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Control of Linear Processes with Distributed Lags Using Dynamic Programming from First Principles[1]

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Abstract. A simple dynamic programming argument is presented for the quadratic-cost controller synthesis problem for discrete-time linear processes with delay. Distributed delays are allowed in both state and control. The solution obtained has a discrete-time Riccati difference structure closely analogous to the Riccati differential structure associated with delay problems in continuous time. Extensions are provided for the cases of varying lag-limits, performance criterion dependent on past variables, and the time-invariant regulator problem. A feedback solution is also obtained for a continuous-time problem with distributed delays in the control, by passage to limit from the discrete results.

Key Words. Dynamic programming, linear–quadratic control theory, feedback controller synthesis, time-delay problem, retarded controls, distributed lags.

1. Introduction

In this paper, we consider the quadratic-criterion, controller-synthesis problem for linear processes with distributed delays in the state and control variables. Problems are discussed in both discrete and continuous time.

In discrete time, we examine processes of the type

$$x_{i+1} = A_i x_i + \sum_{i=1}^{k(i)} B_{i,\theta} x_{i-\theta} + C_i u_i + \sum_{i=1}^{h(i)} D_{i,\phi} u_{i-\phi}, \tag{1}$$

with initial data given. It has been known for some time that optimal controls can be obtained for such problems by lengthening or augmenting the state vector to include the delayed variables, thereby transforming the problem

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into a large, but standard, nondelayed problem. Known results may then be applied. While this approach is expedient, it relies on large, sparse matrices and offers little insight into the special structure of the time-lag problem and its solution. This paper presents a direct technique, one that does not appeal to the nonlag theory, based on dynamic programming from first principles and a partitioning of the optimal value matrix. Besides offering a direct argument, the technique has these advantages:

(i) Results are expressed in a concise form; the gain matrix algorithm avoids iteration with sparse matrices.

(ii) The discrete-time results emerge in a form whose structure corresponds closely to that of the known continuous-time results (see, for example, Refs. 1–4); that is, the feedback law depends on the solution of a matrix-Riccati difference system analogous to the matrix-Riccati differential system of the continuous-time problems. The connection between continuous and discrete problems becomes apparent.

(iii) Results for various forms of the continuous-time problem can be generated by passage to the limit as the time increment becomes small. This offers a simple technique for solution of lagged problems in continuous time.

We exploit this last point by using the limiting argument to provide controller synthesis results for the unsolved continuous-time problem with distributed delays in the control. This problem has dynamics

$$\dot{x}(t)=A(t)x(t)+C(t)u(t)+D_0(t)u(t-h)+\int_0^h D_1(t,s)u(t-s)\,ds. \quad (2)$$

Under conventional techniques, for example Carathéodory and maximum-principle–Fredholm approaches, feedback policies are difficult to obtain when the control is retarded. Koivo and Lee (Ref. 5) and Sendaula (Ref. 4) did manage to obtain a feedback solution for the fixed lag case ($D_0 \neq 0$). But the distributed lag problem ($D_1 \neq 0$) has been solved thus far only in open-loop form (Refs. 6–7). Control delays pose no extra difficulties to the passage-to-limit argument proposed here: Section 5 solves the feedback problem under (2) with the addition of distributed and fixed lags in the state.

The results obtained in this paper (optimal feedback policies under state and control distributed lags in both discrete and continuous time) are useful in problems that arise in the control and regulation of engineering transport processes, macroeconomic planning models, and biological and demographic age-dependent regenerative processes (see Ref. 8). The discrete-time results also provide an alternative technique for the numerical solution of continuous-time problems. The latter may be discretized at the outset and the discrete-time results applied.

2. Notation

We introduce a concise notation to simplify later manipulations.

(i) X_{i-1} and U_{i-1} denote the $nk(i)$-dimensional and $mh(i)$-dimensional vectors of past states and past controls:

$$X_{i-1}=\begin{bmatrix} x_{i-1} \\ x_{i-2} \\ \vdots \\ x_{i-k(i)} \end{bmatrix}, \qquad U_{i-1}=\begin{bmatrix} u_{i-1} \\ u_{i-2} \\ \vdots \\ u_{i-h(i)} \end{bmatrix}.$$

The values (x_i, X_{i-1}, U_{i-1}) will be called the *history* of the system at time i.

(ii) It will be convenient to index certain matrices that occur later in blocks of side-dimension n or m (to correspond with the dimensions of x and u). Block indices run from 1 to $k(i)$ or $h(i)$. Thus, $K(1, 1)$ refers to the top left-hand block in matrix K, $K(1, 2)$ to the next block to the right, and so on. $K(1, \cdot)$ denotes the top band of K, of width n or m as clear from the context.

