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About Differentiability of Order One of Quasiconvex Functions on R^n

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Abstract. This paper is devoted to the study of the different kinds of differentiability of quasiconvex functions on R^n. For these functions, we show that Gâteaux-differentiability and Fréchet-differentiability are equivalent; we study the properties of the directional derivatives; and we show that if, for a quasiconvex function, the directional derivatives at x are all finite and two-sided, the function is differentiable at x.

Key Words. Quasiconvex functions, Fréchet-differentiability, Gâteaux-differentiability, quasidifferentiability, generalized convexity.

1. Introduction

Let f be any function from $\mathbb{R}^n$ to $[-\infty, +\infty]$, and let x be a point where f is finite. The (one-sided) directional derivative of f at x with respect to a vector h is defined as the limit

$$f'(x, h) = \lim_{\lambda \to 0_+} (f(x + \lambda h) - f(x)]/\lambda.$$

If the directional derivative $f'(x, h)$ exists for all h of $\mathbb{R}^n$ and verifies

$$-\infty < -f'(x, -h) = f'(x, h) < \infty, \qquad \text{for each } h \in \mathbb{R}^n,$$

then f is said to be weakly Gâteaux-differentiable at x.

If the directional derivative $f'(x, h)$ exist for each h of $\mathbb{R}^n$, and if the function $f'(x, \cdot)$ is linear, i.e., if there exists a vector c such that

$$f'(x, h) = \langle c, h \rangle, \qquad \text{for each } h \in \mathbb{R}^n,$$

then the function f is said to be Gâteaux-differentiable at x; the unique vector c for which the above relation holds is denoted by $f'(x)$ and is called the Gâteaux-derivative of f at x.

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If there exists a vector $d \in \mathbb{R}^n$ such that

$$\lim_{h \to 0}(1/\|h\|)[f(x+h)-f(x)-\langle d, h\rangle]=0,$$

then f is said to be (Fréchet)-differentiable at x. One easily sees that f is also Gâteaux-differentiable at x and that

$$d=f'(x).$$

All norms being equivalent on $\mathbb{R}^n$, we shall choose the norm which will suit us best.

In Section 2 of the present paper, we show that both concepts, Gâteaux-differentiability and Fréchet-differentiability, coincide for quasiconvex functions on $\mathbb{R}^n$. Section 3 is devoted to the study of the function $f'(x, \cdot)$, f being quasiconvex, when it exists. We show that, if $f'(x, h)$ exists and is finite for each h, the function $f'(x, \cdot)$ is a positively homogeneous quasiconvex function and, under a continuity condition, can be expressed as the minimum of two support functions. We show also that, if f is weakly Gâteaux-differentiable at x, it is also Gâteaux-differentiable, and thus Fréchet-differentiable as well.

Throughout this paper, we shall use the following notations.

Let A be a set of $\mathbb{R}^n$. We denote by A^c the complement of A, by $\bar{A}$ the closure of A, by δ_A and δ_A^* the indicator function and the support function of A, respectively, i.e.,

$$\delta_A(x)=\begin{cases}0, & \text{if } x \in A,\\ +\infty, & \text{if } x \notin A,\end{cases}$$

$$\delta_A^*(y)=\sup[\langle x, y\rangle | x \in A].$$

The relative interior of a convex A, denoted by $\mathrm{ri}(A)$, is the interior of the set A, regarded as a subset of its affine hull. The set difference $\bar{A}\backslash\mathrm{ri}(A)$ is called the relative boundary of $\mathbb{R}^n$.

We set $e_1, e_2, \ldots, e_n$ to be the n coordinate vectors of $\mathbb{R}^n$, and we define the norm $\|x\|_1$ of the vector

$$x=(x_1, x_2, \ldots, x_n)$$

as

$$\|x\|_1=\sum_{i=1}^{n}|x_i|.$$

Let f be a function from R^n to $\mathbb{R} \cup \{+\infty\}$. Its domain is the subset of $\mathbb{R}^n$

$$\mathrm{dom}(f)=\{x \in \mathbb{R}^n \,|\, f(x)<+\infty\}.$$

A function f from $\mathbb{R}^n$ to $\bar{\mathbb{R}}$ is said to be quasiconvex iff

$$f(x+t(y-x)) \leq \max[f(x), f(y)],$$

for every $x, y \in \mathbb{R}^n$, and $t \in [0, 1]$, or equivalently if, for every $\lambda \in \mathbb{R}$, the subset of $\mathbb{R}^n$

$$S_\lambda = \{x \mid f(x) \leq \lambda\}$$

is convex.

2. Gâteaux-Differentiation and Fréchet-Differentiation

It is well known and easy to prove that a convex function on $\mathbb{R}^n$ which is Gâteaux-differentiable at x_0 is also Fréchet-differentiable at x_0. Indeed, if the Gâteaux-derivative f' of a function f on $\mathbb{R}^n$ exists in a neighborhood of x_0 and is continuous at x_0, $f'(x_0)$ is also the Fréchet-derivative of f at x_0. A convex function f which is differentiable at x_0 is not necessarily differentiable in a neighborhood of x_0, but its subdifferential ∂f (an extended concept of the derivative) is a multivalued mapping which is upper semicontinuous at x_0. Quasiconvex functions have no similar properties. Indeed, for these functions, the existence of the first derivative in a neighborhood of a point does not imply the continuity of the derivative in this point. In order to establish our theorem, we shall first prove two lemmas.

Lemma 2.1. Let S be a convex set with a nonempty interior. If 0 belongs to the boundary of S, then there exist x^* and $\lambda < 0$ such that $\delta_S^*(x^*) = 0$ and $\lambda x^* \in \operatorname{int}(S)$.

