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Nonlinear Programming in Normed Linear Spaces[1]

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Abstract. Necessary conditions in the form of multiplier rules are given for a function to have a constrained minimum. First-order differentiability conditions are imposed, and various combinations of set, equality, and inequality constraints are considered in arbitrary normed linear spaces.

Key Words. Neustadt derivative, local cone, multiplier rules, constraints, constraint qualification.

1. Introduction

In this paper, necessary conditions in the form of multiplier rules are given for constrained minimization problems in normed linear spaces. The results can be divided into two general types, according to the types of constraint. Set and inequality constraints are first introduced, and then equality constraints are incorporated.

The objective is to establish first-order multiplier rules in normed linear spaces under the weakest conditions possible. The approach taken is to introduce constraint qualifications which involve the cost function (see Theorems 3.2 and 5.2). That the constraint qualifications introduced are not overly restrictive is seen in the fact that they are, in some sense, necessary conditions for the multiplier obtained.

A number of known theorems or refinements thereof are proven to demonstrate the applicability of the constraint qualifications. The proofs given are generally much simpler and shorter than the original proofs. In addition, the method of proof serves to demonstrate the interrelation of these results.

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2. Notation and Preliminary Results

Throughout, it will be assumed that all vector spaces are over the field of reals.

The first definition is generally attributed to Neustadt (Ref. 1). It is stated for normed linear spaces, although it can be stated for topological vector spaces. Most of the theorems in this section and following sections are equally valid for topological vector spaces; but, for simplicity, they are stated for normed linear spaces. The exceptions are when the Fréchet derivative is used.

Definition 2.1. Let X and Y denote normed linear spaces and $f: X \to Y$. For $x_0 \in X$, f is said to be *differentiable at* x_0 *in the sense of Neustadt* if, for all $k \in X$, there exists a vector $Df(x_0)k$ in Y such that

$$[f(x_0+\epsilon h)-f(x_0)]/\epsilon \to Df(x_0)k \qquad \text{as } (\epsilon, h) \to (0^+, k).$$

$Df(x_0)$ is referred to as the *Neustadt derivative of* f *at* x_0.

Remark 2.1. It is easily verified that $Df(x_0)$ is positively homogeneous; furthermore, if f is differentiable at x_0 in the sense of Neustadt, then f is continuous at x_0.

For additional properties of the Neustadt derivative see Tagawa (Ref. 2).

Remark 2.2. To define the Neustadt derivative of f, it is essential only that f be defined on an open subset of X. If this is kept in mind, it is easily seen that, for all of the theorems to be stated involving extrema, the extrema need only be local.

Remark 2.3. It can be verified that, if f has a Fréchet derivative at x_0, then f has a Neustadt derivative at x_0 and the two derivatives coincide.

Definition 2.2. Let X denote a vector space, and let A and B denote subsets of X. Then, A and B can be *separated by a linear functional* if there exists $x' \in X'$ and $p \in R$ such that

$$x'a \le p \qquad \text{for all } a \in A,$$

$$x'b \ge p \qquad \text{for all } b \in B,$$

where X' denotes the algebraic dual of X and R denotes the field of reals.

If X is a normed linear space, then A and B can be *separated by a continuous linear functional* if A and B can be separated by a linear functional which is continuous.

The space of continuous linear functionals on X is denoted by X^*.

It is well known that, if A and B are convex subsets of a normed linear space such that

$$\operatorname{int} A \neq \emptyset, \qquad \operatorname{int} A \cap B = \emptyset,$$

then A and B can be separated by a continuous linear functional, where int A denotes the interior of A. Theorem 2.3, which will be stated presently, gives sufficient conditions for separating subsets of a vector space.

Definition 2.3. Let X denote a vector space and B a subset of X. B is a *cone* if, for all $b \in B$ and for all $p \geq 0$,

$$pb \in B.$$

Remark 2.4. It is easily verified that, if B is a cone, then B is convex iff

$$B = B + B.$$

From this, it follows that, if B is a convex cone, then

$$B \subset B - b$$

for all $b \in B$.

Definition 2.4. Let X and Y denote vector spaces, B a convex cone in Y, and

$$f: X \to Y.$$

f is said to be B-convex if, for a and $b \in X$ and $p \in [0, 1]$,

$$f(pa + (1-p)b) - pf(a) - (1-p)f(b) \in B.$$

Remark 2.5. In the preceding definition, B is treated as a negative cone, that is, an order $\leq$ can be defined on Y by

$$a \leq b \qquad \text{if } a - b \in B.$$

If Y is the space of reals and B is the set of nonpositive reals, then $\leq$ reduces to the usual order and B-convex reduces to the common notion of convexity for functions.

The next definition is usually credited to Varaiya (Ref. 3).

Definition 2.5. Let X denote a normed linear space, and assume that

$$x_0 \in A \subseteq X.$$

Then, the *local cone of A at x_0*, denoted by $\mathrm{LC}[A; x_0]$, is the set of all vectors k in X satisfying the following property: there exists sequences (λ_n) of reals and (a_n) of elements of A such that

$$\lambda_n > 0, \qquad \lambda_n \uparrow \infty, \qquad a_n \to x_0, \qquad \lambda_n(a_n - x_0) \to k.$$

The next theorem is a slight variation of a theorem by Varaiya and is due to Tagawa (Ref. 2, Theorem 3.2).

