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Numerical Solution of a Time-Optimal Parabolic Boundary-Value Control Problem[1]

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Abstract. A special time-optimal parabolic boundary-value control problem describing a one-dimensional heat-diffusion process is solved numerically. Using a bang–bang principle recently proved by Lempio, this problem can be transformed in such a way that the variables are jumps of bang–bang controls. A discretization is performed in two steps, and the convergence of the approximate solutions is proved. Finally, an algorithm to solve the discrete problem is developed and some numerical results are discussed.

Key Words. Optimal control, boundary-value problems, discretization, nonlinear programming.

1. Problem

In this paper, we try to solve numerically a time-optimal control problem resulting from a special one-dimensional heat-diffusion process. A thin rod is heated at one endpoint in such a way that a given temperature distribution is to be approximated with a given accuracy as soon as possible. Denote by $y(s, t)$ the temperature at a point $s \in [0, 1]$ at time t. Then, we can describe this process through the following problem.

Problem. Minimize the time T under the restrictions that there is a $u \in L_\infty[0, T]$ with

$$y_t(s, t) - y_{ss}(s, t) = 0,$$

$$y_s(0, t) = 0, \qquad y(s, 0) = 0,$$

$$y(1, t) + \alpha y_s(1, t) = u(t),$$

$$|y(s, T) - k_0(s)| \leq \text{eps},$$

$$-1 \leq u(t) \leq 1.$$

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Here, $t \in [0, T]$; $s \in [0, 1]$; $k_0 \in C[0, 1]$ is the given temperature distribution; eps > 0 the given accuracy; and $\alpha > 0$ a constant heat transfer coefficient. For every $u \in L_\infty[0, T]$, we can determine the solution $y(s, t, u)$ of the above boundary-value problem. Yegorov (Ref. 1) has shown that it is equal to

$$y(s, t, u) = \sum_{j=1}^{\infty} A_j \mu_j^2 \cos(\mu_j s) \int_0^t u(\tau) \exp(-\mu_j^2 (t-\tau))\, d\tau, \tag{1}$$

where

$$A_j := 2 \sin \mu_j / (\mu_j + \sin \mu_j \cos \mu_j), \qquad j = 1, 2, \ldots .$$

and $\{\mu_j\}$ is the sequence of all positive solutions of the equation

$$\mu \tan \mu = 1/\alpha.$$

$y(s, t, u)$ is continuous on $[0, 1]$ as a function of s, and our time-optimal control problem is equivalent to the following problem.

Problem (SP):

$$\min T,$$
$$T, u: \|y(\cdot, T, u) - k_0(\cdot)\|_\infty \leq \text{eps},$$
$$\|u\|_\infty \leq 1,$$
$$u \in L_\infty[0, T].$$

We know from Weck (Ref. 2) that Problem (SP) is solvable and we can assume that the minimal control time T_0 is positive. For the subsequent proofs of the convergence of the discretized problems, we need compact feasible sets. Therefore, let us assume (without loss of generality) that there is a $\mu > 0$ with the following property: each $T > 0$, for which a feasible control with respect to Problem (SP) exists is less than or equal to μ.

Lempio (Ref. 3) has shown that the following bang–bang principle is valid.

Theorem 1.1. Let $T_0 > 0$ be the minimal time and u_0 a time-optimal control of Problem (SP). Then, u_0 is uniquely determined and piecewise constant, having the values -1 and 1 with finitely many jumps on any interval $[0, t]$, $0 < t < T_0$.

This means that the jumps of the optimal bang–bang control accumulate at most in T_0. This property of the optimal solution is fundamental further on, because we can consider now only such feasible solutions of Problem (SP) which possess the above bang–bang character. So we obtain a problem whose variables are jumps of bang–bang solutions. We define, therefore,

$$E := \{(T, t_1, t_2, \dots): T, t_i \in \mathbb{R}, \lim_{i\to\infty} t_i = T\}.$$

E is a linear space and is normed by

$$\|(T, t_1, \dots)\|_E := \max\{|T|, |t_i| : i = 1, 2, \dots\}.$$

Using the previously mentioned bang–bang principle, we see that Problem (SP) is equivalent to the following problem.

Problem (P):

$$\min T,$$
$$(T, t_1, \dots) \in X,$$
$$\varphi(T, t_1, \dots) \in K,$$

where

$$X := \{(T, t_1, \dots) \in E: 0 \leqslant t_1 \leqslant \cdots \leqslant T \leqslant \mu\},$$
$$K := \{v \in C[0, 1]: \|v - k_0\|_\infty \leqslant \text{eps}\},$$

and $\varphi: X \to C[0, 1]$ is defined by

$$\varphi(T, t_1, \dots) := y(\cdot, T, u(T, t_1, \dots)).$$

The function $u(T, t_1, \dots) \in L_\infty[0, T]$ is bang–bang with jumps at $t_1, t_2, \dots$, more precisely,

$$u(T, t_1, \dots)(\tau) := (-1)^{i+1}, \qquad \tau \in [t_{i-1}, t_i), \qquad i \in \mathbb{N}, \qquad t_0 := 0.$$

In the following section, we approximate Problem (P) and prove some convergence theorems. In Section 3, we develop an algorithm to solve the discrete problem; Section 4 contains the numerical results.

