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Cutting Planes for Programs with Disjunctive Constraints[1]

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Abstract. A type of program is considered in which, apart from the usual linear constraints, it is required that at least one variable from each of several sets be equal to zero. Applications include complementary pivot theory and concave minimization problems. Cutting planes are generated for the solution of such programs. A geometrical description of the cutting planes explains their meaning.

1. Introduction

We consider here the nonlinear program which is given by the linear system

$$\text{maximize} \qquad \Sigma c_j x_j , \tag{1}$$

$$\text{subject to} \qquad \Sigma a_{ij} x_j = b_i , \qquad i = 1,\ldots, m, \tag{2}$$

$$x_j \geqslant 0, \tag{3}$$

plus additional constraints the form

$$\begin{array}{c} \prod_{j \in J_1} x_j = 0, \\ \cdots\cdots, \\ \cdots\cdots, \\ \prod_{j \in J_k} x_j = 0, \end{array} \tag{4}$$

etc., where $J_1, J_2, \ldots, J_k$ are subsets of the index set $N = \{1, 2,\ldots, n\}$.

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The constraints (4) are more easily interpreted as disjunctions, i.e., the constraint

$$\prod_{j \in J_1} x_j = 0$$

may be rewritten as follows:

$$\text{there is at least one} \quad j \in J_1 \quad \text{such that} \quad x_j = 0. \tag{5}$$

Many types of programs can be given this form.

Such a problem might occur in the case of a minimization program with concave costs. For example, suppose that the cost associated with a variable x is

$$t(x) = \sqrt{x}.$$

Analytically, the graph $z = \sqrt{x}$, can be defined as the envelope of the tangents which have the equation

$$z = (x + p)/2\sqrt{p},$$

where p is a nonnegative parameter. By concavity, the curve lies below the tangents, and so

$$t(x) = \min_{p \geqslant 0} (x + p)/2\sqrt{p}.$$

Let us, therefore, write

$$y_p = -2\sqrt{p}\, t + x + p,$$

and we know that $y_p \geqslant 0$ for each $p \geqslant 0$, but $y_p = 0$ for at least one value of p. We thus obtain an infinite collection of variables y_p , one of which at least must vanish. In practice, of course, we would choose only a finite number of values for p. This would correspond to a polygonal approximation for the cost function $t = \sqrt{x}$, which can be made as accurate as desired, by choosing sufficiently many values for p.

Additionally, constraints such as these arise in the complementary programming problems, where, from each of several pairs of variables, at least one must vanish. Lemke (Ref. 1) has given algorithms for such programs which, however, do not attempt to maximize a function.

2. Cutting Planes

We will approach this problem by generating cutting planes. In general, let us suppose that we have solved the linear program (1)–(3) by the simplex algorithm. If the solution satisfies the nonlinear constraints (4), then it is of course the solution to the original program.

Suppose, then, the contrary. The linear solution has positive values for the variables $x_j, \ldots, x_k$, one of which (at least) must be zero.

For simplicity of argument, we shall assume that only two variables are concerned: one of x_j and x_k must vanish. The simplex tableau then has the rows

$$\begin{array}{cccc|l} y_1 & y_2 & \cdots \; y_n & 1 & \\ \hline a_{j1} & a_{j2} & \cdots \; a_{jn} & -b_j & = -x_j \\ a_{k1} & a_{k2} & \cdots \; a_{kn} & -b_k & = -x_k, \end{array} \tag{6}$$

where b_j, b_k are both positive.