Theorem 2.1. Let X denote a normed linear space and A a subset of X. Assume that $f: X \to R$ has a Neustadt derivative at $x_0 \in X$. If f has a min on A at x_0, then

$$\mathrm{LC}[A; x_0] \subseteq Df(x_0)^+,$$

where

$$Df(x_0)^+ = \{k : Df(x_0)k \geq 0\}.$$

Remark 2.6. It is apparent that

$$\mathrm{cl}(\mathrm{co}\, \mathrm{LC}[A; x_0]) \subseteq Df(x_0)^+$$

if $Df(x_0)$ is linear, where, given a set S, co S and cl S denote the convex hull and topological closure, respectively, of S.

The next definition appears to be due to Girsanov (Ref. 4).

Definition 2.6. Let X denote a normed linear space, $f: X \to R$, and x_0 an element of X. The *cone of direction of decrease of f at x_0*, denoted by $K[f; x_0]$, is the set of all $k \in X$ which satisfies the following condition: there exists a neighborhood G of k, $\epsilon > 0$, and $\alpha < 0$ such that, for all $\delta \in (0, \epsilon)$ and for all $x \in G$,

$$f(x_0 + \delta x) < f(x_0) + \delta\alpha.$$

The next theorem is due to Das (Ref. 5).

Theorem 2.2. Let A denote a subset of a normed linear space X and $f: X \to R$. For $x_0 \in A$, assume that $K[f; x_0]$ is convex and nonempty and that K is a convex subset of $\mathrm{LC}[A; x_0]$ such that $0 \in K$.

If f has a min on A at x_0, then there exists a nonzero φ in X^* such that

$$\varphi k \leq 0 \qquad \text{for all } k \in K[f; x_0],$$

$$\varphi k \geq 0 \qquad \text{for all } k \in K.$$

With the exception of Theorem 2.6 the remaining definitions and theorems appear to be due primarily to Klee (Refs. 6 and 7). Proofs of all of the remaining theorems can be located in Ref. 8.

Definition 2.7. Let X denote a vector space and C a subset of X. The *affine hull* of C, denoted by aff C, is defined by

$$\text{aff } C = \text{span}(C - c) + c,$$

where c is any element of C and $\text{span}(C-c)$ is the smallest subspace of X containing $C - c$.

The *intrinsic core* of C, denoted by $\text{cr}_{\text{aff}}\, C$, is the set of all z in C such that, for all

$$x \in \text{aff } C \sim z,$$

there exists $y \in (x, z)$ such that

$$[y, z] \subseteq C,$$

where (a, b) and $[a, b]$ denote the closed and open line segments, respectively, connecting a and b.

Theorem 2.3. Let X denote a vector space, and let A and B denote convex subsets of X with nonempty intrinsic cores. If

$$\text{cr}_{\text{aff}}\, A \cap \text{cr}_{\text{aff}}\, B = \varnothing,$$

then A and B can be separated by a linear functional.

Definition 2.8. Let X denote a normed linear space and C a convex subset of X. The *relative interior* of C, denoted by $\text{int}_{\text{aff}}\, C$, is the interior of C relative to the affine hull of C.

Theorem 2.4. Let X denote a normed linear space and C a convex subset of X. If

$$\text{int}_{\text{aff}}\, C \neq \varnothing,$$

then

$$\text{int}_{\text{aff}}\, C = \text{cr}_{\text{aff}}\, C.$$

Theorem 2.5. Let X and Y denote vector spaces, T a linear function from X to Y, and C a convex subset of X. If

$$\text{cr}_{\text{aff}}\, C \neq \varnothing,$$

then

$$\text{cr}_{\text{aff}}\, T(C) \neq \varnothing.$$

Remark 2.7. Despite a similarity between intrinsic core and relative interior, the preceding theorem is false if $\mathrm{cr}_{\mathrm{aff}}$ is replaced by $\mathrm{int}_{\mathrm{aff}}$. For closed convex subsets of locally convex, second-category, topological vector spaces, the intrinsic core and the relative interior coincide.

To complete this section, the following theorem due to Flett (Ref. 9, Theorem 3) is stated.

Theorem 2.6. Let X and Y denote Banach spaces and $f: X \to Y$. Assume that f has a continuous Fréchet derivative Df at $x_0 \in N(f)$ and that $Df(x_0)$ is onto Y. Then,

$$N[Df(x_0)] \subseteq \mathrm{LC}[N(f); x_0],$$

where $N(f)$ denotes the null set of f.

3. Abstract Multiplier Rules

The results of this section are of the general form referred to as multiplier rules, but do not involve explicitly optimization. In subsequent sections, applications will be made to minimization problems.

