JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 27, No. 4, APRIL 1979

# A Comparison of Two Methods for Determining the Weights of Belonging to Fuzzy Sets[1]

A. T. W. CHU,[2] R. E. KALABA,[3] AND K. SPINGARN[4]

**Abstract.** Saaty has solved a basic problem in fuzzy set theory using an eigenvector method to determine the weights of belonging of each member to the set. In this paper, a weighted least-square method is utilized to obtain the weights. This method has the advantage that it involves the solution of a set of simultaneous linear algebraic equations and is thus conceptually easier to understand than the eigenvector method. Examples are given for estimating the relative wealth of nations and the relative amount of foreign trade of nations. Numerical solutions are obtained using both the eigenvector method and the weighted least-square method, and the results are compared.

**Key Words.** Fuzzy sets, eigenvectors, weighted least squares, relative weight matrix.

## 1. Introduction

A basic problem in the theory of fuzzy sets (Ref. 1) is the determination of the degree of belonging of each member to the set. Saaty (Refs. 2 and 3) has shown that this problem can be reduced to a matrix eigenvalue problem. In Ref. 4, an imbedding method was applied to obtain the largest eigenvalue and eigenvector of the matrix. In this paper, a weighted least-square method is utilized, and the results are compared with the eigenvector method.

Let $w_i > 0$, $i = 1, 2, \ldots, n$, be the degrees of belonging of the $n$ members. Forming the matrix of relative weights whose $ij$th element is $w_i/w_j$, Saaty observed that the vector $(w_1, w_2, \ldots, w_n)^T$ is an eigenvector

[1] Partial support of this research was provided by NIH Grant No. GM-23732.

[2] Graduate Student, Department of Economics, University of Southern California, Los Angeles, California.

[3] Professor of Economics and Biomedical Engineering, University of Southern California, Los Angeles, California.

[4] Senior Staff Engineer, Space and Communications Group, Hughes Aircraft Company, Los Angeles, California.



0022-3239/79/0400-0531$03.00/0 

corresponding to the largest eigenvalue (The Perron–Frobenius root). All the other eigenvalues are zero.

The idea is to estimate the matrix of relative weights and then obtain an estimate of the vector $(w_1, w_2, \ldots, w_n)^T$ as an eigenvector corresponding to the largest eigenvalue of the relative weight matrix. To compare a set of $n$ objects in pairs according to their relative weights, Saaty denotes the objects by $A_1, \ldots, A_n$ and their weights by $w_1, \ldots, w_n$. The pairwise comparisons are represented by the matrix

$$
\begin{array}{ccc}
 & \begin{array}{cccc} \quad A_1 \quad & \quad A_2 \quad & \cdots & \quad A_n \quad \end{array} & \\
A = & \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & & & \\ w_n/w_1 & w_n/w_2 & & w_n/w_n \end{bmatrix} & \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_n \end{array}
\end{array} \tag{1}
$$

This matrix, called a reciprocal matrix, has positive entries everywhere and satisfies the reciprocal property

$$a_{ji} = 1/a_{ij}.$$

Multiplying this matrix by the vector $w = (w_1, \ldots, w_n)^T$, we have

$$Aw = nw, \tag{2}$$

or

$$(A - nI)w = 0. \tag{3}$$

This is a system of homogeneous linear equations which has a non-trivial solution iff the determinant of $(A - nI)$ vanishes, that is, $n$ is an eigenvalue of $A$. The matrix $A$ is also consistent; that is,

$$a_{jk} = a_{ik}/a_{ij}.$$

In the general case, the precise values of $w_i/w_j$ are not known and must be estimated. Since the eigenvalues are perturbed by a small perturbation of the coefficients. Eq. (2) becomes

$$A'w' = \lambda_{\max} w', \tag{4}$$

where $\lambda_{\max}$ is the largest eigenvalue of $A'$. To simplify the notation, Eq. (4) is expressed in the form

$$Aw = \lambda_{\max} w, \tag{5}$$

where $A$ is Saaty's matrix of pairwise comparisons. The eigenvector associated with the largest eigenvalue is the desired vector of weights.

Numerical methods for obtaining the largest eigenvalue and associated eigenvector were discussed in Ref. 4. In this paper, the equations for a weighted least-square method are derived. Numerical results are given for several examples and compared with the eigenvector results.

## 2. Weighted Least-Square Method

Consider the elements $a_{ij}$ of Saaty's matrix $A$ in Eq. (5). It is desired to determine the weights $w_i$, such that, given $a_{ij}$,

$$a_{ij} \approx w_i / w_j. \tag{6}$$

The weights can be obtained by solving the constrained optimization problem

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} w_j - w_i)^2, \tag{7}$$

$$\sum_{i=1}^{n} w_i = 1, \tag{8}$$

$$\text{minimize } S. \tag{9}$$

An additional constraint is that $w_i > 0$. However, it is conjectured that the above problem can be solved such that $w_i > 0$ without this constraint. The least-square solution for

$$S_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - w_i / w_j)^2, \tag{10}$$

while more desirable than for the weighted least squares given by Eq. (7), is much more difficult to solve numerically.

