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Resolvent Kernels of Green's Function Kernels and Other Finite-Rank Modifications of Fredholm and Volterra Kernels[1]

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Abstract. Many important Fredholm integral equations have separable kernels which are finite-rank modifications of Volterra kernels. This class includes Green's functions for Sturm–Liouville and other two-point boundary-value problems for linear ordinary differential operators. It is shown how to construct the Fredholm determinant, resolvent kernel, and eigenfunctions of kernels of this class by solving related Volterra integral equations and finite, linear algebraic systems. Applications to boundary-value problems are discussed, and explicit formulas are given for a simple example. Analytic and numerical approximation procedures for more general problems are indicated.

Key Words. Fredholm integral equations, Green's functions, Sturm–Liouville problems, boundary-value problems, eigenvalues, eigenfunctions.

1. A Classical Problem in the Theory of Integral Equations

The equation

$$y(x)-\lambda\int_0^1 K(x,t)y(t)\,dt=f(x),\qquad 0\le x\le 1,\tag{1}$$

for the unknown function $y(x)$ is called a *linear integral equation of second kind.* Equations of this form arise in the solution of initial- and boundary-value problems for ordinary differential equations and in other areas of applied analysis. In (1), the function $f(x)$, the *kernel* $K(x,t)$, and the

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parameter λ are assumed to be given. In case

$$K(x, t)=0 \qquad \text{for } t>x,$$

Eq. (1) is said to be of *Volterra type*, and the interval of integration is actually $0\leq t\leq x$; otherwise, (1) is called an integral equation of *Fredholm type.*

A central problem in the classical theory of linear integral equations of second kind is to determine the values of λ for which the solution $y(x)$ of (1) exists and is unique, and to express this solution in the form

$$y(x)=f(x)+\lambda \int_0^1 R(x, t; \lambda)f(t)\,dt, \qquad 0\leq x\leq 1, \tag{2}$$

where $R(x, t; \lambda)$ is called the *resolvent kernel* of $K(x, t)$ (see Ref. 1). The investigation of the unique solvability of Eq. (1) can thus be reduced to the problem of existence and construction of $R(x, t; \lambda)$.

Using operator notation, the kernel $K(x, t)$ may be taken to define the *linear integral operator* K on the space of functions considered. If I denotes the *identity operator*, then (1) can be written in the form

$$(I-\lambda K)y=f. \tag{3}$$

The solution y of (3) exists and is unique if the operator $I-\lambda K$ is invertible for the given value of λ. To obtain the expression corresponding to (2), the inverse of $I-\lambda K$ is represented in the form

$$(I-\lambda K)^{-1}=I+\lambda R(\lambda), \tag{4}$$

where the *resolvent operator* $R(\lambda)$ of K is the linear integral operator with kernel $R(x, t; \lambda)$. Equation (4) leads directly to the relationships

$$\begin{aligned} R(\lambda)&=K+\lambda KR(\lambda),\\ R(\lambda)&=K+\lambda R(\lambda)K, \end{aligned} \tag{5}$$

by the definition of the inverse operator. These are the so-called *resolvent equations.* In terms of the corresponding kernels, Eqs. (5) become

$$\begin{aligned} R(x, t; \lambda)&=K(x, t)+\lambda \int_0^1 K(x, s)R(s, t; \lambda)\,ds,\\ R(x, t; \lambda)&=K(x, t)+\lambda \int_0^1 R(x, s; \lambda)K(s, t)\,ds. \end{aligned} \tag{6}$$

If $K(x, t)$ is a Volterra kernel, then the intervals of integration in (6) reduce to $t\leq s\leq x$.

In the classical setting of the theory of integral equations, one is concerned with kernels which are bounded and at least square-integrable.

The representation (4) allows one to eliminate the identity operator and obtain relationships (5) between linear integral operators with kernels (6) of this type. This simplifies the analysis considerably, as the identity operator cannot be represented as a linear integral operator with a bounded kernel on the spaces of continuous or square-integrable functions (Refs. 2–4).

2. Volterra Resolvent Kernels

In case that $K(x, t)$ is a Volterra kernel, (1) takes the form

$$y(x)-\lambda \int_0^x K(x, t)y(t)\, dt = f(x), \qquad 0 \le x \le 1. \tag{7}$$

It is well-known (Refs. 1 and 3–5) that the resolvent kernel $R(x, t; \lambda)$ of $K(x, t)$ always exists, under the assumptions of the classical theory, and is given by the *Neumann series*

$$R(x, t; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K^{(n)}(x, t), \qquad 0 \le t \le x \le 1, \tag{8}$$

where

$$K^{(1)}(x, t) = K(x, t),$$

$$K^{(n+1)}(x, t) = \int_t^x K(x, s) K^{(n)}(s, t)\, ds = \int_t^x K^{(n)}(x, s) K(s, t)\, ds, \tag{9}$$

$n = 1, 2, \ldots.$

The convergence of (8) for all λ with finite modulus is easy to establish by mathematical induction for bounded kernels $K(x, t)$. If

$$|K(x, t)| \le M, \qquad 0 \le t \le x \le 1,$$

then

$$|K^{(n)}(x, t)| \le M \cdot [M^{n-1}/(n-1)!], \qquad n = 1, 2, \ldots, \tag{10}$$

from which the desired result follows. The kernels $K^{(2)}(x, t)$, $K^{(3)}(x, t), \ldots$ are sometimes called the *iterated kernels* of $K(x, t)$ (see Ref. 6).

It will be useful later to consider also the *transposed* Volterra integral equation corresponding to (7),

$$z(t) - \lambda \int_t^1 z(x) K(x, t)\, dx = g(t), \qquad 0 \le t \le 1. \tag{11}$$

The solution $z(t)$ of (11) is given in terms of $g(t)$, and the resolvent kernel $R(x, t; \lambda)$ of $K(x, t)$ as

$$z(t)=g(t)+\lambda \int_t^1 g(x)R(x, t; \lambda)\, dx, \qquad 0 \leq t \leq 1. \tag{12}$$

3. Fredholm Resolvent Kernels

For the more general case of a Fredholm kernel $K(x, t)$, an expression for the resolvent kernel is sought in the form

$$R(x, t; \lambda)=N(x, t; \lambda)/\Delta(\lambda), \tag{13}$$

with the numerator and denominator having series expansions

$$N(x, t; \lambda)=\sum_{n=1}^{\infty} \lambda^{n-1}K_n(x, t), \tag{14}$$

and

$$\Delta(\lambda)=1+\sum_{n=1}^{\infty} c_n\lambda^n, \tag{15}$$

respectively, which converge for all λ with finite modulus. The resolvent kernel $R(x, t; \lambda)$ will then exist for all values of λ for which the *Fredholm determinant* $\Delta(\lambda)$ of $K(x, t)$ does not vanish. This is analogous to Cramer's rule for the inversion of a finite-dimensional matrix.