Theorem 3.1. Let X, Y, Z denote normed linear spaces,

$$F: X \to R, \qquad G: X \to Y, \qquad H: X \to Z$$

such that

$$F(0) = 0, \qquad G(0) = 0, \qquad H(0) = 0.$$

Let B denote a subset of Y and K a subset of X for which

$$0 \in K, \qquad g(x_0) \in B,$$

where

$$x_0 \in X, \qquad g: X \to Y.$$

Define subsets S_1 and T_1 of $R \times Y$ and S_2 and T_2 of $R \times Y \times Z$ as follows:

$$S_1 = \{(F(k), G(k)): k \in K\},$$

$$S_2 = \{(F(k), G(k), H(k)): k \in K\},$$

$$T_1 = \{(r, y): r \leq 0, y \in B - g(x_0)\},$$

$$T_2 = \{(r, y, 0): r \leq 0, y \in B - g(x_0)\}.$$

Then, the following results hold:

(a) S_1 and T_1 can be separated by a linear functional iff there exists $\eta \in R$ and $y' \in Y'$, not both zero, such that

(i) $\eta F(k) + y'G(k) \geq 0$ for all $k \in K$,
(ii) $\eta \geq 0$ and $y'y \leq 0$ for all $y \in B - g(x_0)$.

(b) S_2 and T_2 can be separated by a linear functional iff there exists

$$\eta \in R, \qquad y' \in Y', \qquad z' \in Z',$$

not all zero, such that

(i) $\eta F(k) + y'G(k) + z'H(k) \geq 0$ for all $k \in K$,
(ii) $\eta \geq 0$ and $y'y \leq 0$ for all $y \in B - g(x_0)$.

Proof. Only (b) will be proved, since the proof of (a) is similar. First, assume that S_2 and T_2 can be separated. Then, there exist real numbers η and p, $y' \in Y'$, and $z' \in Z'$, where η, y', z' are not all zero, such that

$$\eta F(k) + y'G(k) + z'H(k) \geq p \qquad \text{for all } k \in K, \tag{1}$$

$$\eta r + y'y \leq p \qquad \text{for } r \leq 0,\ y \in B - g(x_0). \tag{2}$$

If

$$k = 0,$$

then by (1),

$$0 \geq p.$$

If

$$r = 0, \qquad y = 0,$$

then by (2),

$$0 \leq p.$$

The conclusion is now evident.

Now assume that (i) and (ii) hold for η, y', z' not all zero. To show that S_2 and T_2 can be separated, it suffices to show that

$$\eta r + y'y \leq 0$$

for

$$r \leq 0, \qquad y \in B - g(x_0),$$

which is obvious.

Remark 3.1. (i) Clearly, S_1 and T_1 can be separated by a continuous linear functional iff continuous linear multipliers, not all zero, exist satisfying (i) and (ii) of (a) and (b).

(ii) If B is a convex cone, then it is not necessary to require that $0 \in K$, nor that

$$F(0)=0, \qquad G(0)=0, \qquad H(0)=0.$$

In this case, (2) alone is sufficient to establish that p may be taken to be zero.

(iii) If B is a convex cone, then

$$y'y \leq 0 \qquad \text{for all } y \in B - g(x_0)$$

is equivalent to

$$y'y \leq 0 \qquad \text{for all } y \in B \text{ and } y'g(x_0)=0.$$

This follows from the fact that

$$B \subseteq B - g(x_0),$$

if B is a convex cone.

In applications, it will generally be the case that F is the derivative of the cost function evaluated at the optimal point and that G and H are the derivatives of the functions defining the inequality and equality constraints, respectively, evaluated at the optimal point.

Norris (Ref. 10) and Weatherwax (Ref. 11), among others, have introduced constraint qualification in nonlinear programming problems in infinite-dimensional spaces. They are modeled after the well-known Kuhn–Tucker constraint qualifications common to finite-dimensional optimization. Such qualifications generally do not involve the cost function which is to be optimized. However, even in the classical Kuhn–Tucker theory, assumptions are made concerning the cost function, in particular differentiability. Thus, qualifications concerning the cost function are generally present, to some extent.

In the next theorem, qualifications are introduced which involve the cost function. It should be noted that the theorem does not apply explicitly to optimization, so that the terminology *cost function* is perhaps inaccurate. The qualification (3) will be shown to be, in some sense, necessary as well as sufficient for the existence of multipliers of the desired form. This is not the case with standard constraint qualifications.

Thus, the qualification to be introduced is not overly restrictive, in the sense of precluding application of the necessary conditions to optimization problems for want of satisfaction of the qualifying conditions.

Theorem 3.2. Let X and Y denote normed linear spaces; let

$$F: X \to R, \qquad G: X \to Y$$

denote linear functions. Assume

$$g: X \to Y$$

such that

$$g(x_0) \in B,$$

where B is a closed convex subset of Y with nonempty interior and $x_0 \in X$. If K is a convex subset of X such that $0 \in K$ and

$$K \cap G^{-1}[\text{int}\, B - g(x_0)] \subseteq F^+,$$

then there exists $\eta \in R$ and $y^* \in Y^*$, not both zero, such that

(i) $\eta F(k) + y^* G(k) \geq 0$ for all $k \in K$,
(ii) $\eta \geq 0$ and $y^* y \leq 0$ for all $y \in B - g(x_0)$.

