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# Optimal Skill Mix: An Application of the Maximum Principle for Systems with Retarded Controls[1]

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**Abstract.** In this paper, we analyze the optimal skill mix in a model with two kinds of imperfectly substitutable labor, skilled and unskilled. The population is characterized by a distribution of innate abilities, and individuals are trained according to optimal rules or market rules (with imperfect expectations); the length of each individual's training period depends upon his innate ability. The market and optimal rules are characterized and compared, and corrective policies are investigated. This model represents a major advance over earlier models, which are based on the following assumptions: (a) either unskilled and skilled labor are perfectly substitutable or training is a necessary condition for employment; (b) individuals are innately identical; (c) in most cases, training occurs either instantaneously or with fixed lag.

**Key Words.** Human capital, optimal control, retarded control problems, labor training, economics, maximum principle.

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## 1. Introduction

A large number of the applications of control theory to economics is concentrated in the area of economic growth.[4] In most of these applications, labor is taken as an homogeneous good, exogenously supplied, and fully employed. Dobell and Ho (Ref. 2) were the first to treat labor training; the state variables in their model are per capita capital stock $k$ and fraction of the population employed $x_0$. Training is a necessary condition for employment; training occurs instantaneously and is assumed to cost the same ($d$ GNP units) for all individuals. Dobell and Ho solve the problem of maximizing the discounted sum of per capita consumption over a finite horizon subject to initial and terminal conditions on the state variables. In a later paper (Ref. 3), Dobell and Ho extend this model by making training cost an increasing function of fractional employment $x_0$. An interesting consequence of this modification is that there does not exist a time-varying interest rate on education loans which adequately reflects the externality of rising training costs. Blackburn (Ref. 4) observed that the Dobell and Ho result rests on several implicit assumptions, and that equally plausible assumptions (within the framework of the Dobell and Ho model) lead to a full employment result. For example, if all individuals are trained to some level of skill, then the economy will have full employment, and the problem is to determine the optimal level of skill. In his reply to Blackburn, Dobell (Ref. 5) develops a model which distinguishes different vintages of labor and notes the level of training appropriate at the time of entry of each vintage into employment. The state $x_0$ is changed to $w$, a productivity-weighted measure of the effective labor force; he also introduces an exponential decay in skills. With this model, Dobell confirms Blackburn's observations.

Tu (Ref. 6) introduced a simple model which has two levels of labor, illiterate and educated, appearing in the production function. His analysis assumes a linear production function and balanced growth where the investment in physical capital is constant and the ratio of illiterates to those undergoing education is constant.

Budelis (Ref. 7, Chapter 3) developed the dynamics of the model described in Dobell (Ref. 5). Since the number of years an individual is in the labor force is fixed, the resulting optimal control problem is one of constant lag. The control variables are the fraction of eligible population to be trained and the level of training. Training is considered to be instantaneous with its cost depending on its level. Budelis uses a linear utility function; consequently, after an initial constrained arc, the optimal path

[4] See Ref. 1, for example.

approaches the equilibrium point along a singular arc. The golden-rule policy is obtained, and its stability characteristics are examined. Since the population is assumed homogeneous, the model is characterized by the absence of unemployment in the economy following an optimal growth path, except in the extreme degenerate case where the optimal path is one of zero level of training, i.e., where all newcomers are left unemployed. Numerical solutions are carried out for finite horizon cases. The percentage gain of the optimal control path over the golden-rule path is computed. The gain is small for high initial capital stock and large for low initial capital stock. In most cases, the gain decreases as the horizon increases.

The effect of the skill deterioration rate is also examined. For short skill deterioration time, the average equilibrium output per worker reaches a maximum at a low retirement age; thus, the model suggests that retraining should be considered as a policy for investment in human capital.

What we have described thus far includes only the models which treat training as instantaneous. The particular level of training is achieved by a proper intensity and quality of training and not by duration of training. In the following, we will discuss models with noninstantaneous training resulting in optimal control problems with delays.

Lele, Jacobson, and McCabe (Ref. 8) extended the Dobell and Ho model (Ref. 3) by considering lags which may occur in the training sector in the adjustment of educational services and also between the demanded labor and the existing labor. In connection with educational services, they postulated a simple relationship involving a delayed adjustment. The actual change in the trained labor force and the desired change has a Koyck distributed lag structure. A golden-rule point requiring full employment does exist for their model. However, their objective function allows full employment to be attained almost instantaneously because of a quadratic penalty placed on deviations from some specified mean capital investment rate. A quadratic cost is also placed on the rate of change of the level of employment. They conclude that some kind of lag structure does exist and that it is qualitatively different from no lag. The policy implications may be the use of on-the-job training which may be considered instantaneous. In this case, one must, of course, have different skill levels in the model. Lele (Ref. 9) has computed the optimal path for the Lele–Jacobson–McCabe model.