Proof. Let

$$K = \{x^* \mid \delta_S^*(x^*) = 0\} = \{x^* \mid \delta_S^*(x^*) \leq 0\};$$

K is a nonempty convex cone. If

$$\operatorname{int}(S) \cap [-K] = \varnothing,$$

there exists $x^* \neq 0$ and α such that

$$\sup[\langle x, x^*\rangle \mid x \in S] \leq \alpha \leq \inf[\langle y, x^*\rangle \mid y \in -K].$$

Clearly, $\alpha = 0$. The first inequality implies that $x^* \in K$, but then the second implies that $\alpha < 0$.

Lemma 2.2. Let θ be a convex function on $\mathbb{R}$ with $\theta(0)=0$ and $\theta(x)>0$ if $x \neq 0$, then

$$\lim_{\mu \to 0_+} \sup[|t| \mid \theta(t) - \mu \leq \mu |t|] = 0.$$

Proof. Let

$$h(\mu) = \sup[|t| \mid \theta(t) - \mu \leq \mu |t|];$$

h is nondecreasing on $\mathbb{R}^+$. Let

$$\bar{t} = \lim_{\mu \to 0_+} h(\mu).$$

Hence,

$$\min[\theta(\bar{t}/2), \theta(-\bar{t}/2)] \leq \mu \bar{t}/2 + \mu, \qquad \forall \mu > 0,$$

implies that $\bar{t} = 0$.

Theorem 2.1. If a quasiconvex function on $\mathbb{R}^n$ is Gâteaux-differentiable at x_0, it is also Fréchet-differentiable at x_0.

Proof. Without loss of generality, we assume that $x_0 = 0$ and $f(x_0) = 0$. Denote by $f'(0)$ the G-derivative of f at 0: We shall distinguish two cases, depending on whether $f'(0)$ is zero or different from zero.

Case C1. where $f'(0) = 0$. Let

$$h_i = e_i \quad \text{and} \quad h_{n+i} = -e_i, \qquad \text{for } i = 1, \ldots, n.$$

For each vector x of $\mathbb{R}^n$, $x \neq 0$, it follows that

$$x = \|x\|_1 \sum_{i=1}^{2n} t_i h_i,$$

where, for $i = 1, 2, \ldots, n$,

$$t_i = x_i/\|x\|_1 \quad \text{and} \quad t_{n+i} = 0, \qquad \text{if } x_i \geq 0,$$

$$t_i = 0 \quad \text{and} \quad t_{n+i} = -x_i/\|x\|_1, \qquad \text{if } x_i < 0.$$

Clearly, x is a convex combination of the $2n$ vectors $\|x\|_1 h_i$, and the quasiconvexity of f implies that

$$f(x) \leq \max_{i=1,\ldots,2n} [f(\|x\|_1 h_i)].$$

Hence,

$$\limsup_{x \to 0} \frac{f(x)}{\|x\|_1} \leq \max_{i=1,\ldots,2n} \left[\lim_{x \to 0} \frac{f(\|x\|_1 h_i)}{\|x\|_1} \right] = 0.$$

It follows that, if f is nonFréchet-differentiable at 0, there exists $\mu < 0$ and a sequence $\{x^l\}$ in $\mathbb{R}^n$ converging to 0, such that

$$f(x^l) < \mu\|x^l\|, \; l \in \mathbb{N}, \qquad \|x^1\| > \|x^2\| > \cdots > \|x^l\| > \cdots > 0.$$

Let

$$S = \{x \mid f(x) < 0\}.$$

We can assume, without loss of generality, that the interior of S is nonempty. Since 0 belongs to the boundary of S, according to Lemma 2.1, there exist x^* and $\alpha < 0$ such that

$$\|x^*\| = 1, \qquad \delta_S^*(x^*) = 0, \qquad \alpha x^* \in \operatorname{int}(S).$$

Without loss of generality, we assume that $x^* = e_1$. Therefore,

(i) $f(x) \geq 0$, if $x_1 > 0$;
(ii) there exists $\beta > 0$ such that

$$m = \max[f(\alpha e_1 + (-1)^i \beta e_j) \mid j = 2, \ldots, n, \; i = 1, 2] < 0.$$

Now, the function

$$\theta(t) = f(te_1)$$

is quasiconvex and, for some $t_0 \in]\alpha, 0[$, it is nondecreasing on $[t_0, 0]$. Furthermore, there exists $l_0 \in \mathbb{N}$ such that

$$\mu\|x^{l_0}\| > \max[m, \theta(t_0)].$$

For each $l > l_0$ there exists $t^l \in]t_0, 0[$ such that

$$\theta(t) \leq \mu\|x^l\| < \theta(t') < 0, \qquad \text{if } t_0 < t < t^l < t' < 0. \tag{1}$$

Let

$$S_1 = \{x \mid f(x) \leq \mu\|x^l\|\}.$$

Since $t^l e_1$ belongs to the relative boundary of the convex set S_1, there exists $y^1 \neq 0$ such that

$$\sup[\langle x, y^l \rangle | x \in S_l] \leq t^l \langle e_1, y^l \rangle.$$

It follows that

$$\alpha(y^l)_1 + (-1)^i \beta(y^l)_j \leq t^l (y^l)_1, \qquad j = 2, \ldots, n, \quad i = 1, 2, \tag{2}$$

$$(x^l)_1 (y^l)_1 + \sum_{j=2}^{n} (x^l)_j (y^l)_j \leq t^l (y^l)_1. \tag{3}$$

Inequality (2) implies that $(y^l)_1 > 0$, and so we can take

$$(y^l)_1 = 1.$$

Choosing the norm

$$\|x\| = \max\left[|x_1|, \sum_{j=2}^{n} |x_j|\right],$$

we obtain

$$|(y^l)_j| \leq (t^l - \alpha)/\beta, \tag{4a}$$

$$-\|x^l\|[1 + (t^l - \alpha)/\beta] \leq t^l, \tag{4b}$$

$$\|x^l\| \geq -\beta t^l/(\beta + t^l - \alpha). \tag{4c}$$

Associate with θ the function $\bar{\theta}$ defined by

$$\bar{\theta}(t) = \begin{cases} \theta(t), & \text{if } t \neq t^l,\ l > l_0, \\ \mu\|x^l\|, & \text{if } t = t^l,\ l > l_0. \end{cases}$$

It follows from (1) that

$$\lim_{t\to 0} \bar{\theta}(t)/t = \lim_{t\to 0} \theta(t)/t = \theta'(0) = 0. \tag{5}$$

On the other hand, from (4c), we have

$$\bar{\theta}(t^l)/t^l = \mu\|x^l\|/t^l > -\beta\mu/(\beta + t^l - \alpha).$$

When $l \to +\infty$, $t^l \to 0$. The limit of the last term is then strictly positive, and this contradicts (5).