Furthermore, if there exists $\eta \in R$ and $y^* \in Y^*$ such that $\eta > 0$ and (i) and (ii) are satisfied, then

$$K \cap G^{-1}[B - g(x_0)] \subseteq F^+.$$

Proof. To prove the second part, assume that

$$k \in K \cap G^{-1}[B - g(x_0)].$$

Then,

$$k \in K, \qquad G(k) \in B - g(x_0).$$

Thus,

$$y^* G(k) \leq 0, \qquad \eta F(k) \geq -y^* G(k) \geq 0.$$

So, since $\eta > 0$, it follows that

$$F(k) \geq 0.$$

To complete the proof, by Theorem 3.1 (a) and Remark 3.1 (i), it suffices to show that S_1 and T_1 can be separated by a continuous linear functional. To this end, observe that S_1 is convex, since K is convex and F and G are linear and T_1 is convex with

$$\text{int}\, T_1 = \{(r, y) : r < 0,\ y \in \text{int}\, B - g(x_0)\}.$$

Thus, it suffices to show that

$$S_1 \cap \text{int}\, T_1$$

is empty. Suppose the contrary. Then, there exists

$$k \in K$$

such that

$$F(k) < 0, \qquad G(k) \in \operatorname{int} B - g(x_0).$$

That is, there exists

$$k \in K \cap G^{-1}[\operatorname{int} B - g(x_0)]$$

such that

$$F(k) < 0.$$

This violates (3), and the proof is complete.

Remark 3.2. If B is taken to be a closed convex cone with nonempty interior, then the preceding theorem remains true even if F and G are not linear, but rather F is convex and G is B-convex. In this case, the condition $0 \in K$ can also be dispensed with [see Remarks 3.1 (ii)].

To establish the theorem under the additional condition that B is a cone, but F and G are convex, one must be careful in that S_1 may fail to be convex. However, it is easily seen that

$$(\operatorname{co} S_1) \cap \operatorname{int} T_1 = \varnothing.$$

4. Applications of Theorem 3.2

The usefulness of Theorem 3.2 will be demonstrated by obtaining results of Nagahisa and Sakawa (Ref. 12) and Das (Ref. 5). In both cases, the proofs presented here are more elementary and considerably shorter than the original proofs.

To establish the desired results, the following lemma is needed.

Lemma 4.1. Let X and Y denote normed linear spaces. Assume that

$$x_0 \in A \subseteq X$$

and

$$g : X \to Y$$

has a Neustadt derivative $Dg(x_0)$ at x_0. Also, assume that B is a closed convex subset of Y for which

$$g(x_0) \in B.$$

Then,

$$\mathrm{LC}[A; x_0] \cap Dg(x_0)^{-1}[\mathrm{int}\, B - g(x_0)] \subseteq \mathrm{LC}[A \cap g^{-1}(B); x_0].$$

Proof. Let

$$k \in \mathrm{LC}[A; x_0] \cap Dg(x_0)^{-1}[\mathrm{int}\, B - g(x_0)].$$

Then,

$$Dg(x_0)k \in \mathrm{int}\, B - g(x_0),$$

and there exist sequences (λ_n) of positive reals and (a_n) of elements of A such that

$$\lambda_n \uparrow \infty, \qquad a_n \to x_0, \qquad \lambda_n(a_n - x_0) \to k,$$

$$\lambda_n[g(a_n) - g(x_0)] = [g(x_0 + (1/\lambda_n)\lambda_n(a_n - x_0)) - g(x_0)]/(1/\lambda_n).$$

As $n \to \infty$, we have

$$1/\lambda_n \to 0^+ \qquad \text{and} \qquad \lambda_n(a_n - x_0) \to k.$$

Thus, as $n \to \infty$, we have

$$\lambda_n[g(a_n) - g(x_0)] \to Dg(x_0)k.$$

Now, $\mathrm{int}\, B - g(x_0)$ is open and

$$Dg(x_0)k \in \mathrm{int}\, B - g(x_0);$$

so, for n large,

$$\lambda_n[g(a_n) - g(x_0)] \in B - g(x_0).$$

This in turn implies that

$$g(a_n) \in (1/\lambda_n)B + (1 - 1/\lambda_n)g(x_0),$$

for n large. B is convex and $1/\lambda_n < 1$ for n large; so, we may conclude that

$$g(a_n) \in B$$

for n large. Thus, for n large,

$$a_n \in A \cap g^{-1}(B);$$

consequently,

$$k \in \mathrm{LC}[A \cap g^{-1}(B); x_0].$$

Similar versions of the preceding lemma were proved by Nagahisa and Sakawa (Ref. 12) and Das (Ref. 5). In both cases, more stringent conditions were imposed upon $Dg(x_0)$; in addition, Nagahisa and Sakawa required B to be a cone. The proof presented here is a slight variation of Nagahisa and Sakawa's original proof.

Theorem 4.1. Let X and Y denote normed linear spaces,

$$x_0 \in A \subseteq X,$$

and B a closed convex cone in Y with nonempty interior. Assume that

$$f: X \to R, \qquad g: X \to Y$$

have Neustadt derivatives at x_0 for which $Df(x_0)$ is convex and $Dg(x_0)$ is B-convex. Let K be a convex subset of $\text{LC}[A; x_0]$. If f has a min on $A \cap g^{-1}(B)$ at x_0, then there exists $\eta \in R$ and $y^* \in Y^*$, not both zero, such that

(i) $\eta Df(x_0)k + y^* Dg(x_0)k \geq 0$ for all $k \in K$,
(ii) $\eta \geq 0$ and $y^* y \leq 0$ for all $y \in B$,
(iii) $y^* g(x_0) = 0$.