Budelis (Ref. 7, Chapter 4) extended the Dobell and Ho model (Ref. 3) by taking the training delay into account. This introduces a delay in the control variable and introduces an additional state for labor in the training pool. A golden-rule point which is globally stable under the golden-rule policy is shown to exist. The optimal path is characterized by using the maximum principle. It is shown that control is optimal at its upper bound until full employment is reached; then, full employment is maintained by a suitable singular control which will take the trajectory to the equilibrium point.[5]

**1.1. Scope of This Paper.** In the next section, we develop a model for analyzing labor dynamics which extends previous work in the following important ways.

(i) Earlier models have assumed that all individuals are innately identical; we assume a distribution of innate abilities.

(ii) Earlier work has either assumed that training is a necessary condition for employment or has assumed that skilled and unskilled labor are perfect substitutes (although not, of course, on a one-to-one basis); we allow imperfect substitutability between skilled and unskilled labor.

Because of our assumption of a distribution of innate abilities, our model leads to a continuous-lag optimal control problem; thus, it does not reduce to the fixed-lag optimal control problem like those of Lele, Jacobson, and McCabe (Ref. 8) and Budelis (Ref. 7) or to a model equivalent to the various vintage capital models in the economics literature. Such models have not been discussed extensively in the control theory literature, and we know of no other economics paper in this framework.

## 2. Model

The model consists of a production function (which relates output to the labor inputs; we abstract from capital in this model), an exogeneous labor supply function, and equations describing the time rate of change of skilled and unskilled labor and trainees.

**2.1. Notational Convention.** Let $z_i(t)$ be the value of the $i$th state variable, $i = 1, \ldots, n$, at time $t$, $t \geq 0$. Then,

$$\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))';$$

$\mathbf{z}_i$ is the trajectory $z_i(t)$, $t \geq 0$, and $\mathbf{z}$ is the trajectory $\mathbf{z}(t)$, $t \geq 0$. Where no confusion arises, we may drop the time argument; thus $z_i = z_i(t)$ but $z_i \neq \mathbf{z}_i$.

**2.2. Labor Supply.** It is assumed that new entrants into the labor force at time $t$ occur exogenously at the rate $\beta L(t)$ and that retirements at

[5] The Budelis model (Ref. 7, Chapter 4) needs some revisions which we carry out in Ref. 10 along with some simplifications and extensions. We do not describe this revised Budelis model here, since what follows is a generalization of it.

time $t$ occur exogenously at the rate $\gamma L(t)$ (we shall at times refer to $\beta$ and $\gamma$ as the birth rate and the death rate, respectively). Letting

$$n = \beta - \gamma,$$

the net rate of increase in the labor force is

$$\dot{L}(t) = nL(t), \qquad L(0) = L_0;$$

or, equivalently,

$$L(t) = L_0 \exp(nt).$$

**2.3. Trainees.** It is assumed that there is a time-independent unimodal density function $h(\xi)$, $\xi \geq 0$, such that

$$H(u) = \int_0^u h(\xi)\, d\xi, \qquad H(0) = 0, \quad H(\infty) = 1, \tag{1}$$

is the proportion of the new entrants into the labor force at any instant that could be trained in no more that $u$ years; we will call the value of $\xi$ associated with each individual his *untrainability index*. Let $u = u(t)$ be the length of time (years) such that, at time $t$, any new entrant into the labor force with an untrainability index no more than $u(t)$ will enter the training program. The number of workers who entered training at time $(t-\tau)$ is thus

$$\beta L(t-\tau) \int_0^{u(t-\tau)} h(\xi)\, d\xi = \beta L(t-\tau) H[u(t-\tau)].$$

The number of those workers who are still in training programs at time $t$ is then

$$\beta L(t-\tau) \exp(-\gamma\tau) \int_\tau^{\phi[\tau, u(t-\tau)]} h(\xi)\, d\xi,$$

where

$$\phi[\tau, u(t-\tau)] = \max[\tau, u(t-\tau)]. \tag{2}$$

Thus the total number of workers in training programs at time $t$ is

$$Y(t) = \int_0^\infty \beta L(t-\tau) \exp(-\gamma\tau) \int_\tau^{\phi[\tau, u(t-\tau)]} h(\xi)\, d\xi\, d\tau.$$