Formulas for the so-called *associated kernels* $K_1(x, t), K_2(x, t), \ldots$ of $K(x, t)$ (Ref. 6) appearing in (14) may be obtained by substituting (13)–(15) into the resolvent equations (6). This gives

$$K_1(x, t)=K(x, t),$$

$$\begin{aligned} K_{n+1}(x, t) &= c_nK(x, t)+\int_0^1 K(x, s)K_n(s, t)\, ds \\ &= c_nK(x, t)+\int_0^1 K_n(x, s)K(s, t)\, ds, \qquad n=1, 2, \ldots, \end{aligned} \tag{16}$$

a result which satisfies (6) formally, independently of the values assigned to $c_1, c_2, \ldots$. For example, if

$$c_1=c_2=\cdots=0,$$

then (14) becomes the Neumann series (8), which does not converge in general for $|\lambda|$ large. In order for (14) and (15) to be entire functions of λ, one chooses

$$c_n=-(1/n)\operatorname{tr} K_n=-(1/n)\int_0^1 K_n(x, x)\, dx, \qquad n=1, 2, \ldots, \tag{17}$$

where the quantity $\operatorname{tr} K_n$ is called the *trace* of the kernel $K_n(x, t)$. The construction (16)–(17) of the resolvent kernel is in the form given by

Lalesco (Ref. 5). Fredholm (Ref. 7) originally obtained the formulas

$$K_n(x,t)=[(-1)^{n-1}/(n-1)!]\int_0^1\cdots\int_0^1 \times\begin{vmatrix} K(s,t) & K(x,s_1) & \cdots & K(x,s_{n-1}) \\ K(s_1,t) & K(s_1,s_1) & \cdots & K(s_1,s_{n-1}) \\ \vdots & \vdots & & \vdots \\ K(s_{n-1},t) & K(s_{n-1},s_1) & \cdots & K(s_{n-1},s_{n-1}) \end{vmatrix} ds_1\cdots ds_{n-1}, \quad (18)$$

and the similar expressions corresponding to (17) for c_n, $n=1,2,\ldots$. The satisfaction of (16) is easily verified by mathematical induction.

Using Fredholm's formulas, the convergence of the series (14) and (15) can be established on the basis of Hadamard's inequality for determinants (see Ref. 4 for an elegant proof) and the ratio test. If

$$|K(x,t)|\le M,$$

then Hadamard's inequality applied to (18) yields the estimates

$$|K_n(x,t)|\leqq M\cdot[M^{n-1}n^{n/2}/(n-1)!], \qquad n=1,2,\ldots, \quad (19)$$

and similar bounds for $|c_1|, |c_2|, \ldots$. The rate of convergence which can be predicted for the series (14) and (15) on the basis of (19) is, of course, much slower than that given by the estimates (10) applied to the Neumann series expansion (8) of a Volterra resolvent kernel. For example, if given Volterra and Fredholm kernels are bounded in absolute value by M, then, for $n=100$, the right-hand side of (19) exceeds that of (10) by a factor of 10^{100}. From a computational point of view, the relationships (16)–(17) would appear to be preferable to the equivalent expressions (18) involving determinants. It has been observed, however, that the formulas corresponding to (16)–(17) for finite-dimensional matrices are unstable numerically (Ref. 8).

Other important relationships which follow from the formulation (13) of the resolvent kernel and the resolvent equations (6) are

$$\begin{aligned} N(x,t;\lambda)&=\Delta(\lambda)K(x,t)+\lambda\int_0^1 K(x,s)N(s,t;\lambda)\,ds, \\ N(x,t;\lambda)&=\Delta(\lambda)K(x,t)+\lambda\int_0^1 N(x,s;\lambda)K(s,t)\,ds. \end{aligned} \quad (20)$$

These are immediately evident for $\Delta(\lambda)\neq 0$. In the framework of the classical theory, they can also be extended to the case that $\lambda=\lambda^*$ is an eigenvalue of the kernel $K(x,t)$; that is,

$$\Delta(\lambda^*)=0.$$

Assuming that $N(x, t; \lambda^*)$ does not vanish identically, a point (ξ, τ) in the square $0 \leq x,\ t \leq 1$ exists such that the functions

$$\begin{aligned} y^*(x) &= N(x, \tau; \lambda^*), \qquad 0 \leq x \leq 1, \\ z^*(t) &= N(\xi, t; \lambda^*), \qquad 0 \leq t \leq 1, \end{aligned} \tag{21}$$

are also not identically zero. Furthermore, $y^*(x)$ satisfies the *homogeneous* integral equation

$$y^*(x) = \lambda^* \int_0^1 K(x, t) y^*(t)\, dt, \qquad 0 \leq x \leq 1, \tag{22}$$

and is said to be a *right eigenfunction* of $K(x, t)$ corresponding to the eigenvalue λ^*. Similarly, the function $z^*(t)$ satisfies the transposed homogeneous integral equation

$$z^*(t) = \lambda^* \int_0^1 z^*(x) K(x, t)\, dx, \qquad 0 \leq t \leq 1, \tag{23}$$

and is called a *left eigenfunction* of $K(x, t)$ corresponding to λ^*.

4. Symmetric Separable Kernels

Attention will now be devoted to the construction of resolvent kernels of a special class of Fredholm kernels. In the symmetric case, a kernel of the form

$$G(x, t) = \begin{cases} u(t)v(x), & 0 \leq t \leq x \leq 1, \\ u(x)v(t), & 0 \leq x \leq t \leq 1, \end{cases} \tag{24}$$

will be called a *simple separable kernel.* It is assumed that the functions $u(x)$ and $v(x)$ are linearly independent; otherwise, $G(x, t)$ would be a *degenerate* kernel of rank one (Ref. 3, pp. 37–40). In general, a *symmetric separable kernel* is a finite sum of linearly independent kernels (24).

Before dealing with the general case, the resolvent kernel of the symmetric simple separable kernel (24) will be constructed. To do this, the Fredholm integral equation

$$y(x) - \lambda \int_0^1 G(x, t) y(t)\, dt = f(x), \qquad 0 \leq x \leq 1, \tag{25}$$

will be solved. This is essentially the approach used by Drukarev (Ref. 9), Brysk (Ref. 10), and Aalto (Ref. 11). Using the definition (24) of $G(x, t)$, Eq. (25) can be written as

$$y(x)-\lambda\int_0^x u(t)v(x)y(t)\,dt=f(x)+\lambda\int_x^1 u(x)v(t)y(t)\,dt. \tag{26}$$

Adding the quantity

$$\lambda\int_0^x u(x)v(t)y(t)\,dt$$

to both sides of (26) gives

$$y(x)-\lambda\int_0^x K(x,t)y(t)\,dt=f(x)+\lambda cu(x), \tag{27}$$

where

$$c=\int_0^1 v(t)y(t)\,dt \tag{28}$$

is to be determined, and $K(x, t)$ is the Volterra kernel

$$K(x,t)=u(t)v(x)-u(x)v(t), \qquad 0\leq t\leq x\leq 1. \tag{29}$$

The system of equations (27)–(29) is easily seen to be equivalent to the original integral equation (25).

As was shown in Section 2, the kernel $K(x, t)$ has the Volterra resolvent kernel $R(x, t; \lambda)$, given by (8) for all λ. Define

$$\begin{aligned} F(x)&=f(x)+\lambda\int_0^x R(x,t;\lambda)f(t)\,dt,\\ U(x)&=u(x)+\lambda\int_0^x R(x,t;\lambda)u(t)\,dt, \end{aligned} \tag{30}$$

where the dependence of $F(x)$ and $U(x)$ on λ has been suppressed for clarity of notation. From (27),

$$y(x)=F(x)+\lambda cU(x); \tag{31}$$

and, from (28),

$$c=\int_0^1 v(t)F(t)\,dt+\lambda c\int_0^1 v(t)U(t)\,dt. \tag{32}$$

Thus, (32) has a unique solution for c if

$$\Delta(\lambda)\equiv 1-\lambda\int_0^1 v(t)U(t)\,dt\neq 0, \tag{33}$$

in which case

$$c=[1/\Delta(\lambda)]\int_0^1 v(t)F(t)\,dt. \tag{34}$$

The expression (34) for c may be written in terms of $f(t)$ by introducing the function

$$V(t)=v(t)+\lambda\int_t^1 v(x)R(x,t;\lambda)\,dx. \tag{35}$$

It then follows from (30) and (34) that

$$c=[1/\Delta(\lambda)]\int_0^1 V(t)f(t)\,dt; \tag{36}$$

and, from (31),

$$y(x)=f(x)+\lambda\int_0^x R(x,t;\lambda)f(t)\,dt+[\lambda U(x)/\Delta(\lambda)]\int_0^1 V(t)f(t)\,dt. \tag{37}$$