(iii) $\bar{I}$ is a matrix whose below-diagonal blocks are $n\times n$ or $m\times m$ identity matrices, other blocks zero. The column of blocks e has top block an identity matrix, others zero. The dimensions of $\bar{I}$ and e are assumed clear from the context.

(iv) A dot indicates the time derivative, a prime the transpose, and I the identity matrix proper.

3. Discrete-Time Problem

We start with the case where the dynamic matrices A, B_θ, C, D_ϕ are time-invariant and k and h are constant. Rewrite (1) as

$$x_{i+1}=Ax_i+BX_{i-1}+Cu_i+DU_{i-1}, \tag{3}$$

with initial data x_0, X_{-1}, U_{-1} given, where A is $n\times n$, B is $n\times nk$, C is $n\times m$, and D is $n\times mh$. The matrix B_θ in (1) now appears as block $B(\theta)$ in (3). We wish to select controls

$$\{u_i\}, \qquad i=0,\ldots,N-1,$$

to minimize the performance criterion

$$J=\sum_{i=1}^{N-1}(x_i'Qx_i+u_i'Ru_i)+x_N'Px_N,$$

where Q and P are positive semidefinite (p.s.d., or ≥ 0) $n \times n$ matrices, and R is a positive definite (p.d., or >0) $m \times m$ matrix. Without loss of generality, Q, R, and P are taken to be symmetric.

Denoting S_i as the minimum attainable cost from time i to the end, we make the following assumption.

Assumption 3.1. S_i exists as a function of the history (x_i, X_{i-1}, U_{i-1}) given by the positive-semidefinite quadratic form

$$S_i = [x_i', X_{i-1}', U_{i-1}']F_i \begin{bmatrix} x_i \\ X_{i-1} \\ U_{i-1} \end{bmatrix}.$$

This assumption will be verified later.

The principle of optimality may be stated as

$$S_i(x_i, X_{i-1}, U_{i-1}) = \min_{u_i}[x_i'Qx_i + u_i'Ru_i + S_{i+1}(x_{i+1}, X_i, U_i)]. \tag{4}$$

If Assumption 3.1 is true, S_{i+1} can be written as

$$S_{i+1} = [x_{i+1}', X_i', U_i']F_{i+1} \begin{bmatrix} x_{i+1} \\ X_i \\ U_i \end{bmatrix}. \tag{5}$$

In the argument below, we shall use the principle of optimality and two successive linear transformations to link F_i with F_{i+1}; this will give a recursive method of computing F_i and the optimal control coefficients.

First, partition F_{i+1} so that

$$S_{i+1} = [x_{i+1}' \,\vdots\, X_i' \,\vdots\, U_i'] \left[\begin{array}{c|c|c} K_0 & K_1 & K_4 \\ \hline K_1' & K_2 & K_3 \\ \hline K_4' & K_3' & K_5 \end{array}\right] \left[\begin{array}{c} x_{i+1} \\ \hline X_i \\ \hline U_i \end{array}\right], \tag{6}$$

where, for the K-matrices, the time subscript $i+1$ is understood. The submatrices of F_{i+1} are of dimensions corresponding to

$$\begin{bmatrix} x_{i+1} \\ X_i \\ U_i \end{bmatrix},$$

and K_0, K_2, and K_5 are square and symmetric, by Assumption 3.1. By modifying F_{i+1}, we can express the optimality equation in the same form.

Let

$$\bar{F}_{i+1} = \left[\begin{array}{c|c|c} K_0 & K_1 & K_4 \\ \hline K_1' & \bar{K}_2 & K_3 \\ \hline K_4' & K_3' & \bar{K}_5 \end{array}\right], \tag{7}$$

where

$$\bar{K}_2(1, 1) = K_2(1, 1) + Q, \qquad \bar{K}_5(1, 1) = K_5(1, 1) + R,$$

and otherwise

$$\bar{K}_2 = K_2, \qquad \bar{K}_5 = K_5.$$

The optimality equation (4) then becomes

$$S_i(x_i, X_{i-1}, U_{i-1}) = \min_{u_i}[x'_{i+1}, X'_i, U'_i]\bar{F}_{i+1}\begin{bmatrix} x_{i+1} \\ X_i \\ U_i \end{bmatrix}. \tag{8}$$