Since

$$L(t-\tau) = \exp[-(\beta-\gamma)\tau] L(t),$$

the fraction of the labor force in training at time $t$ is

$$y(t) \equiv Y(t)/L(t) = \beta \int_0^{\infty} \exp(-\beta\tau) \int_{\tau}^{\phi[\tau, u(t-\tau)]} h(\xi)\, d\xi\, d\tau$$
$$= \beta \int_{-\infty}^{t} \exp[-\beta(t-\alpha)] \int_{t-\alpha}^{\phi[t-\alpha, u(\alpha)]} h(\xi)\, d\xi\, d\alpha. \tag{3}$$

Then, the time rate of change of $y(t)$ is

$$\dot{y}(t) = \beta\{-y(t) - c + H[u(t)]$$
$$+ \int_{-\infty}^{t} \exp[-\beta(t-\tau)] h[\phi(t-\tau, u(\tau))]\phi_1[t-\tau, u(\tau)]\, d\tau\},$$
$$y(t) = \underline{y}(t), \qquad t \in [-\infty, 0], \tag{4}$$

where

$$c = \int_0^{\infty} \exp(-\beta\xi) h(\xi)\, d\xi \tag{5}$$

and

$$\phi_1(x, y) = \partial\phi(x, y)/\partial x, \qquad x \neq y.$$

Note that

$$\phi_1(x, y) = \begin{cases} 1, & \text{if } x > y, \\ 0, & \text{if } x < y. \end{cases} \tag{6}$$

While the control variable $u(t)$ may be highly artificial, it should be noted that it has a simple economic interpretation. First, of course, in a planned economy, the government could issue a decree that everybody capable of being trained in no more than $u(t)$ years should enter a training program. Furthermore, if individuals behave so as to maximize the present value of their expected income streams, then there is a monotonic relationship between $u(t)$ and the wage ratio [whether determined by the market or by the planners, see Eq. (51)]. Thus, implementing the control is relatively straightforward.

**2.4. Unskilled Workers.** The fraction of the new entrants to the labor force at time $t$ that remains unskilled is clearly $1 - H[u(t)]$. Thus, the stock of unskilled workers at time $t$ is

$$L_{\infty}(t) = \int_0^{\infty} \{1 - H[u(t-\tau)]\}\beta L(t-\tau) \exp(-\gamma\tau)\, d\tau.$$

Letting $x_\infty(t)$ represent the fraction of the labor force which is unskilled (and not in training) at time $t$,

$$x_\infty(t) = \beta \int_0^\infty \exp(-\beta\tau)\{1 - H[u(t-\tau)]\}\, d\tau.$$

The time rate of change of $x_\infty(t)$ is

$$\dot{x}_\infty(t) = \beta\{1 - H[u(t)] - x_\infty(t)\}, \qquad x_\infty(t) = \underline{x}_\infty(t), \qquad t \in [-\infty, 0]. \quad (7)$$

**2.5. Skilled Workers.** Let $x_0(t)$ be the fraction of the labor force which is skilled at time $t$. Since

$$x_0(t) + x_\infty(t) + y(t) \equiv 1, \quad (8)$$

it follows that

$$\dot{x}_0(t) = -\dot{x}_\infty(t) - \dot{y}(t)$$
$$= \beta\left\{c - x_0(t) - \int_{-\infty}^{t} \exp[-\beta(t-\tau)]h[\phi(t-\tau, u(\tau))]\phi_1[t-\tau, u(\tau)]\, d\tau\right\}, \quad (9)$$
$$x_0(t) = \underline{x}_0(t), \qquad t \in [-\infty, 0].$$

Please note that the given past histories $\underline{x}_\infty(t)$ and $\underline{x}_0(t)$, $t \in [-\infty, 0]$, are not independent initial conditions. In fact, these can be obtained from any given past history $\underline{u}(t)$, $t \in [-\infty, 0]$. A more convenient way of specifying the initial condition, therefore, is to simply specify

$$u(t) = \underline{u}(t), \qquad t \in [\ -\infty, 0].$$

**2.6. Production Function.** We assume that the rate of output $F(t)$ at time $t$ is given by the production function

$$F(t) = F[L_0(t), L_\infty(t)]. \quad (10)$$

It is assumed that $F$ is linear homogeneous,

$$F(\lambda x, \lambda y) = \lambda F(x, y),$$

concave, and satisfies the usual neoclassical conditions

$$F(0, 0) = 0,$$
$$\partial F(x, y)/\partial x > 0, \qquad \partial F(x, y)/\partial y > 0,$$
$$\partial^2 F(x, y)/\partial x^2 < 0, \qquad \partial^2 F(x, y)/\partial y^2 < 0, \qquad \partial^2 F(x, y)/\partial x\, \partial y > 0.$$