The two rows will give us the equations

$$(a_{j1}/b_j)\, y_1 + (a_{j2}/b_j)\, y_2 + \cdots + (a_{jn}/b_j)\, y_n = 1 - (x_j/b_j), \tag{7}$$

$$(a_{k1}/b_k)\, y_1 + (a_{k2}/b_k)\, y_2 + \cdots + (a_{kn}/b_k)\, y_n = 1 - (x_k/b_k). \tag{8}$$

The disjunctive constraint means that the left-hand side of one of the equations (7) and (8) must be equal to 1; thus, the right-hand side of one of the equations must equal 1, and so

$$\max\{\Sigma(a_{ji}/b_j)\, y_i\, , \Sigma(a_{ki}/b_k)\, y_i\} = 1. \tag{9}$$

Let us set

$$\alpha_i = \max\{a_{ji}/b_j\, , a_{ki}/b_k\}. \tag{10}$$

The nonnegativity of the y_i then guarantees that

$$\alpha_i y_i = \max\{(a_{ji}/b_j)\, y_i\, , (a_{ki}/b_k)\, y_i\},$$

and so

$$\Sigma \alpha_i y_i \geqslant \{\Sigma(a_{ji}/b_j)\, y_i\, , \Sigma(a_{ki}/b_k)\, y_i\},$$

or

$$\Sigma \alpha_i y_i \geqslant 1.$$

We thus obtain the new constraint

$$-\alpha_1 y_1 - \alpha_2 y_2 - \cdots - \alpha_n y_n + 1 = -v, \tag{11}$$

where v must be nonnegative. Thus, the new constraint is given by the row

$$\begin{array}{cccc|l} y_1 & y_2 & \cdots \; y_n & 1 & \\ \hline -\alpha_1 & -\alpha_2 & \cdots \; -\alpha_n & +1 & = -v, \end{array} \tag{12}$$

where α_1 is given by (10).

In the more general case, where J_k contains more than two indices, we will still have the new constraint given by (12), where

$$\alpha_i = \max_{j \in J_k} \{a_{ij}/b_j\}. \tag{13}$$

In any case, it is clear that the tableau with the new row represents an infeasible point, inasmuch as it gives a negative value for the new variable v.

3. Example

Maximize $2x_1 + 3x_2$, subject to

$$x_1 \leqslant 8,$$

$$x_2 \leqslant 5\ ,$$

$$x_1 + x_2 \leqslant 10,$$

$$x_1, x_2 \geqslant 0,$$

$$x_1 x_2 = 0.$$

Solving the linear program (i.e., without the disjunctive constraint) by the simplex algorithm, we obtain the tableau

u_3	u_2	1	
-2	-1	25	$= \lambda$
-1	1	-3	$= -u_1$
0	1	-5	$= -x_2$
1	-1	-5	$= -x_1$.

Here, both x_1 and x_2 are positive. We therefore generate the new row

$-\frac{1}{5}$	$-\frac{1}{5}$	1	$= v_1$.

We solve the new expanded program, and get

u_1	v_1	1	
$-\frac{1}{2}$	$-\frac{15}{2}$	19	$= \lambda$
$-\frac{1}{2}$	$-\frac{5}{2}$	-1	$= -u_3$
$-\frac{1}{2}$	$\frac{5}{2}$	-1	$= -x_2$
1	0	-8	$= -x_1$
$\frac{1}{2}$	$-\frac{5}{2}$	-4	$= -u_2$.