It should be mentioned that the bang–bang principle expressed by Theorem 1.1 is also valid for any L_p-norm, $p \geqslant 1$ (see Glashoff, Ref. 4). So it is possible to formulate Problem (SP) for arbitrary L_p-norms and discretize this problem in a similar way.

Analogous heat-diffusion processes, but with a constant control time T, were solved numerically by Sakawa (Ref. 6), Glashoff and Gustavson (Ref. 5), and Sachs (Ref. 7). Other analyses of such problems can be found in the papers of Butkovskiy (Refs. 8, 9), Wypych (Ref. 10), Weck (Ref. 2), Glashoff (Ref. 4), Friedmann (Ref. 11), and Fattorini (Refs. 12, 13). Fattorini (Ref. 14) also succeeded in developing a bang–bang principle for the exact final-value problem (i.e., for the case eps = 0).

2. Discretization of the Problem

In order to compute a solution of Problem (P) that is at least approximate, we must examine a discretization of the problem. This will be done in two steps. The first discretization will be performed with respect to the number of jumps; that is, we construct Problem (P_k) which permits only k jumps of the feasible controls. For this, we define for $k \in \mathbb{N}$

$$X_k := \{(T, t_1, \ldots, t_k) \in \mathbb{R}^{k+1}: 0 \leq t_1 \leq \cdots \leq t_k \leq T \leq \mu\},$$

$$\varphi_k: X_k \to C[0, 1],$$

$$\varphi_k(T, t_1, \ldots, t_k) := \varphi(T, t_1, \ldots, t_k, T, \ldots).$$

So we get for each k the following problem.

Problem (P_k):

$$\min T,$$
$$(T, t_1, \ldots, t_k) \in X_k,$$
$$\varphi_k(T, t_1, \ldots, t_k) \in K.$$

For the proof of the subsequent convergence theorems, we need the following more restrictive feasibility statement, which could be characterized as a Slater condition.

Theorem 2.1. Let T_0 be the optimal time of Problem (P). Then, there is, for every $T_1 > T_0$, a bang–bang control with jumps at $t_1, t_2, \ldots$, i.e., a $(T_1, t_1, \ldots) \in X$, with

$$\varphi(T_1, t_1, \ldots) \in \mathring{K},$$

where $\mathring{K}$ is the set of all interior points of K.

Proof. Let $(T_0, t_1^0, \ldots) \in X$ be the optimal solution of Problem (P). Then, T_0 and $u_0 := u(T_0, t_1^0, \ldots)$ are optimal for Problem (SP); specifically, we get

$$\|y(\cdot, T_0, u_0) - k_0(\cdot)\|_\infty \leq \text{eps}, \qquad \|u_0\|_\infty \leq 1. \tag{2}$$

Choose a $T_1 > T_0$ and define

$$u_0^*(s) := \begin{cases} 0, & s \in [0, T_1 - T_0), \\ u_0(s - T_1 + T_0), & s \in [T_1 - T_0, T_1]. \end{cases} \tag{3}$$

Since

$$\begin{aligned}\int_0^{T_0} & u_0(\tau)\exp(-\mu_j^2(T_0-\tau))\,d\tau \\ &= \int_{T_1-T_0}^{T_1} u_0(\sigma-T_1+T_0)\exp(-\mu_j^2(T_1-\sigma))\,d\sigma \\ &= \int_0^{T_1} u_0^*(\sigma)\exp(-\mu_j^2(T_1-\sigma))\,d\sigma \end{aligned} \tag{4}$$

for each j, we have

$$y(\cdot, T_0, u_0) = y(\cdot, T_1, u_0^*),$$

and

$$\|y(\cdot, T_1, u_0^*) - k_0(\cdot)\|_\infty \leq \text{eps}, \qquad \|u_0^*\|_\infty \leq 1. \tag{5}$$

Consider now the following minimum-norm problem.

Problem (MN): $\min \|y(\cdot, T_1, u) - k_0(\cdot)\|_\infty,$

$$u: \|u\|_\infty \leq 1,$$

$$u \in L_\infty[0, T_1].$$

Glashoff (Ref. 4) has shown that the optimal solution of this problem is bang–bang, uniquely determined, and that the jumps accumulate at most in T_1. u_0^* is a feasible control for Problem (MN) with

$$\|y(\cdot, T_1, u_0^*) - k_0(\cdot)\|_\infty \leq \text{eps},$$

but is not of bang–bang type. So we conclude that the unique optimal bang–bang solution u_1 of Problem (MN) leads to the estimate

$$\|y(\cdot, T_1, u_1) - k_0(\cdot)\|_\infty < \text{eps}. \tag{6}$$

Let $t_1, t_2, \ldots$ be the jumps of u_1 such that $u_1 = u(T_1, t_1, \ldots)$. So we get finally a $(T_1, t_1, \ldots) \in X$ with

$$\varphi(T_1, t_1, \ldots) \in \mathring{K}.$$

Now, we show that the function φ is continuous on X.

Lemma 2.1. The function $\varphi: X \to C[0, 1]$ is continuous on X.