Proof. By Lemma 4.1 and Theorem 2.1,

$$\text{LC}[A; x_0] \cap Dg(x_0)^{-1}[\text{int}\, B - g(x_0)] \subseteq Df(x_0)^+.$$

In particular,

$$K \cap Dg(x_0)^{-1}[\text{int}\, B - g(x_0)] \subseteq Df(x_0)^+.$$

Now, let

$$F = Df(x_0) \qquad \text{and } G = Dg(x_0),$$

and apply Theorem 3.2. The result follows immediately in view of Remarks 3.2 and 3.1 (iii).

In their original result, Nagahisa and Sakawa required that $Df(x_0)$ and $Dg(x_0)$ be linear and continuous and that zero be an element of K. The present proof consists essentially of showing that two relatively simple sets can be separated. Nagahisa and Sakawa obtained their result by constructing a relatively complicated cone and then showing that it could be supported.

Theorem 4.2. (*Das*). Let X and Y denote normed linear spaces,

$$x_0 \in A \subseteq X,$$

B a closed convex set in Y with nonempty interior,

$$f: X \to R, \qquad g: X \to Y.$$

Assume that g has a linear Neustadt derivative at x_0, and let K be a convex subset of $\mathrm{LC}[A; x_0]$ with $0 \in K$. Assume that $K[f; x_0]$ is nonempty and convex. If f has a min on

$$A \cap g^{-1}(B)$$

at x_0, then there exists $x^* \in X^*$ and $y^* \in Y^*$, not both zero, such that

(i) $x^*k + y^*Dg(x_0)k \geq 0$ for all $k \in K$,
(ii) $x^*x \leq 0$ for all $x \in K[f; x_0]$,
(iii) $y^*y \leq 0$ for all $y \in B - g(x_0)$.

Proof. By Theorem 2.2, there exists a nonzero $\varphi \in X^*$ such that

$$\mathrm{LC}[A \cap g^{-1}(B); x_0] \subseteq \varphi^+,$$

$$\varphi x \leq 0 \qquad \text{for all } x \in K[f; x_0].$$

So, by Lemma 4.1 and the fact that

$$K \subseteq \mathrm{LC}[A; x_0],$$

we have

$$K \cap Dg(x_0)^{-1}[\operatorname{int} B - g(x_0)] \subseteq \varphi^+.$$

Now, apply Theorem 3.2 with

$$F = \varphi, \qquad G = Dg(x_0).$$

The desired result follows with

$$x^* = \eta\varphi,$$

where η is given by Theorem 3.2.

5. Equality, Inequality, and Set Constraints

Theorem 5.1. Let X, Y, and Z denote normed linear spaces; and let

$$f: X \to R, \qquad g: X \to Y, \qquad h: X \to Z$$

have linear Neustadt derivatives at $x_0 \in X$. Let A be a subset of X such that

$$\mathrm{LC}[A; x_0] \cap N[Dh(x_0)] \subseteq \mathrm{LC}[A \cap N(h); x_0]. \tag{3}$$

Assume that B is a closed convex cone with nonempty interior, and let K denote any convex subset of $\mathrm{LC}[A; x_0]$ such that $\mathrm{cr}_{\mathrm{aff}}\, K$ is nonempty. If f has a min on

$$A \cap g^{-1}(B) \cap N(h)$$

at x_0, then there exists

$$\eta \in R, \qquad y^* \in Y^*, \qquad z' \in Z',$$

not all zero, such that

(i) $\eta Df(x_0)k + y^* Dg(x_0)k + z' Dh(x_0)k \geq 0$ for all $k \in K$,
(ii) $\eta \geq 0$ and $y^* y \leq 0$ for all $y \in B$,
(iii) $y^* g(x_0) = 0$.

Proof. In Theorem 3.1(b), let

$$F = Df(x_0), \qquad G = Dg(x_0), \qquad H = Dh(x_0).$$

It will first be shown that S_2 and T_2 of Theorem 3.1 can be separated by a linear functional. By Theorem 2.5,

$$\mathrm{cr}_{\mathrm{aff}}\, S_2 \neq \varnothing.$$

In addition,

$$\mathrm{int}_{\mathrm{aff}}\, T_2 = \{(r, y, 0) : r < 0,\ y \in \mathrm{int}\, B - g(x_0)\}$$

is nonempty, so

$$\mathrm{cr}_{\mathrm{aff}}\, T_2 = \mathrm{int}_{\mathrm{aff}}\, T_2 \neq \varnothing.$$

Thus, to show that S_2 and T_2 can be separated, it suffices to show, by Theorem 2.3, that

$$S_2 \cap \mathrm{cr}_{\mathrm{aff}}\, T_2 \neq \varnothing.$$

To this end, first note that, by Lemma 4.1,

$$Dg(x_0)^{-1}[\mathrm{int}\, B - g(x_0)] \cap \mathrm{LC}[A \cap N(h); x_0] \subseteq \mathrm{LC}[A \cap g^{-1}(B) \cap N(h); x_0].$$

This inclusion, together with (3) and Theorem 2.1, implies that

$$\mathrm{LC}[A; x_0] \cap Dg(x_0)^{-1}[\mathrm{int}\, B - g(x_0)] \cap N[Dh(x_0)] \subseteq Df(x_0)^+. \tag{4}$$

That S_2 and $\mathrm{cr}_{\mathrm{aff}}\, T_2$ are disjoint now follows easily from (4).