Use the dynamics equation (3) to make the linear transformation

$$\left[\begin{array}{c} x_{i+1} \\ \hline X_i \\ \hline U_i \end{array}\right] = \left[\begin{array}{c|c|c|c} A & B & C & D \\ \hline e & \bar{I} & 0 & 0 \\ \hline 0 & 0 & e & \bar{I} \end{array}\right] \left[\begin{array}{c} x_i \\ \hline X_{i-1} \\ \hline u_i \\ \hline U_{i-1} \end{array}\right], \tag{9}$$

where the new partitioned matrix is called G. Equation (8) now becomes

$$S_i = \min_{u_i}[x'_i, X'_{i-1}, u'_i, U'_{i-1}]G'\bar{F}_{i+1}G\begin{bmatrix} x_i \\ X_{i-1} \\ u_i \\ U_{i-1} \end{bmatrix}. \tag{10}$$

Multiplying out $G'\bar{F}_{i+1}G$ and using symmetry, the quadratic form to be minimized contains the following terms in u_i:

$$\begin{aligned} &2x_i'(A'K_0C + A'K_4(1) + K_1'(1)C + K_3(1, 1))u_i \\ &\quad + 2X'_{i-1}(B'K_0C + B'K_4(1) + \bar{I}'K_1'C + \bar{I}'K_3(\cdot, 1))u_i \\ &\quad + u_i'(C'K_0C + C'K_4(1) + K_4'(1)C + \bar{K}_5(1, 1))u_i \\ &\quad + 2U'_{i-1}(D'K_0C + D'K_4(1) + \bar{I}'K_4'C + \bar{I}'\bar{K}_5(\cdot, 1))u_i. \end{aligned} \tag{11}$$

We differentiate the above expression with respect to u_i, and equate the result to zero to obtain the optimal control u_i^* as

$$u_i^* = -T_i^{-1}[V_i x_i + W_i X_{i-1} + Y_i U_{i-1}], \tag{12}$$

assuming that T_i is invertible, where

$$T_i = C'K_0C + K_5(1, 1) + R + C'K_4(1) + K_4'(1)C, \tag{13}$$

$$V_i = C'K_0A + K_4'(1)A + C'K_1(1) + K_3'(1, 1), \tag{14}$$

$$W_i = C'K_0B + K_4'(1)B + C'K_1\bar{I} + K_3'(1, \cdot)\bar{I}, \tag{15}$$

$$Y_i = C'K_0D + K_4'(1)D + C'K_4\bar{I} + \bar{K}_5(1, \cdot)\bar{I}. \tag{16}$$

The optimal control is, therefore, a linear function of the history of the system.

The optimal control provides a second linear transformation

$$\left[\begin{array}{c} x_i \\ \hline X_{i-1} \\ \hline u_i \\ \hline U_{i-1} \end{array}\right] = \left[\begin{array}{c|c|c} I & & \\ \hline & I & \\ \hline -T_i^{-1}V_i & -T_i^{-1}W_i & -T_i^{-1}Y_i \\ \hline & & I \end{array}\right] \left[\begin{array}{c} x_i \\ \hline X_{i-1} \\ \hline U_{i-1} \end{array}\right]. \tag{17}$$

The new matrix is called H_i. S_i can now be written as

$$S_i(x_i, X_{i-1}, U_{i-1}) = [x_i', X_{i-1}', U_{i-1}']H_i'G'\bar{F}_{i+1}GH_i \begin{bmatrix} x_i \\ X_{i-1} \\ U_{i-1} \end{bmatrix}, \tag{18}$$

so that, finally, F_i and F_{i+1} are connected by

$$F_i = H_i'G'\bar{F}_{i+1}GH_i. \tag{19}$$

Equation (19) may be used to recursively compute F_i by partitioning it similarly to F_{i+1} and multiplying out $H_i'G'\bar{F}_{i+1}GH_i$. Matching submatrices, and recognizing that $\bar{I}$, where it occurs, merely shifts the block index by one, yields the matrix Riccati difference system, for $i = 0, \ldots, N-1$,

$$\begin{aligned} K_{0_i} = {} & A'K_{0_{i+1}}A + A'K_{1_{i+1}}(1) \\ & + K_{1_{i+1}}'(1)A + K_{2_{i+1}}(1, 1) + Q - V_i'T_i^{-1}V_i, \end{aligned} \tag{20-1}$$

$$\begin{aligned} K_{1_i}(\theta) = {} & A'K_{0_{i+1}}B(\theta) + A'K_{1_{i+1}}(\theta+1) + K_{1_{i+1}}'(1)B(\theta) \\ & + K_{2_{i+1}}(1, \theta+1) - V_i'T_i^{-1}W_i(\theta), \qquad \theta = 1, \ldots, k, \end{aligned} \tag{20-2}$$