Case C2, where $f'(0) \neq 0$. Without loss of generality, we assume $f'(0) = e_n$.

(A) First, we shall prove that

$$\limsup_{x\to 0} \frac{f(x) - \langle f'(0), x\rangle}{\|x\|} \leq 0.$$

Let

$$h_i = e_i \quad \text{and} \quad h_{n-1+i} = -e_i, \qquad \text{for } i = 1, 2, \ldots, n-1.$$

For $x \in \mathbb{R}^n$, take

$$\theta(x) = \sum_{i=1}^{n-1} |x_i|, \qquad \|x\| = \max[\theta(x), |x_n|].$$

If $\theta(x) \neq 0$, then let

$$t_i = x_i/\theta(x), \quad t_{n-1+i} = 0, \qquad \text{if } x_i \geq 0,$$

$$t_i = 0, \quad t_{n-1+i} = -x_i/\theta(x), \qquad \text{if } x_i < 0,$$

for $i = 1, 2, \ldots, n-1$; and, if $\theta(x) = 0$, then let

$$t_1 = 1, \quad t_i = 0, \qquad \text{for } i = 2, \ldots, 2n-2.$$

Hence,

$$x=\sum_{i=1}^{2n-2} t_i(\theta(x)h_i+x_ne_n).$$

Since x is a convex combination of the $2n-2$ vectors $\theta(x)h_i+x_ne_n$, the quasiconvexity of f implies that

$$\frac{f(x)-\langle f'(0),x\rangle}{\|x\|}\leqslant \max_{i=1,\dots,2n-2}\left[\frac{f(\theta(x)h_i+x_ne_n)-x_n}{\max[\theta(x),|x_n|]}\right].$$

Hence, it is enough to prove that, if g is a quasiconvex function on $\mathbb{R}^2$, Gâteaux-differentiable at (0, 0), and such that

$$g(0,0)=0,\qquad g'(0,0)=(0,1),$$

then

$$\limsup_{(x,y)\to(0,0)}\frac{g(x,y)-y}{\|(x,y)\|}\leqslant 0. \tag{6}$$

Let

$$S=\{(x,y)\mid g(x,y)\leqslant 0\}.$$

(i) Suppose that there exist $\mu\in]0,1[$ and a sequence $\{x^l,y^l\}$ in S converging to (0, 0) such that

$$g(x^l,y^l)-y^l\geqslant\mu\max[|x^l|,|y^l|],\qquad l=1,2,\dots. \tag{7}$$

Then,

$$-y^l\geqslant\mu|x^l|,\qquad l=1,2,\dots,$$

and the quasiconvexity of g implies that

$$\frac{g(x^l,y^l)-y^l}{\|(x^l,y^l)\|}\leqslant\frac{1}{\|(x^l,y^l)\|}\max[g(-y^l/\mu,y^l)-y^l,g(y^l/\mu,y^l)-y^l].$$

Since

$$g'(0,0)=(0,1),$$

the second term tends to 0 when l tends to $+\infty$, and this contradicts (7). Thus,

$$\limsup_{\substack{(x,y)\to(0,0)\\(x,y)\in S}}\left[\frac{g(x,y)-y}{\|(x,y)\|}\right]\leqslant 0. \tag{8}$$

(ii) Let $\varepsilon\in]0,1[$. Since

$$g'(0,0)=(0,1),$$

there exists $\mu > 0$ such that

$$g(\mu, -\mu) < 0, \qquad g(-\mu, -\mu) < 0,$$
$$g(0, \xi) < \xi(1+\epsilon), \qquad \text{for every } \xi \in [0, 2\mu].$$

Denote by T the convex hull of the three points

$$(\mu, -\mu), \qquad (-\mu, -\mu), \qquad (0, \mu),$$

and consider the function θ,

$$\theta(x^*) = \delta^*_{S\cap T}(x^*, 1).$$

Clearly, θ is convex,

$$\theta(0) = 0, \quad \theta(x^*) > 0, \qquad \text{if } x^* \neq 0.$$

For each $(x, y) \in T$, let

$$k(x, y) = \begin{cases} 0, & \text{if } (x, y) \in \operatorname{int}(S), \\ (1+\epsilon) \sup\limits_{x^*}[xx^* + y \mid \theta(x^*) \leq xx^* + y], & \text{otherwise.} \end{cases}$$

For a geometric interpretation of k, note that, if $(x, y) \in T \cap S^c$, the straight line passing through the points (x, y) and $(0, xx^* + y)$ for the optimal value x^* is tangent at $T \cap S$.