In view of Remarks 3.1(ii) and (iii), by Theorem 3.1(b) there exists

$$\eta \in R, \qquad y^* \in Y', \qquad z' \in Z',$$

not all zero, satisfying (i), (ii), and (iii). It remains to show that y^* is continuous. This follows immediately from the fact that, by (ii), y^* is bounded above on an open set, namely $\text{int}\, B - g(x_0)$.

Remark 5.1. (i) In general z' is not continuous [see Example 6.2 and Remarks 6.2(i)].

(ii) The constraint qualification defined by condition (3) cannot be eliminated (see Example 6.1).

(iii) The constraint qualification defined by (3) appears to have been first introduced in infinite-dimensional spaces by Weatherwax (Ref. 11) for handling equality and set constraints.

Two corollaries will now be presented, the first of which is due to Norris (Ref. 10).

Corollary 5.1. Let X, Y, and Z denote normed linear spaces; and let

$$f: X \to R, \qquad g: X \to Y, \qquad h: X \to Z$$

have linear Neustadt derivatives at $x_0 \in X$. Let B be a closed convex cone in Y with nonempty interior, and assume that

$$N[Dh(x_0)] \subseteq \text{LC}[N(h); x_0]. \tag{5}$$

If f has a min on

$$g^{-1}(B) \cap N(h)$$

at x_0, then there exists

$$\eta \in R, \qquad y^* \in Y^*, \qquad z' \in Z',$$

not all zero, such that

(i) $\eta Df(x_0)k + y^*Dg(x_0)k + z'Dh(x_0)k = 0$ for all $k \in X$,
(ii) $\eta \geq 0$ and $y^*y \leq 0$ for all $y \in B$,
(iii) $y^*g(x_0) = 0$.

Proof. In Theorem 5.1, let

$$A = X \qquad \text{and} \qquad K = X.$$

If condition (5) is satisfied, then obviously condition (3) is satisfied. Now, observe that, because (i) of Theorem 5.1 holds on all of X, the inequality may be replaced by equality due to linearity. The result follows immediately.

Remark 5.2. The constraint qualification defined by condition (5) cannot be eliminated (see Example 6.2).

Corollary 5.2. Let X, Y, and Z denote Banach spaces; and let

$$f: X \to R, \qquad g: X \to Y$$

have linear Neustadt derivatives at $x_0 \in X$. Assume that

$$h: X \to Z$$

has a Fréchet derivative which is continuous at x_0 and for which $Dh(x_0)$ has closed range. Let B denote a closed convex cone in Y with nonempty interior. If f has a min on

$$g^{-1}(B) \cap N(h)$$

at x_0, then there exists

$$\eta \in R, \qquad y^* \in Y^*, \qquad z^* \in Z^*,$$

not all zero, such that

(i) $\eta Df(x_0)k + y^* Dg(x_0)k + z^* Dh(x_0)k = 0$ for all $k \in X$,
(ii) $\eta \geq 0$ and $y^* y \leq 0$ for all $y \in B$,
(iii) $y^* g(x_0) = 0$.

Proof. Two cases are considered.

Case I. $Dh(x_0)$ not onto Y. By the Hahn–Banach theorem, there exists a nonzero $z^* \in Z^*$ which is zero on the range of $Dh(x_0)$. For this choice of z^*, and for η and y^* both zero, the conclusion follows.

Case II. $Dh(x_0)$ is onto Y. By Theorem 2.6, condition (5) is satisfied. So, by Corollary 5.2, there exists

$$\eta \in R, \qquad y^* \in Y^*, \qquad z^* \in Z',$$

not all zero, satisfying (i), (ii), (iii). It remains to show that z^* is continuous. By the open mapping theorem, $Dh(x_0)$ is open, and continuity of z^* follows easily.

Remark 5.3. The preceding corollary generalizes a result by Craven and Mond (Ref. 13, Theorem 4). Their necessary conditions are the same; but, in addition to the hypotheses of Corollary 5.2, they also find it necessary to require that, if $Dh(x_0)$ is onto Z, then either the algebraic complement of $N[Dh(x_0)]$ is closed or h is affine.

An abstract multiplier rule intended for application to combinations of set, equality, and inequality constraints will be presented next. A constraint qualification will be introduced, the motivation for which can be found in the proof of Theorem 5.1. A crucial step in the proof of that theorem was showing that

$$K \cap Dg(x_0)^{-1}[\operatorname{int} B - g(x_0)] \cap N[Dh(x_0)] \subseteq Df(x_0)^+.$$

This inclusion is the model for the constraint qualification of the next theorem.