Proof. We define a control operator

$$S_T: L_p[0, T] \to C[0, 1], \qquad T > 0, \qquad p > 2,$$

by

$$S_T u(s) := y(s, T, u) \tag{7}$$

for every $u \in L_p[0, T]$ and $s \in [0, 1]$. Let $t := (T, t_1, \dots) \in X$, and choose an arbitrary $\rho > 0$. Denote $\rho^* := (\rho/2)^p$. We can assume that $T - \rho^*/2$ is not a jump of $u(t)$ and that $T - \rho^*/2 > 0$. Then, there is $k(\rho) \in \mathbb{N}$ with

$$0 \leq t_1 \leq \cdots \leq t_{k(\rho)} < T - \rho^*/2 < t_{k(\rho)+1} \leq \cdots \leq T.$$

Let $t_0 := 0$ and

$$\delta := \min\{\rho^*/2k(\rho),\ T - \rho^*/2 - t_{k(\rho)},\ t_{k(\rho)+1} - T + \rho^*/2,\ \tfrac{1}{2}(t_j - t_{j-1}) : j = 1, \dots, k(\rho)\}.$$

For

$$t^* := (T, t_1^*, \dots) \in X \qquad \text{with } \|t - t^*\|_E < \delta,$$

we get the estimates

$$\begin{aligned} \|u(t) - u(t^*)\|_p^p &\leq \int_0^{T-\rho^*/2} |u(t)(\tau) - u(t^*)(\tau)|^p \, d\tau + 2^p \rho^*/2 \\ &= \Big(\sum_{j=1}^{k(\rho)} |t_j - t_j^*| + \rho^*/2\Big) 2^p \\ &\leq 2^p \rho^*, \end{aligned} \tag{8}$$

or

$$\|u(t) - u(t^*)\|_p \leq 2\rho^{*1/p} = \rho. \tag{9}$$

Here, $\|\cdot\|_p$ denotes the p-norm on $L_p[0, 1]$. Glashoff and Gustafson (Ref. 5) have shown that, for the linear operator S_T, the following relation is valid:

$$\|S_T u\|_\infty \leq \chi \|u\|_p \qquad \text{for all } u \in L_p[0, T]. \tag{10}$$

The constant χ is independent of u and T.

Now, we can prove the continuity of φ. Let $(T, t_1, \dots) \in X$ and $\epsilon > 0$. For $\rho := \epsilon/2\chi$, we get a $\delta > 0$ such that, for every

$$t^* := (T, t_1^*, \dots) \in X \qquad \text{with } \|t - t^*\|_E < \delta,$$

we have

$$\|u(t) - u(t^*)\|_p \leq \epsilon/2\chi. \tag{11}$$

Select

$$t' := (T', t_1', \dots) \in X \qquad \text{with } \|t - t'\|_E < \delta/2.$$

It is easy to see that

$$S_T u(T, t_1, \ldots) = S_{T'} u(T', t_1 + T' - T, \ldots) - S_{T'} u^*(T'), \qquad \text{if } T \leq T',$$
$$S_{T'} u(T', t_1', \ldots) = S_T u(T, t_1' + T - T', \ldots) - S_T u^*(T), \qquad \text{if } T > T',$$

where $u^*(T')(s) = 1$, $s \in [0, T' - T]$, and $u^*(T')(s) = 0$, otherwise. If, for example, $T \leq T'$, we conclude this from

$$\int_0^T u(T, t_1, \ldots)(\tau) \exp(-\mu_j^2 (T - \tau))\, d\tau$$
$$= \int_{T'-T}^{T'} u(T, t_1, \ldots)(\tau - T' + T) \exp(-\mu_j^2 (T' - \tau))\, d\tau$$
$$= \int_{T'-T}^{T'} u(T', t_1 + T' - T, \ldots)(\tau) \exp(-\mu_j^2 (T' - \tau))\, d\tau. \tag{12}$$

Since

$$\|t - t'\|_E < \delta/2,$$
$$\|(T, t_1' + T - T', \ldots) - (T, t_1, \ldots)\|_E \leq \|t' - t\|_E + |T - T'| < \delta,$$

we get from (11) that

$$\begin{aligned} &\|u(T', t_1 + T' - T, \ldots) - u(t')\|_p \\ &\quad = \|u(t) - u(T, t_1' + T - T', \ldots)\|_p \\ &\quad \leq \epsilon/2\chi, \qquad \text{if } T \leq T', \\ &\|u(T, t_1' + T - T', \ldots) - u(t)\|_p \leq \epsilon/2\chi, \qquad \text{if } T > T'. \end{aligned} \tag{13}$$

Now, we establish the estimates

$$\begin{aligned} \|\varphi(t) - \varphi(t')\|_\infty &= \|S_T u(t) - S_{T'} u(t')\|_\infty \\ &= \begin{cases} \|S_{T'} u(T', t_1 + T' - T, \ldots) - S_{T'} u(t') - S_{T'} u^*(T')\|_\infty, & \text{if } T \leq T', \\ \|S_T u(t) - S_T u(T, t_1' + T - T', \ldots) + S_T u^*(T)\|_\infty, & \text{if } T > T', \end{cases} \\ &\leq \begin{cases} \chi \|u(T', t_1 + T' - T, \ldots) - u(t')\|_p + \chi (T' - T)^{1/p} & \text{if } T \leq T', \\ \chi \|u(t) - u(T, t_1' + T - T', \ldots)\|_p + \chi (T - T')^{1/p} & \text{if } T > T', \end{cases} \\ &\leq \epsilon/2 + \chi(\delta/2)^{1/p} \leq \epsilon. \end{aligned}$$

This completes the proof, since we assume that $\chi(\delta/2)^{1/p} \leq \epsilon/2$.