$$\begin{aligned} K_{2_i}(\theta, \phi) = {} & B'(\theta)K_{0_{i+1}}B(\phi) + B'(\theta)K_{1_{i+1}}(\phi+1) + K_{1_{i+1}}'(\theta+1)B(\phi) \\ & + K_{2_{i+1}}(\theta+1, \phi+1) - W_i'(\theta)T_i^{-1}W_i(\phi), \qquad \theta, \phi = 1, \ldots, k, \end{aligned} \tag{20-3}$$

$$K_{3_i}(\theta,\phi)=B'(\theta)K_{0_{i+1}}D(\phi)+B'(\theta)K_{4_{i+1}}(\phi+1)+K'_{1_{i+1}}(\theta+1)D(\phi)$$
$$+K_{3_{i+1}}(\theta+1,\phi+1)-W_i'(\theta)T_i^{-1}Y_i(\phi),\qquad \theta=1,\ldots,k,$$
$$\phi=1,\ldots,h, \tag{20-4}$$

$$K_{4_i}(\theta)=A'K_{0_{i+1}}D(\theta)+A'K_{4_{i+1}}(\theta+1)+K'_{1_{i+1}}(1)D(\theta)$$
$$+K_{3_{i+1}}(1,\theta+1)-V_i'T_i^{-1}Y_i(\theta),\qquad \theta=1,\ldots,h, \tag{20-5}$$

$$K_{5_i}(\theta,\phi)=D'(\theta)K_{0_{i+1}}D(\phi)+D'(\theta)K_{4_{i+1}}(\phi+1)+K'_{4_{i+1}}(\theta+1)D(\phi)$$
$$+K_{5_{i+1}}(\theta+1,\phi+1)-Y_i'(\theta)T_i^{-1}Y_i(\phi),\qquad \theta,\phi=1,\ldots,h. \tag{20-6}$$

In this case, where k and h are constant, $K(k+1)$, $K(h+1)$ are not defined and are taken to be zero. End conditions apply at time N,

$$K_{0_N}=P_N;$$

all other matrices K_N are zero.

The optimal controls are given by

$$u_i^*=-T_i^{-1}\left[V_ix_i+\sum_{\theta=1}^{k}W_i(\theta)x_{i-\theta}+\sum_{\theta=1}^{h}Y_i(\theta)u_{i-\theta}\right], \tag{21}$$

where

$$T_i=C'K_{0_{i+1}}C+K_{5_{i+1}}(1,1)+R+C'K_{4_{i+1}}(1)+K'_{4_{i+1}}(1)C, \tag{22-1}$$

$$V_i=C'K_{0_{i+1}}A+K'_{4_{i+1}}(1)A+C'K_{1_{i+1}}(1)+K'_{3_{i+1}}(1,1), \tag{22-2}$$

$$W_i(\theta)=C'K_{0_{i+1}}B(\theta)+K'_{4_{i+1}}(1)B(\theta)+C'K_{1_{i+1}}(\theta+1)$$
$$+K'_{3_{i+1}}(1,\theta+1),\qquad \theta=1,\ldots,k, \tag{22-3}$$

$$Y_i(\theta)=C'K_{0_{i+1}}D(\theta)+K'_{4_{i+1}}(1)D(\theta)+C'K_{4_{i+1}}(\theta+1)$$
$$+K_{5_{i+1}}(1,\theta+1),\qquad \theta=1,\ldots,h. \tag{22-4}$$

These equations may be computed recursively backward from stage N. At stage i, all K_{i+1} are known, and T_i, V_i, W_i, Y_i are computed from (22). Then, from (20), K_i may be obtained.

If desired, all K_i, T_i, V_i, W_i, and Y_i may be computed beforehand, so that only the optimal control rule (21) need be stored in the system. The history is then fed back at each stage i to determine the optimal control u_i^*.

In practice, many of the terms in (20) and (22) would be zero. In particular, when there are no control lags, the matrices K_3, K_4, K_5 are identically zero; when there are no state lags, K_1, K_2, K_3 are identically zero.

4. Extensions and Remarks for the Discrete-Time Problem

Remark 4.1. When the problem is time-varying, the results remain unchanged, except that A, B, C, D, Q, R, and G must be subscripted by time i as they appear in the above derivation and in (20) and (22).

Remark 4.2. Where the duration of the lags varies with time, the derivation and results carry over if k and h are taken to be the maximum values of

$$k(i), h(i), \qquad i=0, \ldots, N-1.$$

If lags vary, but never lengthen by more than one unit at a time, i.e., if

$$k(i+1) \leq k(i)+1, \qquad h(i+1) \leq h(i)+1,$$

all i, the limits in (20), (21), (22) may be replaced by $k(i)$, $h(i)$. In (20) and (22), if $K_{i+1}(\theta+1)$ is not defined, it is taken as zero.