Theorem 5.2. Let X, Y, Z denote normed linear spaces, and assume that

$$f: X \to R, \qquad g: X \to Y, \qquad h: X \to Z$$

have linear Neustadt derivatives at $x_0 \in X$. Let K denote a convex subset of X and B a closed cone in Y such that K and B have nonempty intrinsic cores and $g(x_0) \in B$. If

$$K \cap Dg(x_0)^{-1}[\operatorname{cr}_{\text{aff}} B - g(x_0)] \cap N[Dh(x_0)] \subseteq Df(x_0)^+, \tag{6}$$

then there exists

$$\eta \in R, \qquad y' \in Y', \qquad z' \in Z',$$

not all zero, such that

(i) $\eta Df(x_0)k + y'Dg(x_0)k + z'Dh(x_0)k \geq 0$ for all $k \in K$,
(ii) $\eta \geq 0$ and $y'y \leq 0$ for all $y \in B$,
(iii) $y'g(x_0) = 0$.

If there exists

$$\eta > 0, \qquad y' \in Y', \qquad z' \in Z'$$

satisfying (i), (ii), (iii), then

$$K \cap Dg(x_0)^{-1}[B - g(x_0)] \cap N[Dh(x_0)] \subseteq Df(x_0)^+.$$

Proof. First, suppose that there exists

$$\eta > 0, \qquad y' \in Y', \qquad z' \in Z'$$

satisfying (i), (ii), (iii). Let

$$k \in K \cap Dg(x_0)^{-1}[B - g(x_0)] \cap N[Dh(x_0)].$$

Then, there exists $k \in K$ such that

$$Dg(x_0)k + g(x_0) \in B, \qquad Dh(x_0)k = 0.$$

It follows that

$$0 \geq y'[Dg(x_0)k + g(x_0)] = y'Dg(x_0)k = y'Dg(x_0)k + z'Dh(x_0)k$$
$$\geq -\eta Df(x_0)k.$$

That is,

$$0 \geq -\eta Df(x_0)k,$$

and hence

$$k \in Df(x_0)^+.$$

To prove the first part of the theorem, in Theorem 3.1 let

$$F = Df(x_0), \qquad G = Dg(x_0), \qquad H = Dh(x_0).$$

In view of Theorem 3.1 and Remarks 3.1(ii) and (iii), it suffices to show that S_2 and T_2 can be separated by a linear functional. S_2 is convex and, by Theorem 2.5, has a nonempty intrinsic core. T_2 is also convex and

$$\mathrm{cr}_{\mathrm{aff}}\, T_2 = \{(r, y, 0) : r < 0,\ y \in \mathrm{cr}_{\mathrm{aff}}\, B - g(x_0)\}.$$

It is easily verified that S_2 and $\mathrm{cr}_{\mathrm{aff}}\, T_2$ are disjoint. Thus, by Theorem 2.3, S_2 and T_2 can be separated.

Remark 5.4. (i) Condition (6) cannot be eliminated (see Example 6.1).

(ii) In general, z' is not continuous (see Example 6.3).

Remark 5.5. (i) If the constraint is of the form

$$g^{-1}(B) \cap N(h),$$

then take

$$K = X$$

and replace the inequality in condition (i) by equality.

(ii) If the constraint is of the form

$$A \cap N(h),$$

delete all reference to B, Y, η, y' in the hypotheses and conclusions of the theorem. To see this, take

$$X = Y = B$$

and g equal to the identity. $Dg(x_0)$ is the identity, and the only choice for y' is zero.

(iii) If the constraint is of the form

$$A \cap g^{-1}(B),$$

then delete all reference to Z, h, z'. To see this, in the proof of the theorem replace S_2 and T_2 by S_1 and T_1, respectively.

The final theorem to be presented is a corollary to Theorem 5.2. It is essentially due to Weatherwax (Ref. 11, Theorem 3.1). Also see Remark 5.6. The proof presented is actually similar to Weatherwax's original proof.

Corollary 5.3. Let X and Z denote normed linear spaces, and assume that

$$f : X \to R \qquad \text{and} \quad h : X \to Z$$

have linear Neustadt derivatives at $x_0 \in X$. Assume that A and K are subsets of X and that K is a convex cone with a nonempty relative interior and

$$K \cap N[Dh(x_0)] \subseteq \text{cl}(\text{co LC}[A \cap N(h); x_0]). \tag{7}$$

If f has a min on

$$A \cap N(h)$$

at x_0, then there exists

$$\eta \geq 0, \qquad z' \in Z',$$

not both zero, such that

$$\eta Df(x_0)k + z'Dh(x_0)k \geq 0 \qquad \text{for all } k \in K.$$

Proof. Let

$$F = Df(x_0), \qquad H = Dh(x_0).$$

Note that $\text{cr}_{\text{aff}}\, K$ is nonempty. By Theorem 5.2 and in view of Remark 5.5(ii), to establish the result it suffices to show that

$$K \cap N[Dh(x_0)] \subseteq Df(x_0)^+. \tag{8}$$

By Theorem 2.1 and the linearity of $Df(x_0)$,

$$\text{cl}(\text{co LC}[A \cap N(h); x_0]) \subseteq Df(x_0)^+.$$

Combining this inclusion with (7), it is seen that (8) holds, and the proof is complete.