By definition, the resolvent kernel $\Gamma(x,t;\lambda)$ of $G(x,t)$ can be obtained directly from (37) as

$$\Gamma(x,t;\lambda)=\begin{cases} R(x,t;\lambda)+[1/\Delta(\lambda)]U(x)V(t), & 0\le t\le x\le 1,\\ [1/\Delta(\lambda)]U(x)V(t), & 0\le x\le t\le 1,\end{cases} \tag{38}$$

provided, of course, that

$$\Delta(\lambda)\neq 0.$$

Another expression for $\Gamma(x,t;\lambda)$ can be obtained by making use of the fact that the resolvent kernel of a symmetric kernel $G(x,t)$ must also be symmetric in x and t; that is,

$$\Gamma(x,t;\lambda)=\Gamma(t,x;\lambda).$$

This gives

$$\Gamma(x,t;\lambda)=\begin{cases} [1/\Delta(\lambda)]U(t)V(x), & 0\le t\le x\le 1,\\ R(t,x;\lambda)+[1/\Delta(\lambda)]U(t)V(x), & 0\le x\le t\le 1,\end{cases} \tag{39}$$

which, when compared with (38), yields

$$\Delta(\lambda)R(x,t;\lambda)=U(t)V(x)-U(x)V(t), \qquad 0\le t\le x\le 1, \tag{40}$$

and finally,

$$\Gamma(x,t;\lambda)=\begin{cases} [1/\Delta(\lambda)]U(t)V(x), & 0\le t\le x\le 1,\\ [1/\Delta(\lambda)]U(x)V(t), & 0\le x\le t\le 1.\end{cases} \tag{41}$$

These results give rise to the following observations.

Remark 4.1. Equation (41) shows that if the resolvent kernel of a symmetric simple separable kernel (24) exists, then it is also a symmetric simple separable kernel.

Remark 4.2. The expression (41) for the resolvent kernel $\Gamma(x, t; \lambda)$ of $G(x, t)$ can be obtained without finding the Volterra resolvent kernel $R(x, t; \lambda)$ of $K(x, t)$ explicitly; one need only solve the Volterra integral equation (7) with

$$f(x) = u(x)$$

for

$$y(x) = U(x),$$

and the transposed Volterra integral equation (11) with

$$g(t) = v(t)$$

for

$$z(t) = V(t).$$

Remark 4.3. An equivalent expression for the Fredholm determinant $\Delta(\lambda)$ of $G(x, t)$ is

$$\Delta(\lambda) = 1 - \lambda \int_0^1 V(x)u(x)\, dx, \tag{42}$$

which can be obtained directly from (33) by interchange of the order of integration and use of the definition (35) of $V(t)$. It is easy to show that the above expressions give expansions of $\Delta(\lambda)$ and $N(x, t; \lambda)$ in powers of λ with coefficients satisfying the relationships (16)–(17) (see Refs. 11–12). The advantage of the present approach is that the rate of convergence of the series for $\Delta(\lambda)$,

$$U(x) = U(x; \lambda),$$

and

$$V(t) = V(t; \lambda)$$

can be predicted on the basis of (10), rather than (19).

Remark 4.4. The definitions of the Volterra kernel $K(x, t)$ and its resolvent kernel $R(x, t; \lambda)$ extend to the entire square $0 \leq x, t \leq 1$, with

$$K(x, t) = 0, \qquad R(x, t; \lambda) = 0, \qquad 0 \leq x \leq t \leq 1. \tag{43}$$

This will result in continuous kernels for $u(x)$, $v(t)$ continuous, as

$$K(x, x)=R(x, x; \lambda)=0, \qquad 0 \le x \le 1.$$

In terms of these extended kernels, one may write

$$G(x, t)=K(x, t)+u(x)v(t),$$
$$\Gamma(x, t; \lambda)=R(x, t; \lambda)+[1/\Delta(\lambda)]U(x)V(t). \tag{44}$$

Thus, the kernel $G(x, t)$ given by (24) is the sum of the Volterra kernel $K(x, t)$ and the degenerate kernel $u(x)v(t)$ of rank one. If

$$\Delta(\lambda) \neq 0,$$

then its resolvent kernel $\Gamma(x, t; \lambda)$ exists and is also a rank-one modification of the Volterra resolvent kernel $R(x, t; \lambda)$ of $K(x, t)$. By symmetry, one also has

$$G(x, t)=K(t, x)+u(t)v(x),$$
$$\Gamma(x, t; \lambda)=R(t, x; \lambda)+[1/\Delta(\lambda)]U(t)V(x). \tag{45}$$

In some applications, the expressions (44) or (45) for $G(x, t)$ arise more naturally than (24).

The next case to be examined is that

$$\lambda = \lambda^*$$

is an eigenvalue of $G(x, t)$; that is,

$$\Delta(\lambda^*)=0.$$

Noting that

$$U(x)=U(x; \lambda) \qquad \text{and} \qquad V(t)=V(t; \lambda)$$

depend on λ, define

$$U^*(x)=U(x; \lambda^*), \qquad 0 \le x \le 1,$$
$$V^*(t)=V(t; \lambda^*), \qquad 0 \le t \le 1. \tag{46}$$

From (40), which can be extended easily to $\lambda = \lambda^*$, it follows that the functions $U^*(x)$ and $V^*(x)$ are linearly dependent. Hence, for

$$\lambda = \lambda^*,$$

the Fredholm numerator $N(x, t; \lambda)$ of $\Gamma(x, t; \lambda)$ becomes

$$N(x, t; \lambda^*)=\alpha U^*(x)V^*(t), \tag{47}$$

where $\alpha \neq 0$ is some constant. Thus, by (21)–(23), $U^*(x)$ will be a right eigenfunction of $G(x, t)$ corresponding to λ^*, and $V^*(t)$ is a corresponding left eigenfunction. As $G(x, t)$ is symmetric, the distinction between left and right eigenfunctions is inconsequential. However, it will be shown later that a similar approach gives these as distinct functions in the nonsymmetric case.

The implications of the above results for the solvability of the integral equation (25) may be summed up in the following familiar language.

Theorem 4.1. *Fredholm Alternative.* If the transposed homogeneous integral equation

$$z(t)-\lambda \int_0^1 z(x)G(x, t)\, dx = 0, \qquad 0 \le t \le 1, \tag{48}$$

has only the *trivial solution* $z(t)=0$, $0 \le t \le 1$, then Eq. (25) has the unique solution

$$y(x)=F(x)+[\lambda U(x)/\Delta(\lambda)] \int_0^1 v(t)F(t)\, dt, \qquad 0 \le x \le 1. \tag{49}$$

On the other hand, if (48) has the nontrivial solution

$$z(t)=V^*(t),$$

then (25) has a solution only if $f(x)$ is *orthogonal* to $V^*(x)$, that is,

$$\int_0^1 V^*(t)f(t)\, dt = 0. \tag{50}$$

If (50) is satisfied, then (25) has the solutions

$$y(x)=F(x)+\alpha U^*(x), \tag{51}$$

where α is arbitrary, and

$$y(x)=U^*(x)$$

is a nontrivial solution of the homogeneous integral equation

$$y(x)-\lambda \int_0^1 G(x, t)y(t)\, dt = 0, \qquad 0 \le x \le 1. \tag{52}$$

As derived above, of course, Theorem 4.1 applies only to simple kernels (24). The same technique, however, applies to general symmetric separable kernels. Suppose that

$$G(x, t)=\sum_{j=1}^{n} G_j(x, t)=\begin{cases} \sum_{j=1}^{n} u_j(t)v_j(x), & 0 \le t \le x \le 1, \\ \sum_{j=1}^{n} u_j(x)v_j(t), & 0 \le x \le t \le 1. \end{cases} \tag{53}$$