Defining output per capita (actually, per member of the labor force) $f(t)$ as

$$f(t) = F(t)/L(t),$$

then the per-capita-production function may be written as

$$f(t) = F[x_0(t), x_\infty(t)].$$

An important class of functions satisfying the assumptions listed above is the class of constant elasticity of substitution (CES) production functions

$$F[x_0(t), x_\infty(t)] = \{\delta[\pi_0 x_0(t)]^{-\sigma} + (1-\delta)[\pi_\infty x_\infty(t)]^{-\sigma}\}^{-1/\sigma}, \qquad (11)$$

where $\delta$ is a distribution parameter independent of $\sigma$ and

$$(1+\sigma)^{-1} \geq 0$$

is the elasticity of substitution between skilled and unskilled workers (Ref. 11); except for the case $\sigma = \infty$, we always assume that the parameters $\pi_0$ and $\pi_\infty$ are equal to unity. When

$$\sigma = -1,$$

skilled and unskilled labor are perfectly substitutable and the production function becomes linear in the labor inputs:

$$f(t) = \delta x_0(t) + (1-\delta) x_\infty(t). \qquad (12)$$

In this case, the contribution of any individual to output is independent of the number or composition of other workers. In other words, the relative unit of production is the individual. To anticipate some results, because of this independence, this is the one case for which a competitive market results in optimal dynamic training decisions.

When $\sigma \to \infty$, the production function becomes

$$f(t) = \min[\pi_0 x_0(t), \pi_\infty x_\infty(t)]. \qquad (13)$$

These two cases are qualitatively different from all other cases, i.e.,

$$-1 < \sigma < \infty.$$

Another case, which deserves mention, because of its extensive treatment in the economics literature, although it is not qualitatively different from the general case for our purposes, is the celebrated Cobb–Douglas production function, obtained by taking the limit of (11) as $\sigma \to 0$.

**2.7. Control Problem.** The problem, then, is to choose the trajectory $u(t)$, the critical untrainability level, such that all new entrants at time $t$ with untrainability indices not greater than $u(t)$ will enter the training program, so as to maximize the present value of future output:

$$\max_{u(t) \in R^+} \int_0^\infty \alpha(t) f(t)\, dt. \qquad (14)$$

The maximization procedure is subject to constraints on the rates of change of unskilled and skilled labor, (7) and (9), initial conditions on the state variables, nonnegativity of $u(t)$, $t \geq 0$, and the inequality constraints

$$x_0(t) \geq 0, \qquad x_\infty(t) \geq 0, \qquad x_0(t) + x_\infty(t) \leq 1, \qquad \forall t. \tag{15}$$

The variable $\alpha(t)$ is a discount factor satisfying the usual convergence assumption

$$\int_0^\infty \alpha(t)\, dt < \infty.$$

If the objective is to maximize the present value of the future stream of per capita GNP, then

$$\alpha(t) = \exp(-rt),$$

where $r > 0$ is the social rate of discount; while, if the goal is to maximize total GNP, then

$$\alpha(t) = L(0) \exp[-(r-n)t], \qquad r > n.$$

For convenience, we assume, with no loss of generality, that

$$L_0 = 1.$$

## 3. Optimal Control

In this section, we discuss the optimal control for the model presented in Section 2.

**3.1. Necessary and Sufficient Conditions.** Before stating the necessary and sufficient conditions for an optimal control, we show that any solution will satisfy the inequality constaints [or, equivalently, nonnegativity restrictions on the three state variables $x_0(t)$, $x_\infty(t)$, and $y(t)$, which must sum to unity] if the initial states are nonnegative.

**Lemma 3.1. Nonnegativity.** If the inequality constraints (15) are satisfied at $t = 0$ by the initial conditions $\underline{x}_0$, $\underline{x}_\infty$, then the solution of the integro-differential equation system (7), (9) will satisfy (15) for all $t$.

**Proof.** When $x_\infty(t) = 0$,

$$\dot{x}_\infty(t) = \beta\{1 - H[u(t)]\} \geq 0;$$

see (1) and (7). Since $\underline{x}_\infty \geq 0$, it follows immediately that $x_\infty(t) \geq 0$ for all $t \geq 0$. Similarly, when $x_0(t) = 0$,

$$\dot{x}_0(t) = \beta \left\{ \int_0^\infty \exp(-\beta\xi) h(\xi)\, d\xi - \int_0^\infty \exp(-\beta\xi) h[\phi(\xi, u(t-\xi))]\phi_1[\xi, u(t-\xi)]\, d\xi \right\}$$

$$= \beta \left( \int_0^\infty \exp(-\beta\xi)\{h(\xi) - h[\phi(\xi, u(t-\xi))]\phi_1[\xi, u(t-\xi)]\}\, d\xi \right);$$

see (5) and (9). If $\xi > u(t-\xi)$,

$$h(\xi) - h[\phi(\xi, u(t-\xi))]\phi_1[\xi, u(t-\xi)] = h(\xi) - h(\xi) \cdot 1 = 0;$$

see (2) and (6); while, if $\xi < u(t-\xi)$,

$$h(\xi) - h[\phi(\xi, u(t-\xi))]\phi_1[\xi, u(t-\xi)] = h(\xi) - h[u(t-\xi)] \cdot 0 = h(\xi) \geq 0;$$

see (2) and (6). Since $\underline{x}_0 \geq 0$, nonnegativity of $x_0(t)$ is assured for $t \geq 0$.