It is easy to see that

$$0 \leq k(x, y) \leq \mu(1+\varepsilon), \qquad \text{for every } (x, y) \in T.$$

Now, let $(x, y) \in T \cap S^c$, and let any λ such that

$$k(x, y) < \lambda(1+\epsilon) < 2\mu(1+\epsilon).$$

The straight line passing through the points (x, y) and $(0, \lambda)$ has a nonempty intersection with $\operatorname{int}(S)$; i.e., there exists $(\bar{x}, \bar{y}) \in \operatorname{int}(S)$ such that (x, y) belongs to the segment joining $(0, \lambda)$ with $(\bar{x}, \bar{y})$. The quasiconvexity of g implies that

$$g(x, y) \leq \max[g(\bar{x}, \bar{y}), g(0, \lambda)] \leq \lambda(1+\epsilon).$$

Since this result is true for all λ such that

$$k(x, y) < \lambda(1+\epsilon),$$

then

$$k(x, y) \geq g(x, y), \qquad \text{on } T \cap S^c.$$

Choosing for norm in $\mathbb{R}^2$

$$\|(x, y)\| = \max[|x|, |y|],$$

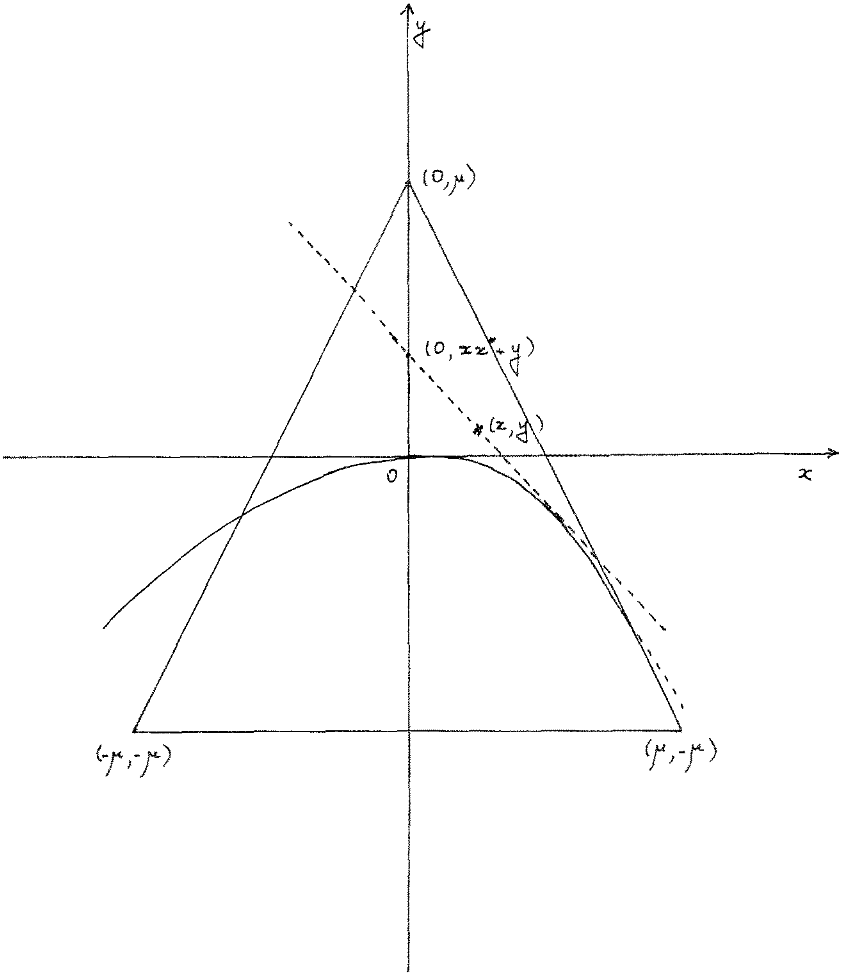

Fig. 1

it follows that

$$A = \limsup_{\substack{(x,y)\to(0,0)\\(x,y)\notin S}} \left[\frac{k(x, y) - y}{\|(x, y)\|}\right] \leqslant \lim_{\mu\to 0_+} (1/\mu) \sup_{\substack{x^*,x,y\\(x,y)\notin S}} [(1+\epsilon)xx^* + \epsilon y \,|\, \theta(x^*) \leqslant xx^*$$

$$+ y, |x| \leqslant \mu, |y| \leqslant \mu],$$

$$A \leqslant \epsilon + \lim_{\mu\to 0_+} \left[\frac{(1+\epsilon)}{\mu} \sup_{x,x^*}\{xx^* \,|\, \theta(x^*) - \mu \leqslant xx^*, |x| \leqslant \mu\}\right],$$

$$A \leqslant \epsilon + (1+\epsilon) \lim_{\mu\to 0_+} \sup[|x^*| \,|\, \theta(x^*) - \mu \leqslant \mu |x^*|].$$

Lemma 2.2 implies that

$$A \leqslant \epsilon.$$

Consequently,

$$\limsup_{\substack{(x,y)\to(0,0)\\(x,y)\notin S}} \left[\frac{g(x,y)-y}{\|(x,y)\|}\right] \leqslant \epsilon, \qquad \text{for every } \epsilon > 0.$$

By combining this result with (8), we see that (6) follows.

(B) Now, we shall prove

$$\liminf_{x\to 0} \frac{f(x) - \langle f'(0), x\rangle}{\|x\|} \geqslant 0. \tag{9}$$

Let

$$v_i = e_i + e_n,\ v_{n-1+i} = -e_i + e_n, \qquad \text{for } i = 1, 2, \ldots, n-1,$$

and let $\epsilon \in]0, \frac{1}{2}[$. Since $f'(0) = e_n$, there exists $\nu_0 > 0$ such that, for each $\nu \in [-\nu_0, \nu_0]$, $\nu \neq 0$, we have

$$-\epsilon < [f(\nu v_j) - \nu]/\nu < \epsilon, \qquad j = 1, 2, \ldots, 2n-2,$$

$$-\epsilon < [f(\nu e_n) - \nu]/\nu < \epsilon.$$

Let

$$\theta(\lambda) = \lambda/(1-\epsilon) \quad \text{and} \quad \theta'(\lambda) = \lambda/(1+\epsilon), \qquad \text{if } \lambda < 0,$$

$$\theta(\lambda) = \lambda/(1+\epsilon) \quad \text{and} \quad \theta'(\lambda) = \lambda/(1-\epsilon), \qquad \text{if } \lambda \geqslant 0,$$

$$\underline{\lambda} = -\nu_0(1+\epsilon), \qquad \bar{\lambda} = \nu_0(1-\epsilon).$$