The continuity property of the function φ will be very important for proving the convergence of the optimal value of Problem (P_k) to the optimal value of Problem (P).

Theorem 2.2. Let $T_0 > 0$ and T_0^k be the minimal times of Problems (P) and (P_k), respectively. Then,

$$\lim_{k\to\infty} T_0^k = T_0.$$

Proof. We choose $k \in \mathbb{N}$ and denote by

$$(T_0^k, t_1^k, \ldots, t_k^k) \in X_k, \qquad (T_0, t_1^0, \ldots) \in X,$$

the optimal controls of Problems (P) and (P_k), respectively.

Since

$$(T_0^k, t_1^k, \ldots, t_k^k, T_0^k) \in X_{k+1}, \qquad (T_0^k, t_1^k, \ldots, t_k^k, T_0^k, \ldots) \in X,$$

and

$$\varphi_{k+1}(T_0^k, t_1^k, \ldots, t_k^k, T_0^k) = \varphi(T_0^k, t_1^k, \ldots, t_k^k, T_0^k, \ldots)$$
$$= \varphi_k(T_0^k, t_1^k, \ldots, t_k^k) \in K,$$

it follows that

$$T_0^k \geq T_0^{k+1} \geq \cdots \geq T_0 \qquad \text{for all } k \in \mathbb{N}. \tag{14}$$

Now, we show that there is a subsequence of $\{T_0^k\}$ converging to T_0. For an arbitrary $T > T_0$, Theorem 2.1 yields a bang–bang control with jumps at $t := (T, t_1, \ldots) \in X$ and with $\varphi(t) \in \mathring{K}$. Since

$$\lim_{k\to\infty} (T, t_1, \ldots, t_k, T, \ldots) = (T, t_1, \ldots),$$

and φ is continuous on X, we obtain $k' \in \mathbb{N}$ such that

$$\varphi_k(T, t_1, \ldots, t_k) = \varphi(T, t_1, \ldots, t_k, T, \ldots) \in K, \tag{15}$$

for all $k \geq k'$. So $(T, t_1, \ldots, t_{k'})$ is feasible for Problem $(P_{k'})$, i.e.,

$$T \geq T_0^{k'} \geq T_0. \tag{16}$$

Because T was chosen arbitrarily, we get a subsequence of $\{T_0^k\}$ converging to T_0. This proves the statement.

It is not possible to solve Problem (P_k) numerically. So we perform a second discretization which allows us to compute approximate solutions of Problem (P_k). Therefore, we consider the following problem.

Problem (P_k^n):

$$\min T,$$
$$(T, t_1, \ldots, t_k) \in X_k,$$
$$\varphi_k^{l_n}(T, t_1, \ldots, t_k) \in K_n,$$

with

$$K_n := \{v \in C[0, 1]: \max_{1 \le i \le n} |v(s_i) - k_0(s_i)| \le \text{eps}, s_i := (i-1)/(n-1)\},$$

$$\varphi_k^{l_n}: X_k \to C[0, 1],$$

$$\varphi_k^{l_n}(t)(s) := \sum_{j=1}^{l_n} \mu_j^2 A_j \cos(\mu_j s) \int_0^T u(t)(\tau) \exp(-\mu_j^2(T-\tau))\, d\tau,$$

for all $t := (T, t_1, \ldots, t_k) \in X_k$. $\{l_n\}$ is a monotone increasing sequence of positive integers with

$$\lim_{n\to\infty} l_n = \infty.$$

As a preparation for the subsequent convergence theorem, we establish two lemmas.

Lemma 2.2. Assume that there is a $v \in C[0, 1]$ and a sequence of functions $v_n \in C[0, 1]$ converging to v uniformly on $[0, 1]$. Then,

$$\lim_{n\to\infty} \max_{1 \le i \le n} v_n(s_i) = \|v\|_\infty,$$

with

$$s_i := (i-1)/(n-1), \qquad i = 1, \ldots, n.$$

Proof. Define $s_0, s_{i_n} \in [0, 1]$, by

$$\max_{0 \le s \le 1} v(s) = v(s_0), \tag{17}$$

$$\max_{1 \le i \le n} v_n(s_i) = v_n(s_{i_n}). \tag{18}$$

Let $s^* \in [0, 1]$ be an accumulation point of $\{s_{i_n}\}$; i.e., there is a set $M \subset \mathbb{N}$, $|M| = \infty$, with

$$\lim_{n \in M} s_{i_n} = s^*.$$

We show that v has a maximum at s^*. For this, we assume that there is an $s' \in [0, 1]$ with