To prove that

$$x_0(t) + x_\infty(t) \leq 1,$$

note that, when

$$x_0(t) + x_\infty(t) = 1,$$

$$\dot{x}_0(t) + \dot{x}_\infty(t) = -\beta H[u(t)] + \beta \int_0^\infty \exp(-\beta\xi) \times \{h(\xi) - h[\phi(\xi, u(t-\xi))]\phi_1[\xi, u(t-\xi)]\}\, d\xi;$$

see (5), (7), and (9). Since

$$y(t) = 0$$

by (8), it follows that the integral above is zero; thus,

$$\dot{x}_0(t) + \dot{x}_\infty(t) = -\beta H[u(t)] \leq 0.$$

Since

$$\underline{x}_0 + \underline{x}_\infty \leq 1,$$

we have

$$x_0(t) + x_\infty(t) \leq 1$$

for all $t \geq 0$, thereby completing the proof of the lemma.

The current-value formulation (Ref. 12) of the Hamiltonian

$$\mathcal{H} = \mathcal{H}(u)$$

is

$$\begin{aligned}\mathcal{H}[u(t)] &= F[x_0(t), x_\infty(t)] + \lambda_\infty(t)\beta[1 - H[u(t)] - x_\infty(t)] + \lambda_0(t)\beta[c - x_0(t)] \\ &\quad -\beta \int_t^\infty \lambda_0(\tau)\alpha(\tau - t)\exp[-\beta(\tau - t)]h\{\phi[\tau - t, u(t)]\}\phi_1[\tau \\ &\qquad - t, u(t)]\,d\tau \\ &= \beta W[u(t)] + F[x_0(t), x_\infty(t)] + \lambda_\infty(t)\beta[1 - x_\infty(t)] + \lambda_0(t)\beta[c - x_0(t)], \end{aligned} \tag{16}$$

where

$$\begin{aligned} W[u(t)] &= -\lambda_\infty(t)H[u(t)] \\ &\quad - \int_t^\infty \lambda_0(\tau)\alpha(\tau - t)\exp[-\beta(\tau - t)]h\{\phi[\tau - t, u(t)]\}\phi_1[\tau \\ &\qquad - t, u(t)]\,d\tau. \end{aligned} \tag{17}$$

The adjoint system is

$$\dot{\lambda}_\infty(t) = (q + \beta)\lambda_\infty(t) - \partial F[x_0(t), x_\infty(t)]/\partial x_\infty(t), \tag{18}$$

$$\dot{\lambda}_0(t) = (q + \beta)\lambda_0(t) - \partial F[x_0(t), x_\infty(t)]/\partial x_0(t), \tag{19}$$

with transversality conditions

$$\lambda_\infty(t) \to \bar{\lambda}_\infty = [1/(q + \beta)][\partial F(\bar{x}_0, \bar{x}_\infty)/\partial \bar{x}_\infty], \tag{20}$$

$$\lambda_0(t) \to \bar{\lambda}_0 = [1/(q + \beta)][\partial F(\bar{x}_0, \bar{x}_\infty)/\partial \bar{x}_0], \tag{21}$$

where $\bar{x}_0$ and $\bar{x}_\infty$ are the steady-state values of $x_0(t)$ and $x_\infty(t)$ and

$$q = -\dot{\alpha}(t)/\alpha(t) = \begin{cases} r - n, & \text{with total GNP maximization,} \\ r, & \text{with per capita GNP maximization;} \end{cases} \tag{22}$$

see Section 2.7.

**Theorem 3.1.** *Necessary Conditions.* If $\mathbf{u}^*$ is an optimal control with corresponding trajectories $\mathbf{x}_\infty^*$ and $\mathbf{x}_0^*$, then the following results hold.

(i) there exist nonzero trajectories $\boldsymbol{\lambda}_\infty^*$ and $\boldsymbol{\lambda}_0^*$ satisfying (18) and (19), respectively;

(ii) $W[\mathbf{u}^*(t)] \geq W[\mathbf{u}(t)]$ for all $\mathbf{u}$ and $t \geq 0$.