Remark 4.3. In the case of a lagged quadratic criterion, that is,

$$J=\sum_{i=1}^{N-1} (X_i' Q X_i + U_i' R U_i) + X_N' P X_N, \tag{23}$$

where Q and P are now of dimension nk, R of dimension mh, all p.s.d., with $R(1, 1)>0$, only $\bar{K}_2$, $\bar{K}_5$ need be modified in the derivation. The systems (20) and (22) are modified accordingly.

Remark 4.4. In many regulator applications, the control interval N is infinite, the dynamics matrices k and h are time-invariant, and P is zero. Providing the system has the controllability property that the history can be returned to the zero vector in a finite number of stages, the results again apply. In this case, an argument similar to that of Ref. 9 shows that the matrices $K_{0_i}, \ldots, K_{5_i}$ become time-invariant, $\hat{K}_0, \ldots, \hat{K}_5$. These matrices can be solved for by starting with arbitrary K_{0_N} and computing (20) and (22) recursively backward indefinitely. As i approaches $-\infty$, $K_{0_i}, \ldots, K_{5_i}$ approach $\hat{K}_0, \ldots, \hat{K}_5$. Other methods for finding limiting matrices are given in Ref. 9.

Remark 4.5. We now prove Assumption 3.1 and the invertibility of T_i, given the conditions that Q, P are p.s.d. and that R is p.d.

At the endpoint, S_N exists as a quadratic form in (x_N, X_{N-1}, U_{N-1}), with

$$K_{0_N}=P,$$

other submatrices zero, and F_N p.s.d. Assume that, at stage $i+1$, S_{i+1} exists as a quadratic form in (x_{i+1}, X_i, U_i) with F_{i+1} p.s.d. Then, from the properties of Q and R, $\bar{F}_{i+1}$ is also p.s.d. Furthermore, $G'F_{i+1}G$ and its submatrices on the diagonal are p.s.d.; in particular, the submatrix

$$(C'K_0C+K_5(1,1)+C'K_4(1)+K_4'(1)C)$$

is p.s.d. Adding $R>0$ to this submatrix yields T_i, so that T_i is p.d. and, therefore, invertible. Since $G'\bar{F}_{i+1}G$ is p.s.d., and T_i is invertible, a unique minimum of the right-hand side of (4) exists; furthermore, H_i is well defined, and the operation of transforming the right-hand side of (4) into a quadratic form in (x_i, X_{i-1}, U_{i-1}), with

$$F_i = H_i'G'\bar{F}_{i+1}GH_i \geq 0,$$

is possible. Assumption 3.1 follows, as does the invertibility of T_i for all i.

A similar argument shows that the above conditions could be replaced by Q, P being p.d. and R being p.s.d., providing C is of full rank.

5. Continuous-Time Distributed-Lag Problem

We now use the discrete-time results to solve a continuous-time problem with distributed lags in the control. For symmetry and completeness of the problem, we add distributed lags in the state and fixed lags in both state and control.

The process is specified by

$$\dot{x}(t)=Ax(t)+B_0x(t-k)+\int_0^k B_1(s)x(t-s)\,ds+Cu(t)+D_0u(t-h)$$
$$+\int_0^h D_1(s)u(t-s)\,ds, \tag{24}$$

with given continuous initial functions

$$x(t)=\xi(t) \qquad \text{on } [t_0-k, t_0],$$
$$u(t)=\eta(t) \qquad \text{on } [t_0-h, t_0).$$

It is assumed that the admissible control functions $u(t)$ belong to $L_2[t_0, t_1]$, the space of square-integrable functions, and that $x(t)$ is an n-vector, $u(t)$ an m-vector as before. $B_1(s)$, $D_1(s)$ are $n\times n$, $n\times m$ matrices, continuous in s, and are nonzero almost everywhere: A and B_0 are $n\times n$; C and D_0 are $n\times m$. These matrices are later allowed to vary with time. We wish to choose an admissible $u(t)$ to minimize

$$J=\int_{t_0}^{t_1}[x'(t)Qx(t)+u'(t)Ru(t)]\,dt+x'(t_1)Px(t_1), \tag{25}$$

where, as before, Q, P are p.s.d. and R is p.d.

To obtain optimal feedback policies for this continuous-time problem, we approximate it by the corresponding discrete-time problem, invoke the discrete solution, and pass to the limit as the stepsize becomes zero. The process is sketched briefly as follows.