Then, the integral equation (25) with the symmetric separable kernel (53) may be reduced to

$$y(x)-\lambda \int_0^x K(x,t)y(t)\,dt = f(x)+\lambda \sum_{j=1}^{n} c_j u_j(x), \tag{54}$$

where the numbers

$$c_j = \int_0^1 v_j(t)y(t)\,dt, \qquad j=1,2,\ldots,n, \tag{55}$$

are to be determined, and $K(x, t)$ is the Volterra kernel

$$K(x,t)=\sum_{j=1}^{n}[u_j(t)v_j(x)-u_j(x)v_j(t)], \qquad 0\le t\le x\le 1. \tag{56}$$

As before, let $R(x, t; \lambda)$ denote the Volterra resolvent kernel of $K(x, t)$, and define

$$\begin{aligned} F(x)&=f(x)+\lambda\int_0^1 R(x,t;\lambda)f(t)\,dt,\\ U_i(x)&=u_i(x)+\lambda\int_0^1 R(x,t;\lambda)u_i(t)\,dt, \qquad i=1,2,\ldots,n. \end{aligned} \tag{57}$$

Equation (54) is then equivalent to

$$y(x)=F(x)+\lambda\sum_{i=1}^{n} c_i U_i(x). \tag{58}$$

Multiplying (58) by $v_1(x), v_2(x), \ldots, v_n(x)$ in turn and integrating with respect to x from 0 to 1 gives the system of equations

$$c_i-\lambda\sum_{j=1}^{n} c_j\int_0^1 v_i(x)U_j(x)\,dx=\int_0^1 v_i(x)F(x)\,dx, \qquad i=1,2,\ldots,n, \tag{59}$$

for the unknowns $c_1, c_2, \ldots, c_n$ defined by (55).

Remark 4.5. It follows from (55), (58), and (59) that the solution of the integral equation (25) with the symmetric separable kernel (53) is equivalent to solving the Volterra integral equation (7) with kernel (56) and right-hand sides $f(x), u_1(x), \ldots, u_n(x)$ for $F(x), U_1(x), \ldots, U_n(x)$, forming the n^2+n coefficients of the system (59), and then solving this system of n linear equations for the n unknowns $c_1, c_2, \ldots, c_n$.

It is also possible to obtain explicit expressions for the Fredholm determinant $\Delta(\lambda)$ of $G(x, t)$ and its resolvent kernel $\Gamma(x, t; \lambda)$ in terms of the determinant and the inverse of the coefficient matrix of the system (59).

It will be convenient to introduce the functions

$$V_i(t)=v_i(t)+\lambda\int_t^1 v_i(x)R(x,t;\lambda)\,dx, \qquad i=1,2,\ldots,n, \tag{60}$$

which are the solutions of the transposed Volterra integral equations (11) with right-hand sides $v_1(t), v_2(t), \ldots, v_n(t)$. Let the coefficients

$$\alpha_{ij}=\int_0^1 v_i(x)U_j(x)\,dx=\int_0^1 V_i(t)u_j(t)\,dt, \qquad i,j=1,2,\ldots,n, \tag{61}$$

define the $n\times n$ matrix

$$A=[\alpha_{ij}].$$

Then, the coefficient matrix of the system (59) has the form

$$I-\lambda A=(\delta_{ij}-\lambda\alpha_{ij}), \tag{62}$$

where δ_{ij} is the *Kronecker delta*,

$$\delta_{ij}=0 \qquad \text{if } i\neq j, \qquad \delta_{ii}=1,$$

and

$$I=[\delta_{ij}]$$

is the $n\times n$ identity matrix. If the determinant

$$\Delta(\lambda)=\det(I-\lambda A) \tag{63}$$

does not vanish, then the inverse of the matrix $I-\lambda A$ exists, and can be written as

$$(I-\lambda A)^{-1}=[1/\Delta(\lambda)]B(\lambda)=[\beta_{ij}(\lambda)/\Delta(\lambda)], \tag{64}$$

by the use of Cramer's rule. In this case, the system (59) has the unique solutions

$$\begin{aligned} c_i&=[1/\Delta(\lambda)]\sum_{j=1}^{n}\beta_{ij}(\lambda)\int_0^1 v_j(x)F(x)\,dx\\ &=[1/\Delta(\lambda)]\sum_{j=1}^{n}\beta_{ij}(\lambda)\int_0^1 V_j(t)f(t)\,dt, \end{aligned} \tag{65}$$

for $i=1,2,\ldots,n$. Thus, the integral equation (25) will have the unique solution $y(x)$ given by (58), which may be written in the form

$$\begin{aligned} y(x)=f(x)&+\lambda\int_0^x R(x,t;\lambda)f(t)\,dt\\ &+\lambda\sum_{i=1}^{n}\sum_{j=1}^{n}U_i(x)[\beta_{ij}(\lambda)/\Delta(\lambda)]\int_0^1 V_j(t)f(t)\,dt. \end{aligned} \tag{66}$$

From (66), it is possible to derive several expressions for the resolvent kernel $\Gamma(x, t; \lambda)$ of the symmetric separable kernel $G(x, t)$, provided that the Fredholm determinant $\Delta(\lambda)$ of $G(x, t)$ is nonzero. Extending the Volterra kernel $K(x, t)$ and its resolvent kernel $R(x, t; \lambda)$ to the entire square $0 \leq x, t \leq 1$, as before, one may write

$$\begin{aligned} G(x, t) &= K(x, t) + \sum_{j=1}^{n} u_j(x) v_j(t), \\ \Gamma(x, t; \lambda) &= R(x, t; \lambda) + [1/\Delta(\lambda)] \sum_{i=1}^{n} \sum_{j=1}^{n} U_i(x) \beta_{ij}(\lambda) V_j(t). \end{aligned} \tag{67}$$

Remark 4.6. The symmetric separable kernel (53) is a rank n modification of the Volterra kernel (56). If

$$\Delta(\lambda) \neq 0,$$

the resolvent kernel $\Gamma(x, t; \lambda)$ is likewise a rank n modification of the Volterra resolvent kernel $R(x, t; \lambda)$ of $K(x, t)$.

Integral operators corresponding to degenerate kernels of rank n are sometimes called *n-term dyads* (Ref. 2). By symmetry,

$$\begin{aligned} G(x, t) &= K(t, x) + \sum_{j=1}^{n} u_j(t) v_j(x), \\ \Gamma(x, t; \lambda) &= R(t, x; \lambda) + [1/\Delta(\lambda)] \sum_{i=1}^{n} \sum_{j=1}^{n} U_i(t) \beta_{ij}(\lambda) V_j(x). \end{aligned} \tag{68}$$

Comparison of (68) and (67) gives

$$\Delta(\lambda) R(x, t; \lambda) = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij}(\lambda) [U_i(t) V_j(x) - U_i(x) V_j(t)], \tag{69}$$

and thus $\Gamma(x, t; \lambda)$ may also be written in the form

$$\Gamma(x, t; \lambda) = \begin{cases} \sum_{i=1}^{n} \sum_{j=1}^{n} [\beta_{ij}(\lambda)/\Delta(\lambda)] U_i(t) V_j(x), & 0 \leq t \leq x \leq 1, \\ \sum_{i=1}^{n} \sum_{j=1}^{n} [\beta_{ij}(\lambda)/\Delta(\lambda)] U_i(x) V_j(t), & 0 \leq x \leq t \leq 1. \end{cases} \tag{70}$$

Remark 4.7. If it exists, the Fredholm resolvent kernel $\Gamma(x, t; \lambda)$ of the symmetric separable kernel (53) is likewise the sum of n symmetric simple separable kernels.