Let

$$S_\lambda = \{x \mid f(x) \leqslant \lambda\},$$

and let

$$\lambda \in [\underline{\lambda}, \bar{\lambda}], \qquad \lambda \neq 0.$$

Hence,

$$-\nu_0 \leqslant \theta(\lambda) < \theta'(\lambda) \leqslant \nu_0;$$

and, since $\theta'(\lambda)e_n \notin S_\lambda$, there exists $x^*(\lambda) \neq 0$ such that

$$\sup[\langle x, x^*(\lambda)\rangle \mid x \in S_\lambda] \leqslant \langle \theta'(\lambda)e_n, x^*(\lambda)\rangle = \theta'(\lambda)x_n^*(\lambda).$$

Furthermore,

$$\theta(\lambda)v_j \in S_\lambda, \qquad \text{for } j = 1, 2, \ldots, 2n-2,$$

and so

$$(-1)^k\theta(\lambda)x_j^*(\lambda)+\theta(\lambda)x_n^*(\lambda)\leq\theta'(\lambda)x_n^*(\lambda), \qquad j=1,\ldots,n-1,\ k=1,2.$$

Clearly, $x_n^*(\lambda)>0$, and so we can assume that $x_n^*(\lambda)=1$. It follows that:

(i) if $\lambda\in[\underline{\lambda},0[$,

$$|x_j^*(\lambda)|\leq 2\epsilon/(1+\epsilon)\leq 2\epsilon/(1-\epsilon), \qquad \text{for } j=1,2,\ldots,n-1,$$

and so, for all $x\in S_\lambda$,

$$-[2\epsilon/(1-\epsilon)]\sum_{j=1}^{n-1}|x_j|+x_n\leq\lambda/(1+\epsilon);$$

then, define

$$T_\lambda=\{x\mid-[2\epsilon/(1-\epsilon)]\sum|x_j|+x_n\leq\lambda/(1+\epsilon)\};$$

(ii) if $\lambda\in]0,\bar{\lambda}]$,

$$|x_j^*(\lambda)|\leq 2\epsilon/(1-\epsilon), \qquad \text{for } j=1,2,\ldots,n-1,$$

and so, for all $x\in S_\lambda$,

$$-[2\epsilon/(1-\epsilon)]\sum_{j=1}^{n-1}|x_j|+x_n\leq\lambda/(1-\epsilon);$$

then, define

$$T_\lambda=\{x\mid-[2\epsilon/(1-\epsilon)]\sum|x_j|+x_n\leq\lambda/(1-\epsilon)\}.$$

Let also

$$T_\lambda=T_{\underline{\lambda}}, \qquad \text{for } \lambda<\underline{\lambda},$$

$$T_\lambda=\mathbb{R}^n, \qquad \text{for } \lambda>\bar{\lambda},$$

and define a function g from the sets T_λ by

$$g(x)=\inf[\lambda\mid x\in T_\lambda].$$

Since $S_\lambda\subset T_\lambda$ for each λ, then $g\leq f$. Computing g, we obtain, in a neighborhood of 0,

$$g(x)=\begin{cases}(1+\epsilon)x_n-2\epsilon[(1+\epsilon)/(1-\epsilon)]\sum_{j=1}^{n-1}|x_j|, & \text{if } (1-\epsilon)x_n-2\epsilon\sum_{j=1}^{n-1}|x_j|\leq 0,\\ (1-\epsilon)x_n-2\epsilon\sum_{j=1}^{n-1}|x_j|, & \text{otherwise.}\end{cases}$$

Choose, for norm in $\mathbb{R}^n$,

$$\|x\|=\max\left[|x_n|,\sum_{j=1}^{n-1}|x_j|\right].$$

Then,

$$g(x)-x_n \geq \begin{cases} -\epsilon\|x\|[1+2(1+\epsilon)/(1-\epsilon)], & \text{if } (1-\epsilon)x_n - 2\epsilon \sum_{j=1}^{n-1} |x_j| \leq 0, \\ -3\epsilon\|x\|, & \text{otherwise.} \end{cases}$$

Recall that

$$0<\epsilon<\tfrac{1}{2}.$$

Then,

$$1+2(1+\epsilon)/(1-\epsilon)\leq 7,$$

and thus

$$\liminf_{x\to 0} \frac{f(x)-x_n}{\|x\|} \geq \lim_{x\to 0} \frac{g(x)-x_n}{\|x\|} \geq -7\epsilon.$$

This implies (9) and completes the proof.

3. Directional Derivatives and Positively Homogeneous Quasiconvex Functions

If f is a convex function and if x_0 belongs to int(dom f), the function $f'(x_0, \cdot)$ is a positively homogeneous convex function from $\mathbb{R}^n$ to $\mathbb{R}$ and can be related to the subdifferential of f at x_0. If f is a quasiconvex function, then x_0 can belong to int(dom f) and $f'(x_0, \cdot)$ may not exist; however, if we assume the existence of $f'(x_0, h)$ for all $h \in \mathbb{R}^n$, then $f'(x_0, \cdot)$ has some interesting properties.

Theorem 3.1. Let f be a quasiconvex function on $\mathbb{R}^n$, and assume that $f'(x_0, h)$ exists for every $h \in \mathbb{R}^n$. Then, the function $f'(x_0, \cdot)$ is a positively homogeneous quasiconvex function.