Approximate the problem (24)–(25) by the corresponding discrete problem with stepsize Δ, where Δ divides h, k and $t_1 - t_0$ exactly. The time t is taken in discrete integral steps of Δ. The discrete-time version becomes: choose

$$\{u_t\}, \qquad t = t_0, t_0+\Delta, t_0+2\Delta, \ldots, t_1-\Delta,$$

to minimize

$$J = \sum_{t=t_0}^{t_1-\Delta} (x_t' Q x_t + u_t' R u_t) + x_{t_1}' P x_{t_1}, \tag{26}$$

subject to

$$x_{t+\Delta} = (I + \Delta A + \Delta^2 B_1(0)) x_t + \Delta B_2 X_{t-\Delta} + (\Delta C + \Delta^2 D_1(0)) u_t + \Delta D_2 U_{t-\Delta}, \tag{27}$$

with initial data x_0, $X_{-\Delta}$, $U_{-\Delta}$ given. The notation $X_{t-\Delta}$ and $U_{t-\Delta}$ denotes the vectors

$$X_{t-\Delta} = \begin{bmatrix} x_{t-\Delta} \\ x_{t-2\Delta} \\ \vdots \\ x_{t-k} \end{bmatrix}, \qquad U_{t-\Delta} = \begin{bmatrix} u_{t-\Delta} \\ u_{t-2\Delta} \\ \vdots \\ u_{t-h} \end{bmatrix}.$$

B_2 is the $n \times (nk/\Delta)$ matrix

$$(\Delta B_1(\Delta), \Delta B_1(2\Delta), \ldots, \Delta B_1(k-\Delta), \Delta B_1(k) + B_0),$$

and D_2 is the $n \times (mh/\Delta)$ matrix

$$(\Delta D_1(\Delta), \Delta D_1(2\Delta), \ldots, \Delta D_1(h-\Delta), \Delta D_1(h) + D_0).$$

As before, matrices will be block-indexed by θ and ϕ in discrete integral steps of Δ. Normalize the problem by rewriting the K-matrices as

$$E_0(t) = K_{0_t}, \qquad E_1(t, \theta) = (1/\Delta) K_{1_t}(\theta),$$

$$E_2(t, \theta, \phi) = (1/\Delta^2) K_{2_t}(\theta, \phi), \qquad E_3(t, \theta, \phi) = (1/\Delta^2) K_{3_t}(\theta, \phi),$$

$$E_4(t, \theta) = (1/\Delta) K_{4_t}(\theta), \qquad E_5(t, \theta, \phi) = (1/\Delta^2) K_{5_t}(\theta, \phi).$$

Note that E_0, E_2, and E_5 are symmetric.

Using the discrete-time results from Section 3, the optimal controls for this problem may be written as

$$u_t^* = -T_t^{-1}[V_t x_t + W_t X_{t-\Delta} + Y_t U_{t-\Delta}], \tag{28}$$

where

$$T_t = R\Delta + O(\Delta)^2, \tag{29-1}$$

$$V_t = \Delta C' E_0(t+\Delta) + \Delta E_4'(t+\Delta, \Delta) + O(\Delta^2), \tag{29-2}$$

$$W_t(\theta) = \Delta^2 C' E_1(t+\Delta, \theta+\Delta) + \Delta^2 E_3'(t+\Delta, \Delta, \theta+\Delta) + O(\Delta^3),$$
$$\theta = \Delta, \ldots, k-\Delta, \tag{29-3}$$

$$W_t(k) = \Delta^2 C' E_0(t+\Delta) B_0 + \Delta^2 E_4'(t+\Delta, \Delta) B_0 + O(\Delta^3), \tag{29-4}$$

and so on. Now, using the system (20), the submatrix relations for this problem may be written as

$$\begin{aligned} E_0(t) = {} & E_0(t+\Delta) + \Delta A' E_0(t+\Delta) + \Delta E_0(t+\Delta) A + \Delta E_1(t+\Delta, \Delta) \\ & + \Delta E_1'(t+\Delta, \Delta) + Q\Delta - \Delta E_0(t+\Delta) C R^{-1} C' E_0(t+\Delta) \\ & - \Delta E_4(t+\Delta, \Delta) R^{-1} E_4'(t+\Delta, \Delta) \\ & - \Delta E_0(t+\Delta) C R^{-1} E_4'(t+\Delta, \Delta) \\ & - \Delta E_4(t+\Delta, \Delta) R^{-1} C' E_0(t+\Delta) + O(\Delta^2), \end{aligned} \tag{30-1}$$