For $n > 1$, there are a number of ways in which the kernel (70) may be written in the form (53). For example, defining the functions

$$\Psi_j(x) = \sum_{i=1}^{n} U_i(x)\beta_{ij}(\lambda), \qquad j = 1, 2, \ldots, n,$$
$$\Phi_i(t) = \sum_{j=1}^{n} \beta_{ij}(\lambda)V_j(t), \qquad i = 1, 2, \ldots, n, \tag{71}$$

one obtains the equivalent representations

$$\Gamma(x, t; \lambda) = \begin{cases} [1/\Delta(\lambda)] \sum_{j=1}^{n} \Psi_j(t)V_j(x), & 0 \le t \le x \le 1, \\ [1/\Delta(\lambda)] \sum_{j=1}^{n} \Psi_j(x)V_j(t), & 0 \le x \le t \le 1, \end{cases} \tag{72}$$

and

$$\Gamma(x, t; \lambda) = \begin{cases} [1/\Delta(\lambda)] \sum_{i=1}^{n} U_i(t)\Phi_i(x), & 0 \le t \le x \le 1, \\ [1/\Delta(\lambda)] \sum_{i=1}^{n} U_i(x)\Phi_i(t), & 0 \le x \le t \le 1, \end{cases} \tag{73}$$

for the resolvent kernel of $G(x, t)$.

Suppose now that

$$\lambda = \lambda^*$$

is an eigenvalue of the kernel (53), that is,

$$\Delta(\lambda^*) = 0.$$

By the same method as used in the nonhomogeneous case, the homogeneous equation (52) can be shown to be equivalent to the system

$$y(x) = \lambda^* \sum_{i=1}^{n} c_i U_i(x),$$
$$c_i - \lambda^* \sum_{j=1}^{n} \alpha_{ij} c_j = 0, \qquad i = 1, 2, \ldots, n, \tag{74}$$

corresponding to

$$F(x) = 0$$

in (58) and (59). As

$$\Delta(\lambda^*) = 0,$$

the homogeneous system

$$(I-\lambda^*A)c=0 \tag{75}$$

has $m \leq n$ linearly independent solutions

$$c^{(k)}=(c_1^{(k)}, c_2^{(k)}, \dots, c_n^{(k)})^T, \qquad k=1,2,\dots,m. \tag{76}$$

Corresponding to these solutions, which are right eigenvectors of the matrix

$$A=[\alpha_{ij}]$$

corresponding to the reciprocal eigenvalue λ^*, one obtains m linearly independent eigenfunctions

$$y_k^*(x)=\lambda^* \sum_{i=1}^n c_i^{(k)} U_i(x), \qquad k=1,2,\dots,n, \tag{77}$$

of the kernel $G(x, t)$ corresponding to the eigenvalue λ^*. Similarly, the transposed homogeneous equation (48) for $\lambda=\lambda^*$ has m linearly independent solutions

$$z_k^*(t)=\lambda^* \sum_{i=1}^n d_j^{(k)} V_j(t), \qquad k=1,2,\dots,m, \tag{78}$$

corresponding to the m linearly independent solutions

$$d^{(k)}=(d_1^{(k)}, d_2^{(k)}, \dots, d_n^{(k)}), \qquad k=1,2,\dots,m, \tag{79}$$

of the transposed homogeneous system

$$d(I-\lambda^*A)=0, \tag{80}$$

that is,

$$d_j-\lambda^* \sum_{i=1}^n d_i\alpha_{ij}=0, \qquad j=1,2,\dots,n. \tag{81}$$

It follows from (59) that the nonhomogeneous integral equation (25) will have no solutions unless the orthogonality conditions

$$\sum_{j=1}^n d_j^{(k)} \int_0^1 v_j(x)F(x)\,dx=\int_0^1 \sum_{j=1}^n d_j^{(k)} V_j(t)f(t)\,dt=\int_0^1 z_k^*(t)f(t)\,dt=0 \tag{82}$$

hold for $k=1,2,\dots,m$; that is, the right-hand side of (25) must be orthogonal to all solutions of the transposed homogeneous equation (48) with

$$\lambda=\lambda^*.$$

If (82) holds, then (25) is satisfied by the family of solutions

$$y(x)=f(x)+\lambda^* \int_0^x R(x, t; \lambda^*)f(t)\, dt + \sum_{k=1}^m \gamma_k y_k^*(x), \tag{83}$$

with $\gamma_1, \gamma_2, \ldots, \gamma_m$ arbitrary.

The above results can be stated as the corresponding generalization of the Fredholm alternative theorem to kernels (53) with $n>1$. Another method for computing resolvent kernels of separable kernels (symmetric or not) will be indicated in a later section.

5. Applications to Boundary-Value Problems

Symmetric separable kernels appear frequently as *Green's functions* for two-point boundary-value problems for ordinary differential operators (Ref. 13). More precisely, suppose that $L[\cdot]$ is a linear ordinary differential operator, and a solution $y(x)$ of the differential equation

$$L[y(x)]=h(x) \tag{84}$$

is sought which, together with its derivatives of lower order than the order of $L[\cdot]$, satisfies given conditions at

$$x=0 \qquad \text{and } x=1.$$

If this boundary-value problem has a unique solution $y(x)$ which can be represented as

$$y(x)=\int_0^1 G(x, t)h(t)\, dt \tag{85}$$

for all functions $h(x)$ from some class such as continuous functions, then $G(x, t)$ is said to be the *Green's function* for $L[\cdot]$ corresponding to the given boundary conditions. If

$$h(x)=\lambda y(x)+g(x), \tag{86}$$

then the boundary-value problem for the differential equation (84) is equivalent to the Fredholm integral equation

$$y(x)-\lambda \int_0^1 G(x, t)y(t)\, dt=f(x), \qquad 0\le x\le 1, \tag{87}$$

where

$$f(x)=\int_0^1 G(x, t)g(t)\, dt, \qquad 0\le x\le 1. \tag{88}$$

Thus, the techniques of Section 4 apply to the solution of (87) if $G(x, t)$ is a symmetric separable kernel; they also apply to finding eigenvalues and eigenfunctions in the homogeneous case

$$g(x) \equiv 0.$$

In many cases, it is possible to proceed directly from the boundary-value problem to representations for the Green's function, its Fredholm determinant, resolvent kernel, and eigenfunctions. For example, consider the *Sturm–Liouville* operator (Ref. 13)

$$L[y(x)] = -(p(x)y'(x))', \tag{89}$$

with $p(x) > 0$, subject to the boundary conditions

$$\begin{aligned} &ap(0)y'(0) + by(0) = 0, \\ &cp(1)y'(1) + dy(1) = 0. \end{aligned} \tag{90}$$

A simple way to find the Green's function for this problem is to start directly from the differential equation

$$(p(x)y'(x))' = -h(x). \tag{91}$$

One integration gives

$$p(x)y'(x) = p(0)y'(0) - \int_0^x h(t)\, dt. \tag{92}$$

Dividing (92) by $p(x)$ and integrating again yields

$$y(x) = y(0) + p(0)y'(0) \int_0^x [1/p(t)]\, dt - \int_0^x \int_0^s [h(t)/p(s)]\, dt\, ds. \tag{93}$$

By defining

$$F(x) = \int_0^x [1/p(t)]\, dt, \tag{94}$$

and noting that change of order of integration results in

$$\int_0^x \int_0^s [h(t)/p(s)]\, dt\, ds = \int_0^x \left\{ \int_t^x [1/p(s)]\, ds \right\} h(t)\, dt = \int_0^x [F(x) - F(t)]h(t)\, dt, \tag{95}$$

one obtains

$$y(x) = y(0) + p(0)y'(0)F(x) - \int_0^x [F(x) - F(t)]h(t)\, dt \tag{96}$$

from (93). The boundary conditions (90) will now be used to express $y(0)$ and $p(0)y'(0)$ in terms of integral transforms of $h(x)$. From (92) and (96),

$$\begin{aligned} p(1)y'(1) &= p(0)y'(0) - \int_0^1 h(t)\,dt, \\ y(1) &= y(0) + p(0)y'(0)F(1) - \int_0^1 [F(1)-F(t)]h(t)\,dt. \end{aligned} \tag{97}$$