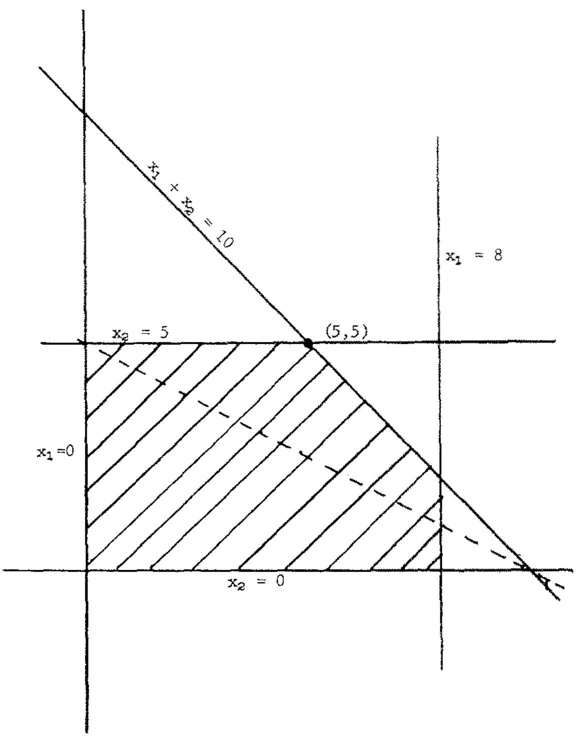

Fig. 1. The original constraint set. The trial solution is at (5, 5). The dotted line represents the constraint $v_1 \geqslant 0$.

We find that x_1 and x_2 are still both positive, and generate the row

$-\frac{1}{8}$	$-\frac{5}{2}$	1

$= -v_2$.

With this new constraint, the program is solved to give the result $x_1 = 8$, $x_2 = 0$, $\lambda = 16$. This satisfies the nonlinear constraint and is therefore the solution of the original program.

It may be of interest to look at the geometry of the situation. In essence, the idea is to extend the edges which begin at the trial solution until they meet one of the hyperplanes $x_j = 0$, $j \in J_k$. In the general n-dimensional case, there are n such edges, which will give us n points. These n points determine a hyperplane, which is the new constraint. For the example above, the linear constraint set is shown in Fig. 1. The trial solution is (5,5), and the two edges starting here meet the x_1- and x_2-axes at (10,0) and at (0,5). The constraint $v_1 \geqslant 0$ is equivalent to $x_1 + 2x_2 \leqslant 10$, which is determined by the line through (0,5) and (10,0). The second trial solution (Fig. 2) is at (8,1), and the two rays

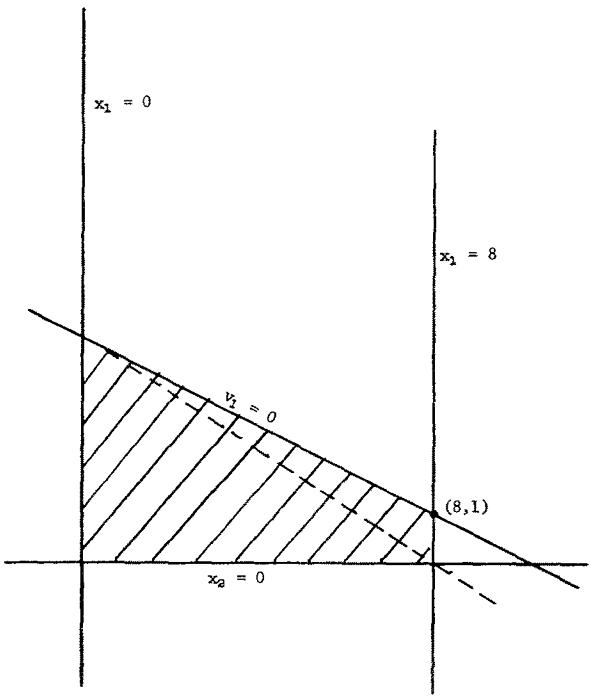

Fig. 2. The constraint set after the addition of the constraint $v_1 \geqslant 0$. The new trial solution is at (8, 1). The dotted line represents the constraint $v_2 \geqslant 0$.

through (8,1) give us the points (8,0) and (0,5). The new constraint $v_2 \geqslant 0$ is equivalent to $5x_1 + 8x_2 \leqslant 40$, which is determined by the points (8,0) and (0,5).

Admittedly, the algorithm given here is quite primitive; no finiteness proof can be given, and it would of course be of interest to test its effectiveness as against, say, an integer program.

Similar results have been published by Glover (Ref. 2), Balas (Ref. 3), and Young (Ref. 4). The discussion in Young (Ref. 5) is to the point and could be repeated here.

Finally, it has been pointed out by a referee that Glover and Klingman in Ref. 6 obtain a more general class of cuts, for which the the cuts considered in this paper are a special example.

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