$$\begin{aligned} \Delta E_1(t, \theta) = {} & \Delta E_1(t+\Delta, \theta+\Delta) + \Delta^2 E_0(t+\Delta) B_1(\theta) + \Delta^2 A' E_1(t+\Delta, \theta+\Delta) \\ & + \Delta^2 E_2(t+\Delta, \Delta, \theta+\Delta) - \Delta^2 E_0(t+\Delta) C R^{-1} C' E_1(t+\Delta, \theta+\Delta) \\ & - \Delta^2 E_0(t+\Delta) C R^{-1} E_3'(t+\Delta, \Delta, \theta+\Delta) \\ & - \Delta^2 E_4(t+\Delta, \Delta) R^{-1} C' E_1(t+\Delta, \theta+\Delta) \\ & - \Delta^2 E_4(t+\Delta, \Delta) R^{-1} E_3'(t+\Delta, \Delta, \theta+\Delta) + O(\Delta^3), \\ & \qquad \theta = \Delta, \ldots, k-\Delta, \end{aligned} \tag{30-2}$$

$$\Delta E_1(t, k) = \Delta E_0(t+\Delta) B_0 + O(\Delta^2), \tag{30-3}$$

and so on. Finally, expand the E-matrices, with arguments $t+\Delta$, $\theta+\Delta$, $\phi+\Delta$, by Taylor series to first order, and let Δ approach zero. In the limit, t, θ, and ϕ become continuous, and the E-matrices become functions defined over t, θ, and ϕ, where $t \in [t_0, t_1]$ and $\theta, \phi \in [0, h]$ or $[0, k]$. The system (30) becomes the following set of generalized Riccati partial differential equations, for $t \in [t_0, t_1]$:

$$\begin{aligned} 0 = {} & \partial E_0/\partial t + A' E_0(t) + E_0(t) A + E_1(t, 0) + E_1'(t, 0) + Q \\ & - (E_0(t) C + E_4(t, 0)) R^{-1} (C' E_0(t) + E_4'(t, 0)), \end{aligned} \tag{31-1}$$

$$0 = \partial E_1/\partial t + \partial E_1/\partial \theta + A'E_1(t, \theta) + E_0(t)B_1(\theta) + E_2(t, 0, \theta)$$
$$- (E_0(t)C + E_4(t, 0))R^{-1}(C'E_1(t, \theta) + E_3'(t, 0, \theta)), \qquad \theta \in [0, k), \quad (31\text{-}2)$$

$$0 = \partial E_2/\partial t + \partial E_2/\partial \theta + \partial E_2/\partial \phi + B_1'(\theta)E_1(t, \phi) + E_1'(t, \theta)B_1(\phi)$$
$$- (E_1'(t, \theta)C + E_3(t, \theta, 0))R^{-1}(C'E_1(t, \phi) + E_3'(t, 0, \phi)), \qquad \theta, \phi \in [0, k), \quad (31\text{-}3)$$

$$0 = \partial E_3/\partial t + \partial E_3/\partial \theta + \partial E_3/\partial \phi + B_1'(\theta)E_4(t, \phi) + E_1'(t, \theta)D_1(\phi)$$
$$- (E_1'(t, \theta)C + E_3(t, \theta, 0))R^{-1}(C'E_4(t, \phi) + E_5(t, 0, \phi)),$$
$$\theta \in [0, k), \phi \in [0, h), \qquad (31\text{-}4)$$

$$0 = \partial E_4/\partial t + \partial E_4/\partial \theta + E_0(t)D_1(\theta) + A'E_4(t, \theta) + E_3(t, 0, \theta)$$
$$- (E_0(t)C + E_4(t, 0))R^{-1}(C'E_4(t, \theta) + E_5(t, 0, \theta)), \qquad \theta \in [0, h), \quad (31\text{-}5)$$

$$0 = \partial E_5/\partial t + \partial E_5/\partial \theta + \partial E_5/\partial \phi + D_1'(\theta)E_4(t, \phi) + E_4'(t, \theta)D_1(\phi)$$
$$- (E_4'(t, \theta)C + E_5(t, \theta, 0))R^{-1}(C'E_4(t, \phi) + E_5(t, 0, \phi)), \qquad \theta, \phi \in [0, h), \quad (31\text{-}6)$$

