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TECHNICAL NOTE

An Implicit Function Theorem

K. JITTORNTRUM[1]

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Abstract. Suppose that $F: D \subset R^n \times R^m \to R^n$, with $F(x^0, y^0) = 0$. The classical implicit function theorem requires that F is differentiable with respect to x and moreover that $\partial_1 F(x^0, y^0)$ is nonsingular. We strengthen this theorem by removing the nonsingularity and differentiability requirements and by replacing them with a one-to-one condition on F as a function of x.

Key Words. Implicit function theorem, nonlinear programming.

1. Introduction

In nonlinear programming, several conditions that are imposed on the problem can be identified as the appropriate conditions for some implicit function theorem. The classical theorem for continuously differentiable functions and its proof can be found in most advanced books (for example, Ref. 1). However, if we are only interested in $x(y)$ as a *continuous function* of y such that

$$F(x(y), y) = 0,$$

both the nonsingularity and differentiability requirements are obviously too strong. Therefore, it is of interest to find a theorem with the weakest requirement. The theorem that we will attempt to prove here is such a theorem.

[1] Research Assistant, Computing Research Group, Australian National University, Canberra, Australia.

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2. Results

Theorem 2.1. *Implicit Function Theorem.* Suppose that $F: D \subset R^n \times R^m \to R^n$ is a continuous mapping with

$$F(x^0, y^0) = 0.$$

Assume that there exist open neighborhoods $A \subset R^n$ and $B \subset R^m$ of x^0 and y^0, respectively, such that, for all $y \in B$, $F(\cdot, y): A \subset R^n \to R^n$ is locally one-to-one. Then, there exist open neighborhoods $A_0 \subset A$ and $B_0 \subset B$ of x^0 and y^0, respectively, such that, for all $y \in B_0$, the equation

$$F(x, y) = 0$$

has a unique solution

$$x = Hy \in A_0,$$

and the mapping $H: A_0 \to R^n$ is continuous.

To prove the above theorem, we first prove the following lemma.

Lemma 2.1. *Invariance of Domain.* Let $U \subset R^n$ be open connected; if $F: U \to R^n$ is a one-to-one continuous map, then $F(U)$ is open connected and F is a homeomorphism onto $F(U)$.

Proof. For $n = 1$, the statement that F is one-to-one and continuous on U means that F is strictly monotone on U, and hence the result is obvious. For $n \geq 2$, this is a direct consequence of the Jordan–Brouwer separation theorem [see Corollary (18.9) of Ref. 2 for proof].

We are now ready to prove the main theorem.

Proof of Theorem 2.1. Let us define

$$\Psi: A \times B \subset R^n \times R^m \to R^n \times R^m,$$

$$\Psi: (x, y) \mapsto (F(x, y), y).$$

It is obvious from the definition that Ψ is one-to-one and continuous. Thus, by the above lemma, $\Psi(A \times B)$ is open and there is an inverse

$$\Phi: \Psi(A \times B) \to A \times B,$$

$$\Phi: (\xi, \eta) \mapsto (\Phi_1(\xi, \eta), \Phi_2(\xi, \eta)),$$

where

$$\Phi_1(\xi, \eta) \in R^n, \qquad \Phi_2(\xi, \eta) \in R^m.$$

By local homeomorphism, we have

$$I = \Psi\Phi;$$

hence,

$$(0, y) = \Psi\Phi(0, y) = (F(\Phi_1(0, y), \Phi_2(0, y)), \Phi_2(0, y)),$$

that is,

$$\Phi_2(0, y) = y \qquad \text{and} \qquad F(\Phi_1(0, y), y) = 0$$

or $\Phi_1(0, y)$ is the required function H.

Remark 2.1. The above proof is also valid for the more general theorem in which we want to satisfy

$$F(x, y) = z,$$

where $z \in C_0$ an open neighborhood of 0. In this case,

$$x = \Phi_1(z, y)$$

is the required solution.

Remark 2.2. This version of the theorem is strongest in the sense that the one-to-one condition is obviously necessary as well as sufficient.

Remark 2.3. The one-to-one property of a function is a necessary condition for other properties, such as the strictly monotone property (5.4.3) of Ref. 1; hence, it should not be too difficult to prove. In fact, the original motivation for the new theorem is that, in solving some nonlinear programming problems, the function F turns out to be the gradient $\partial_1 L$ of a generalized Lagrangian $L: R^n \times R^m \to R^1$. For some problems, the classical theorem fails to hold because $\partial_{11}^2 L$ may become singular; in some cases, $\partial_{11}^2 L$ may not exist at all. But, in using the new theorem, from (3.4.5) of Ref. 1, we need only to show that $L(x, y)$ is strictly convex with respect to x.

References

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