The proof of this theorem follows from Lemma 3.1 and Refs. 12, 13, and 14.[6]

[6] The proof in Ref. 14 assumes the existence of an optimal solution.

**Lemma 3.2.** *Concavity.* The function

$$\mathcal{H}^0[\mathbf{x}(t), \boldsymbol{\lambda}(t), t] = \max_{u(t)} \mathcal{H}[u(t)],$$

defined by (16), is concave in $\mathbf{x}(t)$ for any given $\boldsymbol{\lambda}(t)$, $t$.

**Proof.** From (16), $\mathcal{H}$ is separable in $\mathbf{x}(t)$ and $u(t)$. The production function is concave in $\mathbf{x}(t)$; all other terms in (16) are either linear in $\mathbf{x}(t)$ or independent of $\mathbf{x}(t)$. Thus $\mathcal{H}^0$ is concave in $\mathbf{x}(t)$ for given $\boldsymbol{\lambda}(t)$ and $t$.

**Theorem 3.2.** *Sufficient Conditions.* If $\{\mathbf{u}^*, \mathbf{x}^*, \boldsymbol{\lambda}^*\}$ is a Pontryagin path [i.e., if they satisfy the state transition equations (7) and (9) and the conditions of Theorem 3.1] with $\boldsymbol{\lambda}^*$ satisfying the transversality conditions (20) and (21), then $\mathbf{u}^*$ is an optimal control.

**Proof.** This proof follows immediately from Section 5 of Ref. 15 and Lemma 3.2. Note that the term Pontryagin path comes from Ref. 1.

### 3.2. Properties of the Optimal Control and Adjoint System

**Theorem 3.3.** *Necessary and Sufficient Conditions for Local Maximization of $\mathcal{H}$.* For any trajectories $\lambda_\infty(t)$ and $\lambda_0(t)$ satisfying (18) and (19), respectively, for given trajectories $x_\infty(t)$ and $x_0(t)$, the control $u^*(t)$ maximizes $W[u(t)]$ iff it satisfies the static efficiency condition

$$\alpha[u^*(t)]\exp[-\beta u^*(t)] = \begin{cases} \lambda_\infty(t)/\lambda_0[t+u^*(t)], & \lambda_\infty(t) < \lambda_0(t), \\ 1, & \lambda_\infty(t) \geq \lambda_0(t). \end{cases} \tag{23}$$

**Proof.** Since

$$\alpha(t) = \exp(-qt),$$

condition (23) can be expressed equivalently as

$$u^*(t) = \begin{cases} (\beta+q)^{-1}\{\log\{\lambda_0[t+u^*(t)]\} - \log[\lambda_\infty(t)]\}, & \lambda_\infty(t) < \lambda_0(t), \\ 0, \quad \lambda_\infty(t) \geq \lambda_0(t). \end{cases} \tag{24}$$

A necessary condition for $u$ to maximize $W$ is that

$$\partial W[u^*(t)]/\partial u(t) = 0, \qquad \text{if } u^*(t) > 0, \tag{25}$$

$$\partial W[u^*(t)]/\partial u(t) \leq 0, \qquad \text{if } u^*(t) = 0. \tag{26}$$

Now,

$$\partial W[u(t)]/\partial u(t) = -\lambda_\infty(t)h[u(t)] + \lambda_0[t+u(t)]\alpha[u(t)]\exp[-\beta u(t)]h[u(t)]. \tag{27}$$

Clearly, a positive solution to (25) for $u^*(t)$ exists if

$$\lambda_\infty(t) < \lambda_0(t).$$

Furthermore, since

$$\alpha[u(t)] \exp[-\beta u(t)] = 1$$

for $u(t) = 0$, (27) is negative for

$$\lambda_\infty(t) > \lambda_0(t),$$

thereby establishing the necessity of (23) or (24).

To prove sufficiency, we examine the second derivative of $W$ with respect to $u$:

$$d^2W/du^2 = -\lambda_\infty(t)h'(u) + \dot{\lambda}_0(t+u)\alpha(u)\exp(-\beta u)h(u) + \lambda_0(t+u)\alpha'(u)\exp(-\beta u)h(u) - \beta\lambda_0(t+u)\alpha(u)\exp(-\beta u)h(u) + \lambda_0(t+u)\alpha(u)\exp(-\beta u)h'(u).$$