In order to minimize $S$, form the sum

$$S' = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} w_j - w_i)^2 + 2\lambda \sum_{i=1}^{n} w_i, \tag{11}$$

where $\lambda$ is the Lagrange multiplier. Differentiating Eq. (11) with respect to $w_m$, the following set of equations is obtained:

$$\sum_{i=1}^{n} (a_{im} w_m - w_i) a_{im} - \sum_{j=1}^{n} (a_{mj} - w_m) + \lambda = 0, \qquad m = 1, 2, \ldots, n. \tag{12}$$

Equations (12) and (8) form a set of $n+1$ inhomogeneous linear equations with $n+1$ unknowns. For example, for $n=2$, the equations are

$$(1+a_{21}{}^2)w_1-(a_{12}+a_{21})w_2+\lambda=0, \tag{13}$$

$$-(a_{12}+a_{21})w_1+(1+a_{12}{}^2)w_2+\lambda=0, \tag{14}$$

$$w_1+w_2=1. \tag{15}$$

Given the coefficients $a_{ij}$, Eqs. (13)–(15) can be solved for $w_1$, $w_2$, $\lambda$ using a standard FORTRAN subroutine for solving simultaneous linear equations. In this simple case, however, an analytical solution is possible:

$$w_1=[(1+a_{12}{}^2)+a_{12}+a_{21}]/[(1+a_{12})^2+(1+a_{21})^2], \tag{16}$$

$$w_2=[(1+a_{21}{}^2)+a_{12}+a_{21}]/[(1+a_{12})^2+(1+a_{21})^2]. \tag{17}$$

Equations (16) and (17) show that, since the $a_{ij}$'s$>0$, then the $w_i$'s$>0$.

In general, Eqs. (12) and (8) can be expressed in the matrix form

$$Bw=m, \tag{18}$$

where

$$w=(w_1, w_2, \ldots, w_n, \lambda)^T, \tag{19}$$

$$m=(0, 0, \ldots, 0, 1)^T, \tag{20}$$

$$B=(n+1)\times(n+1) \text{ matrix with elements } b_{ij}, \tag{21}$$

$$b_{ii}=(n-1)+\sum_{j\neq i}^{n} a_{ji}{}^2, \qquad i, j=1, \ldots, n, \tag{22}$$

$$b_{ij}=-a_{ij}-a_{ji}, \qquad i, j=1, \ldots, n, \tag{23}$$

$$b_{k,n+1}=b_{n+1,k}=1, \qquad k=1, \ldots, n, \tag{24}$$

$$b_{n+1,n+1}=0. \tag{25}$$

## 3. Wealth-of-Nations Matrix

Numerical results were obtained for Saaty's wealth-of-nations matrix given in Ref. 2 and repeated in Table 1. Saaty made estimates of the relative wealth of nations and showed that the eigenvector corresponding to the matrix agreed closely with the GNP. The power method for obtaining the eigenvector was utilized in Ref. 4 and compared to Saaty's results. Table 2 compares these results with the weighted least squares results. It is seen that the sums $S$ and $S_1$, defined by Eqs. (7) and (10), are less for the weighted

Table 1. Wealth-of-nations matrix.

| Country | US | USSR | China | France | UK | Japan | W. Germany |
|---|---|---|---|---|---|---|---|
| US | 1 | 4 | 9 | 6 | 6 | 5 | 5 |
| USSR | 1/4 | 1 | 7 | 5 | 5 | 3 | 4 |
| China | 1/9 | 1/7 | 1 | 1/5 | 1/5 | 1/7 | 1/5 |
| France | 1/6 | 1/5 | 5 | 1 | 1 | 1/3 | 1/3 |
| UK | 1/6 | 1/5 | 5 | 1 | 1 | 1/3 | 1/3 |
| Japan | 1/5 | 1/3 | 7 | 3 | 3 | 1 | 2 |
| W. Germany | 1/5 | 1/4 | 5 | 3 | 3 | 1/2 | 1 |

least-square method than for the power method. The sums $S_1$ were computed for comparison, even though the minimizations were made with respect to the sums $S$.