Multiplying the first equation of (97) by c, the second by d, and adding the results gives

$$0 = dy(0) + \gamma p(0)y'(0) - \int_0^1 [\gamma - dF(t)]h(t)\,dt, \tag{98}$$

by the second of the boundary conditions (90), where

$$\gamma = c + dF(1). \tag{99}$$

Now, multiplying (98) by a and using the first boundary condition of (90) results in

$$0 = (ad - b\gamma)y(0) - \int_0^1 a[\gamma - dF(t)]h(t)\,dt; \tag{100}$$

or, if

$$\delta = ad - b\gamma \neq 0, \tag{101}$$

then

$$y(0) = \int_0^1 (a/\delta)[\gamma - dF(t)]h(t)\,dt. \tag{102}$$

Similarly, multiplication of (98) by b and use of the first boundary condition gives

$$p(0)y'(0) = -\int_0^1 (b/\delta)[\gamma - dF(t)]h(t)\,dt. \tag{103}$$

Equations (102) and (103) may be substituted into (96) to obtain

$$y(x) = -\int_0^x [F(x)-F(t)]h(t)\,dt + \int_0^1 (1/\delta)[a - bF(x)][\gamma - dF(t)]h(t)\,dt. \tag{104}$$

Comparison of (104) with (85) gives the following results.

Remark 5.1. If $\delta \neq 0$, then the Green's function $G(x, t)$ of the Sturm–Liouville problem (89)–(90) is the rank-one modification

$$G(x, t) = K(x, t) + u(x)v(t) \tag{105}$$

of the Volterra kernel

$$K(x, t) = -[F(x) - F(t)], \qquad 0 \leq t \leq x \leq 1, \tag{106}$$

with

$$\begin{aligned} u(x) &= (1/\delta)[a - bF(x)], \\ v(t) &= \gamma - dF(t). \end{aligned} \tag{107}$$

Remark 5.2. If $\delta \neq 0$, then the Green's function (105) is the symmetric simple separable kernel

$$G(x, t) = \begin{cases} (1/\delta)[a - bF(t)][\gamma - dF(x)], & 0 \leq t \leq x \leq 1, \\ (1/\delta)[a - bF(x)][\gamma - dF(t)], & 0 \leq x \leq t \leq 1. \end{cases} \tag{108}$$

This follows directly from (105)–(106) and the simple calculation

$$[a - bF(x)][\gamma - dF(t)] - \delta F(x) + \delta F(t) = [a - bF(t)][\gamma - dF(x)]. \tag{109}$$

Thus, the evaluation of the Fredholm determinant $\Delta(\lambda)$ of $G(x, t)$ and the calculation of its resolvent kernel and eigenfunctions depends only on being able to calculate the functions $U(x)$, $V(t)$ either in terms of the Volterra resolvent kernel $R(x, t; \lambda)$ of the kernel (106), or directly by solving the Volterra integral equations cited in Remark 4.2.

For certain simple examples, it is possible to give explicit formulas for these results. Taking

$$p(x) \equiv 1$$

gives

$$L[y(x)] = -y''(x), \tag{110}$$

and hence

$$F(x) = x, \qquad K(x, t) = -(x - t), \tag{111}$$

from which

$$R(x, t; \lambda) = -\sin[\sqrt{\lambda}(x - t)]/\sqrt{\lambda}. \tag{112}$$

The boundary conditions

$$y(0) = 0, \qquad y'(1) = 0 \tag{113}$$

correspond to (90) with

$$a=0, \qquad b=1, \qquad c=1, \qquad d=0,$$

and hence

$$\gamma=1, \qquad \delta=-1,$$

and thus

$$u(x)=x, \qquad v(t)=1. \tag{114}$$

It follows that the Green's function $G(x, t)$ for the differential operator (110) with boundary conditions (113) is

$$G(x, t)=-(x-t)_{+}+x, \tag{115}$$

where

$$(x-t)_{+}=0 \qquad \text{for } 0\leq x\leq t\leq 1,$$

or

$$G(x, t)=\begin{cases} t, & 0\leq t\leq x\leq 1, \\ x, & 0\leq x\leq t\leq 1. \end{cases} \tag{116}$$

Using (112) and (114), one obtains

$$U(x)=\sin(\sqrt{\lambda}x)/\sqrt{\lambda}, \qquad V(t)=\cos[\sqrt{\lambda}(1-t)], \tag{117}$$

from which the Fredholm determinant $\Delta(\lambda)$ of $G(x, t)$ is seen to be

$$\Delta(\lambda)=\cos\sqrt{\lambda}. \tag{118}$$

If

$$\Delta(\lambda)\neq 0,$$

then the resolvent kernel $\Gamma(x, t; \lambda)$ of $G(x, t)$ may be written as

$$\Gamma(x, t; \lambda)=-\sin[\sqrt{\lambda}(x-t)_{+}]/\sqrt{\lambda}+\sin(\sqrt{\lambda}x)\cos[\sqrt{\lambda}(1-t)]/\sqrt{\lambda}\cos\sqrt{\lambda}, \tag{119}$$

corresponding to (115), or as the symmetric simple separable kernel

$$\Gamma(x, t; \lambda)=\begin{cases} \sin(\sqrt{\lambda}t)\cos[\sqrt{\lambda}(1-x)]/\sqrt{\lambda}\cos\sqrt{\lambda}, & 0\leq t\leq x\leq 1, \\ \sin(\sqrt{\lambda}x)\cos[\sqrt{\lambda}(1-t)]/\sqrt{\lambda}\cos\sqrt{\lambda}, & 0\leq x\leq t\leq 1, \end{cases} \tag{120}$$

of the same form as (116). It also follows immediately from (118) that the eigenvalues of $G(x, t)$ are

$$\lambda_n^*=\{[(2n-1)/2]\pi\}^2, \qquad n=1, 2, 3, \ldots, \tag{121}$$

and the corresponding eigenfunctions are proportional to

$$\sqrt{\lambda_n^*}U_n^*(x) = \sin[(2n-1)\pi x/2], \qquad n = 1, 2, 3, \ldots. \tag{122}$$

It is also possible to write down explicit formulas for the more general second-order boundary-value problem

$$\begin{aligned} L[y(x)] &= -y''(x), \\ ay'(0) + by(0) &= 0, \\ cy'(1) + dy(1) &= 0, \end{aligned} \tag{123}$$

by the use of (107), (111), and (112). The Green's function $G(x, t)$ is obtained from

$$\begin{aligned} u(x) &= (1/\delta)[a - bx], \\ v(t) &= c + d(1-t), \end{aligned} \tag{124}$$

provided

$$\delta = ad - b(c+d) \neq 0,$$

the resolvent kernel $\Gamma(x, t; \lambda)$ of $G(x, t)$ from

$$\begin{aligned} U(x) &= (1/\delta)[a\cos(\sqrt{\lambda}x) - (b/\sqrt{\lambda})\sin(\sqrt{\lambda}x)], \\ V(t) &= c\cos[\sqrt{\lambda}(1-t)] + (d/\sqrt{\lambda})\sin[\sqrt{\lambda}(1-t)], \end{aligned} \tag{125}$$

and the Fredholm determinant is

$$\Delta(\lambda) = (1/\delta)[(ad-bc)\cos\sqrt{\lambda} - (ac\sqrt{\lambda} + bd/\sqrt{\lambda})\sin\sqrt{\lambda}]. \tag{126}$$