Proof. Let $h_1, h_2 \in \mathbb{R}^n$, and let $\lambda \in]0, 1[$. Since f is quasiconvex,

$$\frac{f[x_0+t(\lambda h_1+(1-\lambda)h_2)]-f(x_0)}{t} \leq \max\left[\frac{f(x_0+th_1)-f(x_0)}{t}, \frac{f(x_0+th_2)-f(x_0)}{t}\right],$$

for every $t>0$. If we take the limit when t tends to 0, then

$$f'(x_0, \lambda h_1+(1-\lambda)h_2)\leq \max[f'(x_0, h_1), f'(x_0, h_2)].$$

Positively homogeneous quasiconvex functions which are either positive or negative on their domain are well known; see, for instance, Refs. 1 and 2. Recall the following result.

Proposition 3.1. Let g be a positively homogeneous quasiconvex function from $\mathbb{R}^n$ to $R \cup \{+\infty\}$. Denote by D the domain of g, and assume D to be nonempty. If g is lower semicontinuous at every point of D, and if one of the two following conditions is true:

$$(C_1)\ g(x) \geq 0, \qquad \forall x \in \mathbb{R}^n,$$

$$(C_2)\ g(x) < 0, \qquad \forall x \in \mathrm{ri}(D),$$

then g is convex.

Let θ be a lower semicontinuous positively homogeneous quasiconvex function from $\mathbb{R}^n$ to $\mathbb{R}$, and let

$$D = \{x/\theta(x) < 0\}.$$

D is a convex cone, and $\theta(x) \leq 0$ on the relative boundary of D. Define the functions θ^+ and θ^- as follows:

$$\theta^-(x) = \begin{cases} \theta(x), & \text{if } x \in \bar{D}, \\ +\infty, & \text{if } x \notin \bar{D}, \end{cases}$$

$$\theta^+(x) = \begin{cases} 0, & \text{if } x \in \bar{D}, \\ \theta(x), & \text{if } x \notin \bar{D}. \end{cases}$$

Clearly,

$$\theta = \min[\theta^-, \theta^+].$$

The functions θ^+ and θ^- are positively homogeneous quasiconvex functions and verify the assumptions of the above proposition. Hence, θ^+ and θ^- are convex functions, and θ is the minimum of two l.s.c. convex functions. Furthermore, there exist two closed convex sets C^+ and C^- of which θ^+ and θ^- are the support functions. Note that C^+ is a compact set, since θ^+ is defined on the whole space $\mathbb{R}^n$ and that, if D is empty, then $C^- = \mathbb{R}^n$. Let

$$C = C^+ \cap C^- \quad \text{and} \quad K = \bigcup_{\lambda \geq 0} \lambda C^-,$$

i.e., K is the convex cone generated by C^-. We shall now study the relationships between C^+, C^-, C, K.

Proposition 3.2. Under the above assumptions and, if D is nonempty, we have

(a) $0 \in C^+, 0 \notin C^-$;

(b) $C^+ = \bigcup_{\lambda \in [0,1]} \lambda C^+$, $C^- = \bigcup_{\lambda \geq 1} \lambda C^-$;

(c) $C^+ \subset \bar{K}$.

Proof. By definition,

$$\delta_{C^+}(x) = \sup_y [\langle x, y\rangle - \theta^+(y)], \qquad \theta^+(y) = \sup_x [\langle y, x\rangle \,|\, x \in C^+],$$

$$\delta_{C^-}(x) = \sup_y [\langle x, y\rangle - \theta^-(y)], \qquad \theta^-(y) = \sup_x [\langle y, x\rangle \,|\, x \in C^-].$$

(a) If $0 \in C^-$, then we should have $\theta^-(y) \geq 0$ for every y; and, if $0 \notin C^+$, there should exist some y such that $\theta^+(y) < 0$.

(b) Let $\lambda \in]0, 1]$. Then,

$$\theta^+(\lambda y) = \lambda\theta^+(y) \leq \theta^+(y),$$

and so

$$\delta_{C^+}(\lambda y) = \sup_x [\langle \lambda y, x\rangle - \theta^+(x)] \leq \sup_x [\lambda\langle y, x\rangle - \lambda\theta^+(x)] = \lambda\delta_{C^+}(y).$$

Now, let $\lambda \in [1, +\infty[$. Then,

$$\theta^-(\lambda y) = \lambda\theta^-(y) \leq \theta^-(y), \qquad \text{for every } y \in \bar{D},$$

and so

$$\delta_{C^-}(\lambda y) \leq \lambda\delta_{C^-}(y).$$

(c) Let $y \in C^+$, $y \neq 0$. We have

$$\delta_{C^+}(y) = \sup_x [\langle x, y\rangle - \theta^+(x)] \geq \sup_{x \in \bar{D}} [\langle x, y\rangle - \theta^+(x)] = \sup_{x \in \bar{D}} [\langle x, y\rangle].$$

Furthermore, δ_K is the positively homogeneous convex function generated by δ_{C^-}. And it follows from a theorem of Rockafellar (Ref. 3, Theorem 13.5) that

$$\sup_{x \in \bar{D}} [\langle x, y\rangle] = \sup_x [\langle x, y\rangle \,|\, \theta^-(x) \leq 0] = \delta_{\bar{K}}(y).$$

Example 3.1. Let $\mu > 0$. Consider the following function from $\mathbb{R}^2$ to $\mathbb{R}$:

$$\theta(x, y) = \begin{cases} -\mu\sqrt{(xy)}, & \text{if } x \geq 0,\ y \geq 0, \\ -x, & \text{if } x \leq 0, \text{ and } y \geq x, \\ -y, & \text{if } y \leq 0, \text{ and } x \geq y. \end{cases}$$

The function θ is continuous, positively homogeneous, and quasiconvex. C^+ and C^- are defined as follows:

$$C^- = \{(x^*, y^*) | x^* \leq 0, y^* \leq 0, x^* + y^* \geq \mu^2/4\},$$

$$C^+ = \{(x^*, y^*) | x^* \leq 0, y^* \leq 0, x^* + y^* \geq -1\}.$$

Note that we have $C^+ \subset \bar{K}$, but not $C^+ \subset K$; and note that, for a convenient value μ, the set $C^+ \cap C^-$ is empty.