$$v(s') > v(s^*).$$

Define

$$\delta := \tfrac{1}{2}(v(s') - v(s^*)).$$

The uniform convergence of $\{v_n\}$ and the continuity of v implies that

$$\lim_{n \in M} v_n(s_{i_n}) = v(s^*).$$

So we have

$$v(s^*) > v_n(s_{i_n}) - \delta \qquad \text{for all } n > n^*,\ n \in M.$$

For s', we get a sequence

$$s_{j_n} := (j_n - 1)/(n-1), \qquad 1 \leq j_n \leq n, \qquad n \in M,$$

with

$$\lim_{n \in M} s_{j_n} = s'.$$

Since

$$\lim_{n \in M} v_n(s_{j_n}) = v(s'),$$

there is an $n' \geq n^*$ such that

$$v(s') < v(s_{j_n}) + \delta \tag{19}$$

for all $n \in M$, $n > n'$. Therefore,

$$v_n(s_{j_n}) + \delta > v(s') = v(s^*) + 2\delta > v_n(s_{i_n}) + \delta,$$

or

$$v_n(s_{j_n}) > v_n(s_{i_n}) = \max_{1 \leq i \leq n} v_n(s_i). \tag{20}$$

This is a contradiction; we get

$$\lim_{n \in M} v(s_{i_n}) = \|v\|_\infty.$$

It is easy to see that now

$$\lim_{n\to\infty} v_n(s_{i_n}) = \|v\|_\infty.$$

Lemma 2.3. Let $g, g_n: X_k \to \mathbb{R}$ be defined by

$$g(t) := \|\varphi_k(t) - k_0\|_\infty,$$

$$g_n(t) := \max_{1\le i\le n} |\varphi_k^{l_n}(t)(s_i) - k_0(s_i)|,$$

$$s_i := (i-1)/(n-1), \qquad i = 1, \ldots, n.$$

Then, the following relation is valid:

$$\lim_{n\to\infty} \sup_{t\in X_k} |g(t) - g_n(t)| = 0.$$

Proof. g and g_n are continuous on X_k, because the functions φ_k and $\varphi_k^{l_n}$ are continuous on X_k. Since X_k is compact, there exists, for every $n \in \mathbb{N}$,

$$t_n := (T_n, t_1^n, \ldots, t_k^n) \in X_k$$

with

$$\max_{t\in X_k} |g(t) - g_n(t)| = |g(t_n) - g_n(t_n)|. \tag{21}$$

Let $t^* \in X_k$ be an accumulation point of $\{t_n\}$, i.e., there is a set $M \subset \mathbb{N}$, $|M| = \infty$, with

$$\lim_{n\in M} t_n = t^*.$$

Define

$$v(s) := |\varphi_k(t^*)(s) - k_0(s)|,$$

$$v_n(s) := |\varphi_k^{l_n}(t_n)(s) - k_0(s)|,$$

$$s \in [0, 1], \qquad n \in \mathbb{N}.$$

From

$$\begin{aligned}|v(s) - v_n(s)| &\le |\varphi_k(t^*)(s) - \varphi_k^{l_n}(t_n)(s)| \\ &\le |\varphi_k(t^*)(s) - \varphi_k(t_n)(s)| + |\varphi_k(t_n)(s) - \varphi_k^{l_n}(t_n)(s)| \\ &\le \|\varphi_k(t^*) - \varphi_k(t_n)\|_\infty \\ &\quad + \sum_{j=l_n+1}^{\infty} |A_j|\mu_j^2 \left| \int_0^{T_n} u(t_n)(\tau) \exp(-\mu_j^2(T_n - \tau))\, d\tau \right| \end{aligned} \tag{22}$$

and the existence of $\sum_{j=1}^{\infty} |A_j|$ follows the uniform convergence of $\{v_n\}_{n\in M}$ to v.

Lemma 2.2 shows that

$$\lim_{n\in M} \max_{1\leq i\leq n} v_n(s_i) = \|v\|_\infty,$$

$$\lim_{n\in M} g_n(t_n) = g(t^*).$$

Further,

$$\sup_{t\in X_k} |g(t) - g_n(t)| = |g(t_n) - g_n(t_n)|$$

$$\leq |g(t_n) - g(t^*)| + |g(t^*) - g_n(t_n)| \xrightarrow{n\in M} 0. \tag{23}$$

Since this is valid for each accumulation point of $\{t_n\}$, the statement of the lemma is proved.

Now, we are able to show the convergence of the optimal value of Problem (P_k^n) to the optimal value of Problem (P).

Theorem 2.3. Let T_0^{kn} be the minimal time of Problem (P_k^n), with $n, k \in \mathbb{N}$, and T_0 be the minimal time of Problem (P). Then,

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}}\, T_0^{kn} = \lim_{k\to\infty} \underline{\lim_{n\to\infty}}\, T_0^{kn} = T_0. \tag{24}$$