If $u^* > 0$ we can replace $\lambda_\infty(t)$ with $\alpha(u^*)\exp(-\beta u^*)\lambda_0(t+u^*)$; see (23). Thus,

$$d^2W(u^*)/du^2 = \exp(-\beta u^*)\alpha(u^*)h(u^*)[\dot{\lambda}_0(t+u^*) - (\beta+q)\lambda_0(t+u^*)], \tag{28}$$

since

$$q = -\dot{\alpha}/\alpha.$$

Since

$$\exp(-\beta u^*)\alpha(u^*)h(u^*) > 0 \qquad \text{for } u^* > 0,$$

one has

$$\text{sign}[d^2W(u^*)/du^2] = \text{sign}[\dot{\lambda}_0(t+u^*) - (\beta+q)\lambda_0(t+u^*)] = \text{sign}\{-\partial F[x_0(t+u^*), x_\infty(t+u^*)]/\partial x_0(t+u^*)\} < 0,$$

since $F$ was assumed to have positive first partial derivatives; see (19). But this result is the second-order condition for a relative maximum. This completes the proof of the theorem.

The important result (23) or (24) has a simple economic interpretation. If the object is to maximize the discounted stream of total GNP, then

$$\alpha(t) = \exp[-(r-n)t],$$

and (23) may be written as

$$\lambda_\infty(t) = \exp[-(r+\gamma)u^*(t)]\lambda_0[t+u^*(t)]. \tag{29}$$

Since the adjoint variables are current-value multipliers, $\lambda_\infty(t)$ is the present value of an extra unskilled worker at time $t$, while $\lambda_0[t+u^*(t)]$ is the present value at time $t+u^*(t)$ of an extra skilled worker at time $t+u^*(t)$. However, because of the assumed exponential retirement process, one worker today reduces to $\exp(-\gamma u)$ workers in $u$ years. Thus, the present value at time $t+u^*(t)$ of a worker who enters training at time $t$ and completes it (if he survives) at time $t+u^*(t)$ is only $\exp[-\gamma u^*(t)]\lambda_0[t+u^*(t)]$. Discounting this back to time $t$, the present value at time $t$ of a worker who enters a training program at time $t$ which is to be completed at time $t+u^*(t)$ is $\exp[-(r+\gamma)u^*(t)]\lambda_0[t+u^*(t)]$. Condition (23) states that $u^*(t)$ should be chosen such that society is indifferent to whether or not an individual with an untrainability index of $u^*(t)$ chooses (or is forced) to become trained.

**Lemma 3.3.** *Upper Bound on Rate of Increase of* $\lambda(t)$. Along any optimal path,

$$\lambda_i(t+\delta)<\exp[(q+\beta)\delta t]\lambda_i(t), \qquad \forall t, \forall \delta t>0, \qquad i=0,\infty. \tag{30}$$

**Proof.** From (18) and (19),

$$\begin{aligned}\lambda_i(t+\delta t) &= \exp[(q+\beta)(t+\delta t)] \\ &\quad\times\left[\lambda_i(0)-\int_0^{t+\delta t}[\partial F[x_0(\tau),x_\infty(\tau)]/\partial x_i(\tau)]\exp[-(q+\beta)\tau]\,d\tau\right] \\ &= \exp[(q+\beta)\,\delta t]\lambda_i(t) \\ &\quad -\exp[(q+\beta)(t+\delta t)]\int_t^{t+\delta t}\partial F[x_0(\tau),x_\infty(\tau)]/\partial x_i(\tau) \\ &\quad \times\exp[-(q+\beta)\tau]\,d\tau.\end{aligned}$$

Since the last term on the right is positive, the lemma follows.

This result also has a simple economic interpretation. As noted before, $\lambda_i(t+\delta t)$ is the present value at $t+\delta t$ of the discounted stream of future earnings of a worker of type $i$, $i=0,\infty$, at time $t+\delta t$. Due to the assumed exponential retirement process, $\exp(\gamma\,\delta t)$ workers at time $t$ reduce to one worker at time $t+\delta t$. Now, suppose that we take $\exp(\gamma\,\delta t)$ workers at time $t$. Their marginal product for the next $\delta t$ year must exceed zero. Furthermore, the present value at time $t$ of their marginal product from $t+\delta t$ to $\infty$ must be $\exp(-r\,\delta t)\lambda_i(t+\delta t)$. Thus,

$$\exp(\gamma\,\delta t)\lambda_i(t)>\exp(-r\,\delta t)\lambda_i(t+\delta t),$$

whence

$$\lambda_i(t+\delta t) < \exp[(r+\gamma)\,\delta t]\lambda_i(t) = \exp[(q+\beta)\,\delta t]\lambda_i(t).$$

There is an alternative simple interpretation of this lemma. Define $\lambda_{\delta t}(t)$ as the present value at time $t$ of a trainee with $\delta t$ more years of training remaining. Clearly,

$$\lambda_{\delta t}(t) = \exp[-(q+\beta)\,\delta t]\lambda_0(t+\delta t).$$

But, given the nature of the retirement process, obviously

$$\lambda_{\delta t}(t) < \lambda_0(t).$$

Thus,

$$\exp[-(q+\beta)\,\delta t]\lambda_0(t+\delta t) < \lambda_0(t), \tag{31}$$

which is the result given by (30). In fact, the result (31) is also equivalent to

$$\lambda_{\delta t_1}(t) > \lambda_{\delta t_2}(t), \qquad \forall t,\ \delta t_1 < \delta t_2,$$

which implies that trainees with less time until completion of the training program are worth more than trainees with more time.