Conversely, we have the following proposition.

Proposition 3.3. Let C_1, C_2 be two closed convex sets of $\mathbb{R}^n$ such that:

(a) $0 \in C_1$, $0 \notin C_2$, C_1 is compact;

(b) $C_1 = \bigcup_{\lambda \in [0,1]} \lambda C_1$, $C_2 = \bigcup_{\lambda \geq 1} \lambda C_2$;

(c) C_1 is contained in the closure of the convex cone generated by C_2.

If we define the function θ by

$$\theta(x) = \min[\delta^*_{C_1}(x), \delta^*_{C_2}(x)],$$

then θ is a l.s.c. positively homogeneous quasiconvex function from $\mathbb{R}^n$ to R, and the sets C^+ and C^- defined from θ as above coincide respectively with C_1 and C_2.

Proof. Let

$$\theta_i = \delta^*_{C_i}, \qquad i = 1, 2.$$

The functions θ_1 and θ_2 are positively homogeneous and lower semicontinuous; θ_1 is nonnegative on $\mathbb{R}^n$; and θ_2 is strictly negative on ri(dom θ_2). In order to show that θ is quasiconvex, it is enough to show that

$$\theta_1(x) = 0, \qquad \text{if } x \in \text{dom}(\theta_2);$$

this follows from the already quoted theorem of Rockafellar.

We shall now study conditions on C^+ and C^- which are related to the convexity of the function θ.

Proposition 3.4. Let the function θ and the sets C^+, C^-, C, D, K be defined as above; let D be nonempty. The function θ is convex iff we have

$$C^+ = \bigcup_{\lambda \in [0,1]} \lambda C, \qquad C^- = \bigcup_{\lambda \geq 1} \lambda C. \tag{10}$$

Furthermore, if θ is convex, then $C^+ \subset K$ and $\theta = \delta^*_C$.

Proof. Assume θ to be convex. From the expression of the conjugate function of θ,

$$\theta^*(x^*) = \sup_x[\langle x, x^*\rangle - \min[\delta^*_{C^+}(x), \delta^*_{C^-}(x)]],$$

it follows that

$$\theta^*(x^*) = \max[\delta_{C^+}(x^*), \delta_{C^-}(x^*)] = \delta_{C^+\cap C^-}(x^*).$$

Since the domain of θ is the whole space $\mathbb{R}^n$, θ is continuous and so coincides with its biconjugate function θ^{**}; hence, θ is the support function of C.

Let $y \in C^+$. Then,

$$\delta_{C^+}(y) \leq 0;$$

and, since θ^+ is nonnegative and positively homogeneous, it follows that

$$\sup_x[\langle y, x\rangle \mid \theta(x) \leq 1] \leq 1;$$

and, by convex duality,

$$\min_{\lambda\geq 0}\sup_x[\langle y, x\rangle + \lambda - \lambda\theta(x)] \leq 1,$$

$$\min_{\lambda\geq 0}[\lambda + \sup_x[\langle y, x\rangle - \theta(\lambda x)]] \leq 1,$$

$$\min_{\lambda\geq 0}[\lambda + \theta^*(y/\lambda)] = \min_{\lambda\geq 0}[\lambda + \delta_C[y/\lambda)] \leq 1.$$

Consequently, there exists $\lambda \in [0, 1]$ such that $y \in \lambda C$; on the other hand, $C \subset C^+$.

Let $y \in C^-$. Then,

$$\delta_{C^-}(y) \leq 0.$$

Since θ is convex, then

$$\bar{D} = \{x \mid \theta(x) \leq 0\},$$

and so

$$\sup_x[\langle y, x\rangle - \theta^-(x) \mid \theta^-(x)] \leq 0] = \sup_x[\langle y, x\rangle - \theta(x) \mid \theta(x) \leq 0] \leq 0,$$

$$\min_{\lambda\geq 0}\sup_x[\langle y, x\rangle - (1+\lambda)\theta(x)] \leq 0,$$

$$\min_{\mu\geq 1}[\theta^*(y/\mu)] = \min_{\mu\geq 1}[\delta_C(y/\mu)] \leq 0.$$

So, there exists $\mu \geq 1$ such that $y \in \mu C$, and consequently

$$C^- = \bigcup_{\lambda \geq 1} \lambda C.$$

It follows that $C^+ \subset K$.

Conversely, assume that (10) holds. Define $\hat{\theta}$ to be the support function of the nonempty closed convex set C; $\hat{\theta}$ is obviously a positively homogeneous lower semicontinuous quasiconvex function. Associate with it the convex sets $\hat{C}^+$ and $\hat{C}^-$. These sets coincide, respectively, with C^+ and C^-; consequently, $\hat{\theta}$ coincides with θ, which is then convex.

A quasiconvex positively homogeneous function from $\mathbb{R}^n$ to $\mathbb{R}$ is not necessarily lower semicontinuous. Consider, for example, the function $\theta: \mathbb{R}^2 \to \mathbb{R}$,

$$\theta(x, y) = \begin{cases} -2\sqrt{(xy)}, & \text{if } 0 \leqslant x < y, \\ -y, & \text{if } 0 \leqslant x = y, \\ 0, & \text{otherwise.} \end{cases}$$

θ is not lower semicontinuous or upper semicontinuous at (1, 1). However, the following result holds.