## 4. Taiwan Trade Matrices

One of the authors was a student in the College of Chinese Culture in Taiwan. Using Saaty's scales (Ref. 2), she estimated the relative strengths of belonging of the US, Japan, S. America, and Europe to the fuzzy set of important trading partners with Taiwan. This was done with regard to exports, imports, and total trade (Ref. 5). Both methods for determining the

Table 2. Comparison of numerical results for wealth-of-nations matrix.

| Country | Saaty's eigenvector ($\lambda_{\max} = 7.61$) | Power method eigenvector ($\lambda_{\max} = 7.60772$) | Weighted least-square method |
|---|---|---|---|
| US | 0.429 | 0.427115 | 0.486711 |
| USSR | 0.231 | 0.230293 | 0.175001 |
| China | 0.021 | 0.0208384 | 0.0299184 |
| France | 0.053 | 0.0523856 | 0.0593444 |
| UK | 0.053 | 0.0523856 | 0.0593444 |
| Japan | 0.119 | 0.122719 | 0.10434 |
| W. Germany | 0.095 | 0.0942627 | 0.0853411 |
| $S = \sum_i \sum_j (a_{ij} w_j - w_i)^2$ | | $S = 0.458232$ | $S = 0.288071$ |
| $S_1 = \sum_i \sum_j (a_{ij} - w_i/w_j)^2$ | | $S_1 = 187.898$ | $S_1 = 124.499$ |

Table 3. Taiwan trade matrices.

| | Country | US | Japan | S. America | Europe |
|---|---|---|---|---|---|
| Exports | US | 1 | 3 | 9 | 5 |
| | Japan | 1/3 | 1 | 9 | 1/2 |
| | S. America | 1/9 | 1/5 | 1 | 1/2 |
| | Europe | 1/5 | 3 | 3 | 1 |
| Imports | US | 1 | 1 | 9 | 3 |
| | Japan | 1 | 1 | 7 | 3 |
| | S. America | 1/9 | 1/9 | 1 | 1/7 |
| | Europe | 1/3 | 1/2 | 7 | 1 |
| Total trade | US | 1 | 3 | 9 | 3 |
| | Japan | 1/4 | 1 | 7 | 3 |
| | S. America | 1/9 | 1/7 | 1 | 1/5 |
| | Europe | 1/5 | 1/2 | 5 | 1 |

relative weights were used, and comparisons were made with published trade data (Ref. 6) for the year 1975. Both methods yielded good agreement with those data. Table 3 gives the Taiwan trade matrices, and Table 4 gives a comparison of the numerical results. The sums $S$ and $S_1$ are again lower for the weighted least-square method than for the power method in all cases, except for the imports sum $S_1$. In all cases, the dominant weight tends to be larger for the weighted least-square method.

## 5. Discussion

The numerical results tend to show that either the eigenvector or the weighted least-square method can be used to obtain the weights. For the examples used in this paper, the eigenvector method appeared to give answers closer to the expected values. However, the sum $S$ and $S_1$ were generally smaller for the weighted least-square method. Also, the weighted least-square method, which involves the solution of a set of simultaneous linear algebraic equations, is conceptually easier to understand than the eigenvector method. Using the eigenvector method, it can be proved that the weights $w_i$ are all greater than zero (Ref. 2). While we do not know whether such a theorem exists for the weighted least-square method, the numerical results given here indicate that the $w_i$'s obtained by this method are also greater than zero and are comparable to those obtainable by the eigenvector method.

Table 4. Comparison of numerical results for Taiwan trade matrices.

| Country | Fraction of exports | Power method eigenvector ($\lambda_{max} = 4.93$) | Weighted least-square method |
|---|---|---|---|
| US | 0.525 | 0.540 | 0.641 |
| Japan | 0.204 | 0.193 | 0.160 |
| S. America | 0.038 | 0.052 | 0.047 |
| Europe | 0.233 | 0.215 | 0.152 |
| | | $S = 0.526$<br>$S_1 = 41.23$ | $S = 0.277$<br>$S_1 = 59.56$ |
| **Country** | **Fraction of imports** | **Power method eigenvector ($\lambda_{max} = 4.14$)** | **Weighted least-square method** |
| US | 0.394 | 0.399 | 0.414 |
| Japan | 0.418 | 0.382 | 0.396 |
| S. America | 0.014 | 0.036 | 0.042 |
| Europe | 0.174 | 0.183 | 0.148 |
| | | $S = 0.081$<br>$S_1 = 22.75$ | $S = 0.040$<br>$S_1 = 18.71$ |
| **Country** | **Fraction of total trade** | **Power method eigenvector ($\lambda_{max} = 4.09$)** | **Weighted least-square method** |
| US | 0.452 | 0.533 | 0.575 |
| Japan | 0.323 | 0.279 | 0.232 |
| S. America | 0.024 | 0.041 | 0.049 |
| Europe | 0.200 | 0.147 | 0.144 |
| | | $S = 0.179$<br>$S_1 = 19.94$ | $S = 0.126$<br>$S_1 = 19.92$ |

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