The eigenvalues of $G(x, t)$ may thus be found by solving the simple transcendental equation

$$\tan\sqrt{\lambda}/\sqrt{\lambda} = (ad-bc)/(\lambda ac + bd), \tag{127}$$

for $\lambda_1, \lambda_2, \ldots$, and the results substituted into (125) to obtain the corresponding eigenfunctions. To simplify these expressions, one may introduce the angles

$$\phi = \phi(\lambda) = \tan^{-1}(b/a\sqrt{\lambda}), \qquad \theta = \theta(\lambda) = \tan^{-1}(c\sqrt{\lambda}/d), \tag{128}$$

which gives

$$\begin{aligned} U(x) &= (1/\delta)\sqrt{(a^2 + b^2/\lambda)}\cos(\sqrt{\lambda}x + \phi), \\ V(t) &= \sqrt{(c^2 + d^2/\lambda)}\sin[\sqrt{\lambda}(1-t) + \theta], \end{aligned} \tag{129}$$

and

$$\Delta(\lambda) = (1/\delta)\sqrt{[(a^2\lambda + b^2)(c^2\lambda + d^2)/\lambda]}\cos(\sqrt{\lambda} + \phi + \theta). \tag{130}$$

Thus,

$$\Gamma(x,t;\lambda)=\begin{cases}\cos(\sqrt{\lambda}t+\phi)\sin[\sqrt{\lambda}(1-x)+\theta]/\sqrt{\lambda}\cos(\sqrt{\lambda}+\phi+\theta), & 0\le t\le x\le 1,\\ \cos(\sqrt{\lambda}x+\phi)\sin[\sqrt{\lambda}(1-t)+\theta]/\sqrt{\lambda}\cos(\sqrt{\lambda}+\phi+\theta), & 0\le x\le t\le 1,\end{cases} \tag{131}$$

provided, of course, that

$$\Delta(\lambda)\neq 0.$$

From (130), the eigenvalues $\lambda_1, \lambda_2, \ldots$ must satisfy

$$\sqrt{\lambda_n}+\phi(\lambda_n)+\theta(\lambda_n)=[(2n-1)/2]\pi, \qquad n=1,2,\ldots, \tag{132}$$

or

$$\sqrt{\lambda_n}+\cot^{-1}[(ad-bc)\sqrt{\lambda_n}/(\lambda_n ac+bd)]=[(2n-1)/2]\pi, \qquad n=1,2,\ldots, \tag{133}$$

which is equivalent to (127).

The method given above extends readily to two-point boundary-value problems of arbitrary order. In general, the Volterra kernel $K(x, t)$ is obtained by integrating (84) as an initial-value problem with zero initial conditions; for example, for

$$L[y(x)]=-y^{\text{iv}}(x), \tag{134}$$

one obtains

$$K(x,t)=-(x-t)^3/3!, \qquad 0\le t\le x\le 1. \tag{135}$$

The functions $u_1(x), \ldots, u_n(x)$ and $v_1(t), \ldots, v_n(t)$ which give the Green's function $G(x, t)$ as a finite-rank modification of $K(x, t)$ are then found by solving for the initial conditions, in terms of integral transforms of $h(x)$, to satisfy the given boundary conditions, it being assumed that the resulting system of equations has a unique solution. Of course, only self-adjoint boundary-value problems give rise to symmetric Green's functions. In the next section, nonsymmetric separable kernels will be discussed.

For *nonlinear boundary-value problems,*

$$h(x)=f(x, y(x)), \tag{136}$$

the use of the Green's function $G(x, t)$ leads to a *Hammerstein integral equation*

$$y(x)-\int_0^1 G(x,t)f(t,y(t))\,dt=0. \tag{137}$$

If $G(x, t)$ is of the form (24), then (137) is equivalent to the nonlinear system

$$y(x)-\int_0^x K(x, t)f(t, y(t))\,dt=\alpha u(x),$$
$$\alpha=\int_0^1 v(t)f(t, y(t))\,dt. \tag{138}$$

The first equation of (138) is a nonlinear Volterra integral equation. If this can be solved for

$$y(x)=y(x;\alpha), \tag{139}$$

then substitution into the second equation gives the single nonlinear scalar equation

$$\alpha=\int_0^1 v(t)f(t, y(t;\alpha))\,dt=\phi(\alpha) \tag{140}$$

for α. In general, this procedure for solution cannot be carried out explicitly, as in the linear case, so various approximation methods, usually based on iterations, have been studied for this problem. In particular, it may happen that

$$\alpha=y'(0),$$

in which case the iterative determination of α, and hence $y(x)$, is sometimes referred to as a *shooting method* (Ref. 14). If $G(x, t)$ is of the form (53), then a similar construction leads to a single nonlinear Volterra integral equation for

$$y(x)=y(x;\alpha_1,\ldots,\alpha_n)$$

and the nonlinear scalar system

$$\alpha_i=\int_0^1 v_i(t)f(t, y(t;\alpha_1,\ldots,\alpha_n))\,dt, \qquad i=1,2,\ldots,n, \tag{141}$$

for $\alpha_1, \alpha_2, \ldots, \alpha_n$. Further discussion of nonlinear boundary-value problems is outside the scope of this paper.

6. Nonsymmetric Case

In certain problems, one may have to deal with separable kernels which are nonsymmetric, such as finite linear combinations of kernels of the form $r(x)G(x, t)w(t)$, where $G(x, t)$ is a symmetric, simple separable kernel (24). As a prototype of separable kernels in the general case, consider the simple kernel

$$G(x, t)=\begin{cases} u(t)v(x), & 0\leq t\leq x\leq 1, \\ p(x)q(t), & 0\leq x\leq t\leq 1, \end{cases} \tag{142}$$

subject to the condition

$$u(x)v(x)=p(x)q(x), \qquad 0\leq x\leq 1, \tag{143}$$

which ensures that the traces of $G(x, t)$ and its associated kernels are uniquely defined. Formulas will now be developed for the Fredholm determinant $\Delta(\lambda)$ and the resolvent kernel $\Gamma(x, t; \lambda)$ of the simple separable kernel (142). The general separable kernel, which is a finite linear combination of kernels of the form (142), can then be handled by the technique given in Section 4, or by the method to be discussed in Section 7.

Writing the Fredholm integral equation (25) as the system

$$\begin{aligned} y(x)-\lambda\int_0^x K(x, t)y(t)\,dt&=f(x)+\lambda cp(x), \\ c&=\int_0^1 q(t)y(t)\,dt, \end{aligned} \tag{144}$$

where

$$K(x, t)=u(t)v(x)-p(x)q(t), \tag{145}$$

one obtains, as before,

$$\Delta(\lambda)=1-\lambda\int_0^1 q(x)P(x)\,dx=1-\lambda\int_0^1 Q(t)p(t)\,dt, \tag{146}$$

with the functions $P(x)$, $Q(t)$ given by

$$\begin{aligned} P(x)&=p(x)+\lambda\int_0^x R(x, t; \lambda)p(t)\,dt, \\ Q(t)&=q(t)+\lambda\int_t^1 q(x)R(x, t; \lambda)\,dx, \end{aligned} \tag{147}$$

in terms of the Volterra resolvent kernel $R(x, t; \lambda)$ of $K(x, t)$. If

$$\Delta(\lambda)\neq 0,$$

then Eq. (25) has the unique solution

$$y(x)=f(x)+\lambda\int_0^x R(x, t; \lambda)f(t)\,dt+[\lambda/\Delta(\lambda)]\int_0^1 P(x)Q(t)f(t)\,dt, \tag{148}$$

from which it follows that the Fredholm resolvent kernel of the kernel (142) is

$$\Gamma(x, t; \lambda) = \begin{cases} R(x, t; \lambda) + P(x)Q(t)/\Delta(\lambda), & 0 \le t \le x \le 1, \\ P(x)Q(t)/\Delta(\lambda), & 0 \le x \le t \le 1. \end{cases} \tag{149}$$

Alternatively, one may write

$$y(x) + \lambda \int_x^1 K(x, t)y(t)\, dt = f(x) + \lambda\, dv(x), \qquad d = \int_0^1 u(t)y(t)\, dt. \tag{150}$$

This leads to the expressions

$$\Delta(\lambda) = 1 - \lambda \int_0^1 u(x)V(x)\, dx = 1 - \lambda \int_0^1 U(t)v(t)\, dt \tag{151}$$

for the Fredholm determinant of $G(x, t)$ in terms of the functions

$$V(x) = v(x) - \lambda \int_x^1 R(x, t; \lambda)v(t)\, dt, \qquad U(t) = u(t) - \lambda \int_0^t u(x)R(x, t; \lambda)\, dx; \tag{152}$$

and, if

$$\Delta(\lambda) \neq 0,$$

this leads to the unique solution

$$y(x) = f(x) - \lambda \int_x^1 R(x, t; \lambda)f(t)\, dt + [\lambda/\Delta(\lambda)] \int_0^1 V(x)U(t)f(t)\, dt \tag{153}$$

of (25). Thus,

$$\Gamma(x, t; \lambda) = \begin{cases} U(t)V(x)/\Delta(\lambda), & 0 \le t \le x \le 1, \\ -R(x, t; \lambda) + U(t)V(x)/\Delta(\lambda), & 0 \le x \le t \le 1, \end{cases} \tag{154}$$

provided that the Fredholm determinant does not vanish. In (152)–(154), use is made of the extension of the function $R(x, t; \lambda)$, defined by (8), to the triangle $0 \le x \le t \le 1$.