Proof. Choose a $k \in \mathbb{N}$, and assume that there is an accumulation point T_k^* of $\{T_0^{kn}\}_{n\in\mathbb{N}}$ with $T_k^* < T_0^k$, where T_0^k is the minimal control time of Problem (P_k). Then, there is a set $M \subset \mathbb{N}$, $|M| = \infty$, with

$$\lim_{n\in M} T_0^{kn} = T_k^* < T_0^k.$$

For every $n \in M$, we determine an optimal bang–bang control of Problem (P_k^n), with jumps at

$$t_0^{kn} := (T_0^{kn}, t_1^{kn}, \ldots, t_k^{kn}) \in X_k.$$

X_k is compact. Therefore, a set $M' \subset M$, $|M'| = \infty$, and a

$$t' := (T', t_1', \ldots, t_k') \in X_k$$

exist with

$$\lim_{n\in M'} t_0^{kn} = t'.$$

From Lemma 2.3, we see that

$$\lim_{n \in M'} g_n(t_0^{kn}) = g(t'). \tag{25}$$

Since

$$g_n(t_0^{kn}) \in [0, \text{eps}] \qquad \text{for all } n \in M',$$

it follows that

$$g(t') \in [0, \text{eps}].$$

So t' is feasible for Problem (P_k). This implies a contradiction, because

$$\lim_{n \in M'} T_0^{kn} = T' \geq T_0^k > T_k^* = \lim_{n \in M} T_0^{kn}.$$

We conclude that, for every $k \in \mathbb{N}$,

$$\delta_k := \underline{\lim}_{n \to \infty} T_0^{kn} \geq T_0^k \geq T_0. \tag{26}$$

Let $T > T_0$ be chosen arbitrarily. Theorem 2.1 shows that there is

$$t := (T, t_1, \ldots) \in X \qquad \text{with } \varphi(t) \in \mathring{K}.$$

Define

$$\bar{t}^k := (T, t_1, \ldots, t_k, T, \ldots) \in X,$$
$$t^k := (T, t_1, \ldots, t_k) \in X_k.$$

From $\lim_{k \to \infty} \bar{t}^k = t$ and the continuity of φ it follows that there is $k(T) \in \mathbb{N}$ with

$$\varphi_k(t^k) = \varphi(\bar{t}^k) \in \mathring{K} \qquad \text{for all } k \geq k(T). \tag{27}$$

Now, choose $k \geq k(T)$. Since $g(t^k) \in [0, \text{eps})$, we conclude from Lemma 2.3 that there is an $n(T, k) \in \mathbb{N}$ with

$$g_n(t^k) \in [0, \text{eps})$$

for all $n \geq n(T, k)$. So t^k is feasible for Problem (P_k^n), that is,

$$T \geq T_0^{kn}$$

for all $n \geq n(T, k)$. This implies that

$$T \geq \gamma_k := \overline{\lim}_{n \to \infty} T_0^{kn}. \tag{28}$$

The statement of the theorem follows now from the estimates

$$T \geq \gamma_k \geq \delta_k \geq T_0^k \geq T_0 \tag{29}$$

for all $k \geq k(T)$ and the fact that T could be chosen arbitrarily.

3. Algorithm

In the last section, we succeeded in reducing the original problem (P) to a nonlinear optimization problem (P_k^n) in $\mathbb{R}^{k+1}$. We have seen that the optimal values of Problem (P_k^n) are approximations of the optimal value of Problem (P).

But, for the subsequent algorithm, it is necessary to replace Problem (P_k^n) by a slightly modified problem with one more variable.

Problem ($\bar{\mathrm{P}}_{k+1}^n$):

$$\min T,$$

$$(T, t_0, t_1, \ldots, t_k) \in X_{k+1},$$

$$\bar{\varphi}_{k+1}^{l_n}(T, t_0, \ldots, t_k) \in K_n,$$

with

$$\bar{\varphi}_{k+1}^{l_n}(t)(s) := \sum_{j=1}^{l_n} \mu_j^2 A_j \cos(\mu_j s) \int_0^T \bar{u}(t)(\tau) \exp(-\mu_j^2 (T-\tau))\, d\tau$$

and

$$\bar{u}(t)(\tau) := \begin{cases} 0, & \text{for } \tau \in [0, t_0), \\ (-1)^{i+1}, & \text{for } \tau \in [t_{i-1}, t_i), \qquad i = 1, 2, \ldots, k, \\ (-1)^k, & \text{for } \tau \in [t_k, T], \end{cases}$$

where

$$t := (T, t_0, t_1, \ldots, t_k) \in X_{k+1}.$$

Let us first show that Problems (P_k^n) and ($\bar{P}_{k+1}^n$) are equivalent.

Lemma 3.1. The optimal control times of Problems (P_k^n) and ($\bar{\mathrm{P}}_{k+1}^n$) are identical.

Proof. Let

$$t_0 := (T^0, t_1^0, \ldots, t_k^0) \in X_k$$

be an optimal solution of Problem (P_k^n). Then,

$$t^0 := (T^0, 0, t_1^0, \ldots, t_k^0) \in X_{k+1}$$

is feasible for Problem $(\bar{P}_{k+1}^n)$, i.e.,

$$T_0 \geq \bar{T}_0,$$

where $\bar{T}_0$ is the minimal control time of Problem $(\bar{P}_{k+1}^n)$. Now, let

$$\bar{t}^0 := (\bar{T}_0, \bar{t}_0^0, \ldots, \bar{t}_k^0) \in X_{k+1}$$

be an optimal solution of Problem $(\bar{P}_{k+1}^n)$. We assume that $\bar{t}_0^0 > 0$ and define

$$\bar{s}^0 := (\bar{T}_0 - \bar{t}_0^0, 0, \bar{t}_1^0 - \bar{t}_0^0, \ldots, \bar{t}_k^0 - \bar{t}_0^0) \in X_{k+1}.$$