Proposition 3.5. Let θ be a positively homogeneous quasiconvex function from $\mathbb{R}^n$ to $\mathbb{R}$, and let

$$D = \{x \mid \theta(x) < 0\}.$$

Then, θ is continuous on $(D)^c$ and lower semicontinuous on $\mathrm{ri}(D)$; furthermore, if $\mathrm{int}(D)$ is nonempty, then θ is also continuous on $\mathrm{int}(D)$.

Proof. Let ξ be the greatest lower semicontinuous function which is bounded above by θ. ξ is also a positively homogeneous quasiconvex function. Associate with θ and ξ the functions

$$\theta^-(x) = \begin{cases} \theta(x), & \text{if } x \in D, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\theta^+(x) = \begin{cases} 0, & \text{if } x \in D, \\ \theta(x), & \text{otherwise,} \end{cases}$$

$$\xi^-(x) = \begin{cases} \xi(x), & \text{if } x \in \bar{D}, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\xi^+(x) = \begin{cases} 0, & \text{if } x \in \bar{D}, \\ \xi(x), & \text{otherwise.} \end{cases}$$

The functions ξ^- and ξ^+ are the greatest lower semicontinuous functions which are bounded above by θ^- and θ^+, respectively. Hence, the closures of the epigraph of θ^- and θ^+ are convex sets. The restrictions of the functions θ^+ and θ^- to the relative interior of their respective domain are convex, and so lower semicontinuous on the relative interior. Moreover, if this coincides with the interior, the function is also continuous on it.

Proposition 3.6. Let θ be a positively homogeneous quasiconvex function from $\mathbb{R}^n$ to $\mathbb{R}$ satisfying

$$\theta(h)+\theta(-h)=0, \qquad \text{for every } h \in \mathbb{R}^n.$$

Then, θ is linear.

Proof. Excluding the trivial case where θ is the null function on $\mathbb{R}^n$, let

$$D=\{x/\theta(x)<0\}.$$

The function θ is also quasiconcave. Hence, D^c is also a convex cone, and $\bar{D}$ is a half space. The function θ is continuous, convex, and concave such that

$$\theta(x)=\langle x, x^*\rangle, \qquad \text{if } x \in \operatorname{int}(D)\cup(-\operatorname{int}(D)).$$

It easily follows that $\theta(x)=0$ on the boundary of D.

Now, we return to the study of the function $f'(x_0, \cdot)$. Assume this function to be lower semicontinuous. Let

$$\theta=f'(x_0, \cdot),$$

and define $\partial^- f(x_0)$ and $\partial^+ f(x_0)$ as

$$\partial^- f(x_0)=C^-, \qquad \partial^+ f(x_0)=C^+.$$

If f is convex, then the function $f'(x_0, \cdot)$ is known to be the support function of the subgradient of f at x_0; but, if f is quasiconvex, we have just shown that $f'(x_0, \cdot)$ must be expressed as the minimum of two support functions. Moreover, if the function f is convex,

$$\partial f(x_0)=\partial^- f(x_0)\cap\partial^+ f(x_0).$$

If f is a convex function which admits partial derivatives at x_0, i.e.,

$$-\infty\langle f'(x_0, e_i)=-f'(x_0, -e_i)<+\infty, \qquad \text{for } i=1,2,\ldots,n, \tag{11}$$

then f is differentiable at x_0. Indeed, x_0 belongs to the interior of the domain of f, and so $\partial f(x_0)$ exists; it is enough to note that $\partial f(x_0)$ is a singleton.

For quasiconvex functions, things are not the same. First, even if the existence of all partial derivatives at x_0 implies the upper semicontinuity of f at x_0, it does not imply the lower semicontinuity of f at x_0 (for a counterexample, see Ref. 4, p. 15) and, *a fortiori*, the differentiability of f at x_0. Moreover, even if $f'(x_0, h)$ exists and is lower semicontinuous for all h, the differentiability of f at x_0 is not ensured. To see this, consider the sets of $\mathbb{R}^2$

$$C^+ = \{(x, y) \mid 0 \leq x = y \leq 1\},$$

$$C^- = \{(x, y) \mid 1 \leq x \leq y \leq 2x\},$$

and the positively homogeneous quasiconvex function θ defined from C^+ and C^-. The function θ admits directional derivatives and partial derivatives at (0, 0), but is not differentiable at this point.

However, the following proposition establishes that the differentiability at x_0 of the restriction of a quasiconvex function f to each line passing through x_0 implies the differentiability of f at x_0.

Proposition 3.7. Let f be a quasiconvex function which is weakly Gâteaux-differentiable at x. Then, f is Gâteaux-differentiable and Fréchet-differentiable at x.

Proof. Apply Proposition 3.6.

Finally, we conclude this study on differentiability of quasiconvex functions by relating it with the quasidifferentiability developed by Pshenichnyi (Ref. 5) and Borwein (Ref. 6). If $f'(x_0, \cdot)$ is a convex function from $\mathbb{R}^n$ to $\mathbb{R}$ (f is then said to be quasidifferentiable at x_0), then $f'(x_0, \cdot)$ is the support function of the set $\partial^- f(x_0) \cap \partial^+ f(x_0)$, which is the so-called quasigradient of f at x_0.

References

1. NEWMAN, P., *Some Properties of Concave Functions*, Journal of Economic Theory, Vol. 1, pp. 291–314, 1969.
2. CROUZEIX, J. P., *Conditions for Convexity of Convex Functions*, Mathematics of Operations Research, Vol. 5, pp. 120–125, 1980.
3. ROCKAFELLAR, R. T., *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
4. CROUZEIX, J. P., *Contribution à l'Etude des Fonctions Quasiconvexes*, Université de Clermont 2, Thèse de Docteur des Sciences, 1977.
5. PSHENICHNYI, B. N., *Necessary Conditions for an Extremum*, Marcel Dekker, New York, New York, 1971.
6. BORWEIN, J. M., *Fractional Programming without Differentiability*, Mathematical Programming, Vol. 11, pp. 283–290, 1976.