Comparison of (154) and (149) yields

$$\Delta(\lambda)R(x, t; \lambda) = U(t)V(x) - P(x)Q(t), \qquad 0 \le x, t \le 1, \tag{155}$$

and finally,

$$\Gamma(x, t; \lambda) = \begin{cases} U(t)V(x)/\Delta(\lambda), & 0 \le t \le x \le 1, \\ P(x)Q(t)/\Delta(\lambda), & 0 \le x \le t \le 1. \end{cases} \tag{156}$$

Theorem 6.1. If $\Delta(\lambda) \neq 0$, then the simple separable kernel (156) is the Fredholm resolvent kernel of the simple separable kernel (142).

If

$$\Delta(\lambda^*) = 0,$$

then it follows from (155) that

$$U^*(t)V^*(x) = P^*(x)Q^*(t), \tag{157}$$

where

$$\begin{aligned} U^*(t) &= U(t; \lambda^*), & Q^*(t) &= Q(t; \lambda^*), \\ V^*(x) &= V(x; \lambda^*), & P^*(x) &= P(x; \lambda^*) \end{aligned} \tag{158}$$

are obtained from (147) and (152) with

$$\lambda = \lambda^*.$$

It follows from (157) that $V^*(x)$ and $P^*(x)$ are linearly dependent, and either may be taken as a right eigenfunction of $G(x, t)$ corresponding to the eigenvalue λ^*; similarly, $U^*(t)$ and $Q^*(t)$ are proportional and furnish a corresponding left eigenfunction of the kernel (142).

Of course, the functions $U(t)$, $V(x)$, $P(x)$, $Q(t)$ may be found by solving the appropriate Volterra integral equations if it is desired to avoid the explicit computation of $R(x, t; \lambda)$.

7. Alternative Computational Method

Suppose that $G(x, t)$ is a Fredholm or Volterra kernel with known resolvent kernel $\Gamma(x, t; \lambda)$; and suppose that it is desired to construct the resolvent kernel $\Gamma_n(x, t; \lambda)$ of the modified kernel

$$G_n(x, t) = G(x, t) + \sum_{j=1}^{n} u_j(x)v_j(t). \tag{159}$$

In the special case that

$$G(x, t) = K(x, t)$$

is a Volterra kernel, one form of the solution of this problem is given by (67), which requires the inversion of an $n \times n$ matrix. It is also possible to obtain the resolvent kernel by a step-by-step process similar to an elimination method for matrix inversion. First of all, consider the rank-one modification

$$G_1(x, t) = G(x, t) + u_1(x)v_1(t). \tag{160}$$

By applying the method of Section 4 [solving Eq. (25) with the kernel (160)], one obtains

$$\Delta_1(\lambda) = 1 - \lambda \int_0^1 v_1(x)U_1(x)\,dx = 1 - \lambda \int_0^1 V_1(t)u_1(t)\,dt \tag{161}$$

for the Fredholm determinant of $G_1(x, t)$, where

$$\begin{aligned} U_1(x) &= u_1(x) + \lambda \int_0^1 \Gamma(x, t; \lambda)u_1(t)\,dt, \\ V_1(t) &= v_1(x) + \lambda \int_0^1 v_1(x)\Gamma(x, t; \lambda)\,dx. \end{aligned} \tag{162}$$

Theorem 7.1. If $\Delta_1(\lambda) \neq 0$, then the resolvent kernel $\Gamma_1(x, t; \lambda)$ of the rank-one modification $G_1(x, t)$ of $G(x, t)$ is

$$\Gamma_1(x, t; \lambda) = \Gamma(x, t; \lambda) + U_1(x)V_1(t)/\Delta_1(\lambda), \tag{163}$$

which is a rank-one modification of the resolvent kernel $\Gamma(x, t; \lambda)$.

This result has been exploited a number of times previously; see, for example, (38), (39), (44), (55), (119), (149), and (154). It is analogous to the Sherman–Morrison–Woodbury formula for finite matrices (Ref. 15, pp. 123–124). Now, instead of (159), one may consider the sequence of kernels

$$\begin{aligned} G_0(x, t) &= G(x, t), \\ G_k(x, t) &= G_{k-1}(x, t) + u_k(x)v_k(t), \qquad k = 1, 2, \ldots, n, \end{aligned} \tag{164}$$

each of which is a rank-one modification of the previous kernel. Taking

$$\Gamma_0(x, t; \lambda) = \Gamma(x, t; \lambda), \tag{165}$$

one may construct the corresponding sequence of resolvent kernels

$$\Gamma_k(x, t; \lambda) = \Gamma_{k-1}(x, t; \lambda) + U_k(x)V_k(t)/\Delta_k(\lambda), \qquad k = 1, 2, \ldots, n, \tag{166}$$

where

$$\begin{aligned} U_k(x) &= u_k(x) + \lambda \int_0^1 \Gamma_{k-1}(x, t; \lambda)u_k(t)\,dt, \\ V_k(t) &= v_k(t) + \lambda \int_0^1 v_k(x)\Gamma_{k-1}(x, t; \lambda)\,dx, \end{aligned} \tag{167}$$

and

$$\Delta_k(\lambda) = 1 - \lambda \int_0^1 v_k(x) U_k(x)\, dx = 1 - \lambda \int_0^1 V_k(t) u_k(t)\, dt, \qquad (168)$$

and obtain $\Gamma_n(x, t; \lambda)$, provided, of course, that

$$\Delta_k(\lambda) \neq 0, \qquad k = 1, 2, \ldots, n. \qquad (169)$$

8. Numerical Implications

For a Fredholm integral equation with a separable kernel $G(x, t)$, it has been shown that methods appropriate to Volterra integral equations can be used to obtain the Fredholm determinant, resolvent kernel, and eigenfunctions of $G(x, t)$. As better estimates are available for convergence of the resulting expansions than in the general Fredholm case, effective analytic or approximate computations can be carried out. Although explicit formulas can be obtained only for very simple problems, it may be that the Volterra kernel $K(x, t)$ is a polynomial or other simple function of x and t, in which case a computer can be programmed to find the coefficients in the expansion (8) of the resolvent kernel $R(x, t; \lambda)$ to obtain any desired degree of accuracy. Eigenvalues of $G(x, t)$ can also be obtained by computing zeros of the entire function $\Delta(\lambda)$. This is a more difficult problem, but can once again be done with any desired accuracy.

If the analytic or semi-analytic approach appears fruitless or uneconomical, then strictly numerical methods based on numerical integration may be used. For the Volterra integral equations considered, these lead to lower triangular systems of equations which can be solved quickly and accurately, even if large (Refs. 16–17). As numerical integration is usually much more accurate than numerical differentiation, this technique offers an alternative to finite-difference methods for two-point boundary-value problems for ordinary differential operators.

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