Remark 5.6. In his original statement of the preceding corollary, Weatherwax asserted the existence of a continuous linear multiplier z^*, as opposed to the linear (possibly noncontinuous) multiplier z' given above.

In Example 6.3, it is demonstrated that, under the given hypotheses, the multiplier may fail to be continuous.

6. Examples

Three examples will be presented to demonstrate that various hypotheses in the preceding theorems cannot be omitted and still have the necessary conditions hold. The details are left to the reader as the properties ascribed to the examples are easily verified.

Example 6.1. Let

$$X = R^2, \qquad Z = R, \qquad B = Y = X,$$

and define

$$f: X \to R, \qquad g: X \to Y, \qquad h: X \to Z$$

as follows:

$$f(a, b) = -a, \qquad g(a, b) = (a, b), \qquad h(a, b) = (1/2)a^2 - b,$$

for $(a, b) \in X$. Define $C \subseteq X$ by

$$C = \{(a, b) : a^2 > b > 0 \text{ and } a > 0\},$$

and define $A \subseteq X$ by

$$A = X \sim C.$$

The following properties can be shown to hold:

(i) f has a min on $A \cap N(h) = A \cap g^{-1}(B) \cap N(h)$ at $x_0 = (0, 0)$;
(ii) $\mathrm{LC}[A; x_0] = X$;
(iii) $\mathrm{LC}[A; x_0] \cap N[Dh(x_0)] \not\subseteq Df(x_0)^+$;

thus, by Theorem 2.1,

$$\mathrm{LC}[A; x_0] \cap N[Dh(x_0)] \not\subseteq \mathrm{LC}[A \cap N(h); x_0].$$

(iv) let $K = \mathrm{LC}[A; x_0]$; if $\eta \in R$, $z' \in Z'$, $y^* \in Y^*$ satisfy (i), (ii), (iii) of Theorem 5.1, then each of η, z', y^* is zero.

Remark 6.1. The preceding example serves to demonstrate the following points:

(i) The condition $\operatorname{int} B \neq \varnothing$ cannot be eliminated from Theorem 4.1 nor Theorem 3.2. To see this, treat h as defining an inequality constraint with the constraint cone equal to $\{0\}$.

(ii) The constraint qualification

$$\mathrm{LC}[A; x_0] \cap N[Dh(x_0)] \subseteq \mathrm{LC}[A \cap N(h); x_0]$$

cannot be eliminated from Theorem 5.1.

(iii) The qualification

$$K \cap Dg(x_0)^{-1}[\mathrm{cr}_{\mathrm{aff}}\, B - g(x_0)] \cap N[Dh(x_0)] \subseteq Df(x_0)^+$$

cannot be eliminated from Theorem 5.2.

Example 6.2. Let

$$X = Z = l_2, \qquad Y = l_\infty.$$

Define

$$f: X \to R, \qquad g: X \to Y, \qquad h: X \to Z$$

as follows:

$$f(x) = \sum_n 2x_n/n, \qquad g(x) = x, \qquad h(x) = (x_n^2 - 1/n^2),$$

where

$$x = (x_n).$$

Let B be the set of all $(y_n) \in Y$ such that $y_n \geq 0$ for all n. The following properties can be shown to hold:

(i) f has a min on $g^{-1}(B) \cap N(h)$ at $x_0 = (1/n)$;
(ii) $N[Dh(x_0)] \subseteq \mathrm{LC}[N(h); x_0]$;
(iii) $\mathrm{int}\, B \neq \varnothing$;
(iv) if η, y^*, z' satisfy the conclusion of Corollary 5.2, then z' is not continuous;
(v) the range of $Dh(x_0)$ is not closed.

Remark 6.2. The preceding example serves to illustrate the following points:

(i) The multiplier z' in Corollary 5.1 need not be continuous.

(ii) The multiplier z' in Theorem 5.1 need not be continuous. To see this, let

$$A = X, \qquad K = \mathrm{LC}[A; x_0].$$

(iii) The condition that the range of $Dh(x_0)$ is closed cannot be eliminated from Corollary 5.2.

Example 6.3. Let

$$X = Z = l_2.$$

Let

$$A \subseteq X$$

be the set of all $(x_n) \in X$ such that

$$x_n \geq 0$$

for each n; and let

$$K = \mathrm{LC}[A; x_0].$$

Define

$$f: X \to R, \qquad h: X \to Z$$

as follows:

$$f(x) = \sum_n 2x_n/n, \qquad h(x) = (x_n^2 - 1/n^2),$$

for $x = (x_n)$. The following properties can be shown to hold:

(i) f has a min on $A \cap N(h)$ at $x_0 = (1/n)$;

(ii) K is a convex cone with nonempty relative interior; in fact, $K = X$, so K has a nonempty interior;

(iii) $K \cap N[Dh(x_0)] \subseteq \mathrm{LC}[A \cap N(h); x_0]$;

(iv) for any η and z' satisfying the conclusion of Corollary 5.4, z' is not continuous.

Remark 6.3. The preceding example serves to illustrate the following points:

(i) The multiplier z' of Corollary 5.3 need not be continuous.

(ii) The multiplier z' of Theorem 5.2 need not be continuous. To apply the example to Theorem 5.2, let

$$B = Y = X,$$

and let g be the identity.

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