$\bar{s}^0$ is feasible for Problem $(\bar{P}_{k+1}^n)$, since we can conclude from (4) that

$$\bar{\varphi}_{k+1}^l(\bar{s}^0) = \bar{\varphi}_{k+1}^l(\bar{t}^0).$$

But the estimate

$$\bar{T}_0 - \bar{t}_0^0 < \bar{T}_0$$

leads to a contradiction, because $\bar{T}_0$ is the minimal time of Problem $(\bar{P}_{k+1}^n)$. So we have $\bar{t}_0^0 = 0$, and the control

$$t^0 := (\bar{T}_0, \bar{t}_1^0, \ldots, \bar{t}_k^0) \in X_k$$

is feasible for Problem (P_k^n), i.e.,

$$\bar{T}_0 \geq T_0.$$

Now, we define, for a $T > 0$, with $n, k \in \mathbb{N}$,

$$f_i^T(t) := \bar{\varphi}_k^l(T, t)(s_i),$$
$$k_i := k_0(s_i),$$

with

$$t := (t_1, \ldots, t_k) \in \mathbb{R}^k, \qquad 0 \leq t_1 \leq \cdots \leq t_k \leq T,$$
$$s_i := (i-1)/(n-1), \qquad i = 1, \ldots, n.$$

The computation of a feasible control relative to Problem $(\bar{P}_k^n)$ for a given T is equivalent to the following problem.

Problem (D_T). Determine $t := (t_1, \ldots, t_k) \in \mathbb{R}^k$, with

$$|f_i^T(t) - k_i| \leq \text{eps}, \qquad i = 1, \ldots, n,$$
$$0 \leq t_1 \leq \cdots \leq t_k \leq T. \tag{30}$$

The optimal control time T_0^{kn} of Problem ($\bar{P}_k^n$) can be computed now by the following bisection method.

Algorithm 3.1

Step 0. Start. Select $T_1 > 0$ such that Problem (D_{T_1}) is solvable. Set

$$T_{\max} := T_1, \qquad T_{\min} := 0.$$

For $j = 1, 2, \ldots$, determine T_{j+1} from T_j as follows.

Step 1. If (D_{T_j}) is solvable, set $T_{\max} := T_j$. If (D_{T_j}) is not solvable, set $T_{\min} := T_j$.

Step 2. Set $T_{j+1} := \frac{1}{2}(T_{\max} - T_{\min})$.

We stop the algorithm, if the difference $T_{\max} - T_{\min}$ is small enough.

Theorem 3.1. Let $\{T_j\}$ be any sequence generated by Algorithm 3.1. Then,

$$\lim_{j\to\infty} T_j = T_0^{kn}.$$

Proof. The convergence of $\{T_j\}$ is obvious, if the following statement is valid: If Problem (D_T) is not solvable for a $T > 0$, then

$$T < T_0^{kn}.$$

Let us assume that

$$T > T_0^{kn}.$$

There is an optimal control of Problem ($\bar{P}_k^n$), i.e.,

$$t_0 := (T_0^{kn}, t_1^0, \ldots, t_k^0) \in X_k, \qquad \text{with } \bar{\varphi}_k^l(t_0) \in K_n.$$

Displace t_0 such that the right end of t_0 coincides with T, that is, define

$$t_0^* := (T, t_1^0 + T - T_0^{kn}, \ldots, t_k^0 + T - T_0^{kn}) \in X_k.$$

As shown in the proof of Lemma 3.1, we have

$$\bar{\varphi}_k^l(t_0^*) = \bar{\varphi}_k^l(t_0) \in K_n.$$

So t_0^* is feasible for Problem ($\bar{P}_k^n$). This contradicts the assumption that Problem (D_T) is not solvable.

It should be mentioned that the statement in the beginning of the last proof is in general not valid for Problem (P_k^n).

Let us now construct a method solving Problem (D_T) for a given $T > 0$. We reduce Problem (D_T) to a minimizing problem without restrictions. Therefore, we define the penalty function

$$h(t, r) := \sum_{i=1}^{n} |f_i^T(t) - k_i|^2 - r \sum_{j=1}^{k+1} [1/c_j(t)], \tag{31}$$

where

$$t \in \mathring{X}_k, \qquad r > 0,$$
$$c_1(t) := -t_1,$$
$$c_j(t) := t_{j-1} - t_j, \qquad j = 2, \ldots, k,$$
$$c_{k+1}(t) := t_k - T.$$

$h(t, r)$ is as a function of t continuously differentiable on $\mathring{X}_k$. For sufficiently small $r > 0$, the minimum of $h(t, r)$ is a good approximation of the restricted problem

$$\min \sum_{i=1}^{n} |f_i^T(t) - k_i|^2,$$
$$0 \leq t_1 \leq \cdots \leq t_k \leq T.$$

In order to define $h(t, r)$ on $\mathbb{R}^k$, we set

$$h(t, r) := \sum_{i=1}^{k} [t_i - T/(k + 2 - i)]^2 + 10^4$$

for every $t \notin \mathring{X}_k$.