with boundary conditions

$$E_1(t, k) = E_0(t)B_0, \qquad (32\text{-}1)$$
$$E_2(t, \theta, k) = E_1'(t, \theta)B_0, \qquad \theta \in [0, k], \qquad (32\text{-}2)$$
$$E_3(t, k, \phi) = B_0'E_4(t, \phi), \qquad \phi \in [0, h], \qquad (32\text{-}3)$$
$$E_3(t, \theta, h) = E_1'(t, \theta)D_0, \qquad \theta \in [0, k], \qquad (32\text{-}4)$$
$$E_4(t, h) = E_0(t)D_0, \qquad (32\text{-}5)$$
$$E_5(t, \theta, h) = E_4'(t, \theta)D_0, \qquad \theta \in [0, h], \qquad (32\text{-}6)$$

with $E_1, \ldots, E_5$ zero at $t = t_1$,

$$E_0(t_1) = P,$$

and E_0, E_2, E_5 symmetric.

Providing the above conditions for the E-matrices are satisfied almost everywhere, the optimal control $u^*(t)$ is given by the functional

$$u^*(t) = -R^{-1}\{[C'E_0(t) + E_4'(t, 0)]x(t)$$
$$+ \int_0^k [C'E_1(t, \theta) + E_3'(t, 0, \theta)]x(t - \theta)\, d\theta$$
$$+ \int_0^h [C'E_4(t, \theta) + E_5(t, 0, \theta)]u(t - \theta)\, d\theta\}. \qquad (33)$$

Also, in the limit, S_t becomes the optimal value functional given by

$$\begin{aligned} S(t) = x'(t)E_0(t)x(t) + 2x'(t)\int_0^k E_1(t, \theta)x(t-\theta)\, d\theta \\ + \int_0^k \int_0^k x'(t-\theta)E_2(t, \theta, \phi)x(t-\phi)\, d\phi\, d\theta \\ + 2\int_0^k \int_0^h x'(t-\theta)E_3(t, \theta, \phi)u(t-\phi)\, d\phi\, d\theta \\ + 2x'(t)\int_0^h E_4(t, \theta)u(t-\theta)\, d\theta \\ + \int_0^h \int_0^h u'(t-\theta)E_5(t, \theta, \phi)u(t-\phi)\, d\phi\, d\theta. \end{aligned} \tag{34}$$

Note that, where no state lags are present, the matrices E_1, E_2, E_3 disappear; and, where no control lags are present, E_3, E_4, E_5 disappear. In versions of the problem with k, h constant, for the case of state lags, the results specialize to those of Refs. 1–3; and, for a fixed control lag, the results specialize to those of Ref. 4.

6. Remarks for the Continuous-Time Case

Remark 6.1. The results are once again easily extended to the time-varying case where A, B_0, B_1, C, D_0, D_1, Q, and R are continuous functions of time, by indexing these matrices with respect to the time t, as they appear in the above system.

Remark 6.2. For infinite-horizon, time-invariant problems, assuming that the system history is controllable to the zero function, the E-matrices reach a limit independent of the time t, as in the discrete case. The time subscript can then be dropped in (31) along with $\partial E/\partial t$ terms.

Remark 6.3. With finite horizon, a special caveat is in order. Although, when t is in the end region $(t_1 - k, t_1]$, Eq. (31-2) for the matrix $E_1(t, \theta)$ holds in the two separate regions

$$\{t, \theta | t \in (t_1 - k, t_1], \theta \in (k - (t_1 - t), k]\}$$

and

$$\{t, \theta | t \in (t_1 - k, t_1], \theta \in [0, k - (t_1 - t))\},$$

it is not defined at their boundary, where a discontinuity in $E_1(t, \theta)$ occurs. Similar remarks extend to the other E-matrices. In the special case where B_1 and D_1 are zero,

$$(k-(t_1-t), k), (h-(t_1-t), h)$$

may replace

$$[0, k), [0, h)$$

in (31) when t is in the end regions

$$(t_1-k, t_1] \qquad \text{or } (t_1-h, t_1].$$

Terms in $E(t, 0)$ may then be dropped, since they are undefined. Outside the end regions, (31) applies as written.

7. Conclusions

A simple dynamic programming argument was presented for the discrete-time linear–quadratic problem with distributed delays in both state and control. The technique relies on a partitioning of the optimal value matrix to correspond with the state–control lag structure of the dynamics. For problems with dynamics structured differently, other partitionings might be more effective.

The Riccati system for the discrete-time problem is similar to that for continuous-time problems, but, as in the nonlag case, contains more terms. The discrete-time results were used to solve, by a limiting process, a continuous-time problem with the novelty of distributed lags in the control. The discrete delay process is general enough to approximate many forms of continuous-time process (e.g., integral-equation dynamics), so that feedback solutions for other continuous-time processes could be obtained in the same way.

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