot y < 0.$$

Therefore,

$$[\theta p(1) + (1 - \theta)p(0)] \cdot y < 0,$$

for all $0 < \theta \leq 1$, implying that $X(\theta)$ is nonempty and compact.

According to Lemma 3.2, the mapping $\theta \rightarrow X(\theta)$ is upper semicontinuous on the interval $[\epsilon, 1]$ for any $0 < \epsilon \leq 1$. Consequently, to complete the proof, we must show that this mapping is upper semicontinuous at $\theta = 0$.

This indeed is the case if, for some $\delta > 0$, the set

$$S_\delta \equiv \cup \{X(\theta) : 0 < \theta \leq \delta\}$$

is bounded. For, if the mapping $\theta \rightarrow X(\theta)$ is not upper semicontinuous at $\theta = 0$, then there is an open set G containing $X(0)$ and points $x^j \in X(\theta^j) \setminus G$ for some real numbers $\theta^j > 0$ approaching 0. Since

$$p(\theta^j) \cdot x^j \geq p(\theta^j) \cdot x, \quad \text{for all } x \in X,$$

any limit point x^* (such a limit point exists since the x^j eventually lie in the bounded set S_δ) to the sequence $\{x^j\}_{j \geq 1}$ satisfies $x^* \notin G$ and

$$p(0) \cdot x^* \geq p(0) \cdot x, \quad \text{for all } x \in X.$$

But then

$$x^* \in X(0) \setminus G,$$

contradicting

$$X(0) \subseteq G.$$

Therefore, to establish the theorem, we only need to show that S_δ is bounded for some $\delta > 0$. For notational simplicity, suppose (by translation if necessary) that $0 \in X(0)$. Then, by definition,

$$p(0) \cdot x(\theta) \leq p(0) \cdot x(0) = 0, \tag{18}$$

for any $x(\theta) \in X(\theta)$, $0 < \theta \leq 1$. Since $0 \in X$,

$$[\theta p(1) + (1 - \theta)p(0)] \cdot x(\theta) \geq 0,$$

implying from (18) that

$$p(1) \cdot x(\theta) \geq 0. \quad (19)$$

Now, if S_δ is unbounded for every $0 < \delta \leq 1$, then there are $\theta^j \rightarrow 0$ and points $x(\theta^j) \in X(\theta^j)$ whose Euclidean norms λ_j approach $+\infty$. Since $0 \in X$, any limit point y to the sequence $\{x(\theta^j)\}_{\lambda_j \geq 1}$ is a direction of recession of X . But Ineq. (19) implies that

$$p(1) \cdot y \geq 0,$$

contradicting the hypothesis that $X(1)$ is bounded. Consequently, S_δ must be bounded for some $\delta > 0$, and the proof is complete. \square

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