For the numerical solution of the problem

$$\min_{t \in \mathbb{R}^k} h(t, r),$$

we use a procedure of the program library, which can be characterized as a rank-two quasi-Newton method of Broyden, Fletcher, Goldfarb, and Shanno. This algorithm stops as soon as the relation

$$\max_{1 \leq i \leq n} |f_i^T(t) - k_i| \leq \text{eps}$$

is valid. If the above estimate is not fulfilled after a sufficient number of iteration steps, we suppose that Problem (D_T) has no solution. It has turned out in practice that ten iteration steps are enough.

Table 1. Numerical results for Algorithm 3.1.

k	$T_0^{k,10}$	$t_0^{k,10}$ ($l_{10}=20$)	$T_0^{k,20}$	$t_0^{k,20}$ ($l_{20}=30$)
2	1.59–1.63	0.029, 1.37	1.53–1.56	0.033, 1.32
4	1.69–1.72	0.040, 0.85, 0.87, 1.45	1.56–1.59	0.054, 0.798, 0.801, 1.35
6	1.69–1.75	0.046, 0.61, 0.63, 1.12 1.13, 1.49	1.65–1.70	0.056, 0.60, 0.61, 1.10 1.11, 1.45
8	1.94–1.97	0.072, 0.52, 0.57, 0.97 0.99, 1.41, 1.43, 1.71	1.94–1.97	0.072, 0.51, 0.57, 0.97 0.99, 1.41, 1.43, 1.72

4. Numerical Results

For the numerical solution of Problem ($\bar{P}_k^n$), we choose the following data:

temperature distribution $k_0(s) := 0.5 - 0.5\, s^2$, $\quad s \in [0, 1]$;
accuracy eps $:= 0.01$;
heat-transfer coefficient: $\alpha := 1$

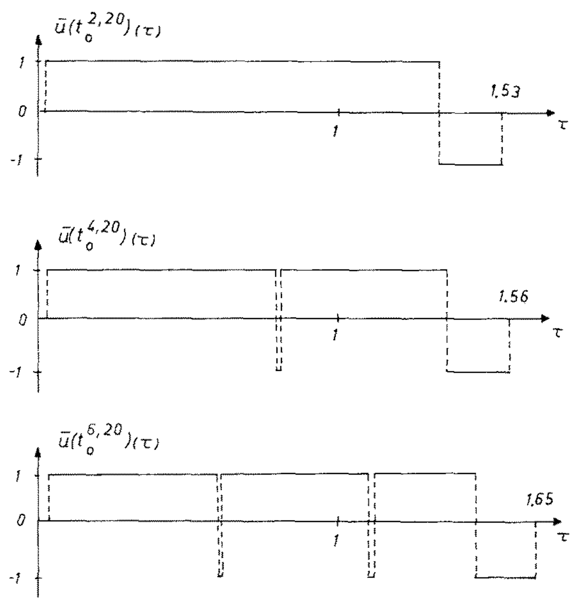

Fig. 1. Optimal controls for $k = 2, 4, 6$ and $n = 20$.

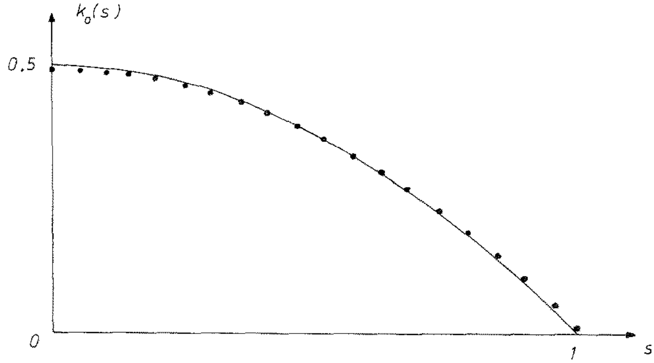

Fig. 2. Temperature distribution $\bar{u}(t_0^{2,20})$ and k_0.

penalty parameter	$r := 10^{-6}$;
initial time	$T_1 := 2$.

The computations have been performed on the computer TR 440 of the computing center of Würzburg University.

Table 1 contains the numerical results for Algorithm 3.1 for $n = 10$ and $n = 20$, respectively. The values in the second column are bounds for T_0^{kn}; t_0^{kn} denotes the jumps of the control computed with respect to the upper bound. Furthermore, Fig. 1 shows the controls for $k = 2, 4, 6$ and $n = 20$. The corresponding temperature distribution $\bar{u}(t_0^{2,20})$, i.e., the computed values for $f_i^{T_0^{2,20}}(t_0^{2,20})$, $i = 1, \ldots, 20$, and the desired temperature distribution k_0 are illustrated in Fig. 2.

Obviously, the control for $k = 2$ is optimal for two reasons. First, for every $k > 2$, we get a greater time. The second reason is that the controls t_0^{kn} for $k > 2$ approximate the control computed for $k = 2$. As expected from Lemma 3.1, the first jump of each control is close to zero.

The increase of the computed time T_0^{kn} with k and the fact that no jumps of the control belonging to it are identical can be traced to the application of the penalty function $h(t, r)$. The function values $h(t, r)$ increase rapidly when approaching the boundary. This causes an error in determining the optimal solutions and is also the reason for which we get no feasible controls for $k = 1, 3, 5, \ldots$.

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