**Theorem 3.4.** Along any optimal path, the control satisfying the static efficiency condition (23) is unique.

**Proof.** First, we prove that, if $u$ satisfies (23), then $u+\delta u$ cannot satisfy (23) for any $\delta u > 0$; this will be proved by contradiction. Given $u$ satisfying (23), suppose that $u+\delta u$ also satisfies (23); then,

$$\exp[-q(u+\delta u)]\exp[-\beta(u+\delta u)] = \lambda_\infty(t)/\lambda_0(t+u+\delta u).$$

From lemma 3.3, we have

$$\lambda_0(t+u+\delta u) < \exp[(q+\beta)\,\delta u]\lambda_0(t+u);$$

thus,

$$\exp[-(q+\beta)(u+\delta u)] > \lambda_\infty(t)/\lambda_0(t+u).$$

But,

$$\exp[-(q+\beta)u] = \lambda_\infty(t)/\lambda_0(t+u)$$

by Theorem 3.3, since $u$ is an optimal control by hypothesis. Consequently,

$$\exp[-(q+\beta)\,\delta u] > 1,$$

which is impossible since $q+\beta$ and $\delta u$ are positive; this contradiction establishes the result that, if $u$ is an optimal control, then $u+\delta u$ cannot be an optimal control for $\delta u > 0$. If $\delta u < 0$, let

$$v = u + \delta u, \qquad \delta v = -\delta u.$$

By the above result, both $v$ and $v + \delta v$ cannot be optimal controls, completing the proof of the theorem.

Note that Theorem 3.4 does not imply that there is a unique optimal control path.

**Theorem 3.5.** *Necessary and Sufficient Condition for Global Maximization of $\mathcal{H}$.* On any optimal path, the static efficiency condition (23) is the necessary and sufficient condition for $u^*$ to maximize globally the Hamiltonian $\mathcal{H}$ in (16).

**Proof.** The proof follows immediately from Theorems 3.3 and 3.4.

**Theorem 3.6.** *Sufficiency.* If $u^*(t)$, $x^*(t)$, $\lambda^*(t)$ satisfy the state dynamics (7) and (9), the adjoint dynamics (18) and (19), the transversality conditions (20) and (21), and the static efficiency condition (23) almost everywhere, then $u^*(t)$ is an optimal control.

**Proof.** The proof follows immediately from Theorems 3.2 and 3.5.

**Theorem 3.7.** On a converging optimal trajectory,

$$\dot{u}^*(t) = \exp[(q + \beta)u^*(t)] \times \left[\frac{\partial F[x_0(t), x_\infty(t)]/\partial x_\infty(t)}{\partial F\{x_0[t + u^*(t)], x_\infty[t + u^*(t)]\}/\partial x_0[t + u^*(t)]}\right] - 1. \tag{32}$$

**Proof.** Differentiating the static efficiency condition (23) with respect to time, we have

$$-(q+\beta)\exp[-(q+\beta)u^*(t)]\dot{u}(t) = \frac{\lambda_0[t+u^*(t)]\dot{\lambda}_\infty(t) - \lambda_\infty(t)\dot{\lambda}_0[t+u^*(t)][1+\dot{u}^*(t)]}{\{\lambda_0[t+u^*(t)]\}^2}.$$

Multiplying both sides by $\lambda_0[t+u^*(t)]$ and subtracting

$$(q+\beta)\exp[-(q+\beta)u^*(t)]\lambda_0[t+u^*(t)]$$

from both sides gives

$$\begin{aligned}-(q+\beta)&\exp[-(q+\beta)u^*(t)]\lambda_0[t+u^*(t)][1+\dot{u}^*(t)]\\ &= -(q+\beta)\exp[-(q+\beta)u^*(t)]\lambda_0[t+u^*(t)] + \dot{\lambda}_\infty(t)\\ &\qquad - \{\lambda_\infty(t)/\lambda_0[t+u^*(t)]\}\dot{\lambda}_0[t+u^*(t)][1+\dot{u}^*(t)].\end{aligned}$$

Applying (23) to the first and third terms on the right and grouping terms in $1+\dot{u}^*(t)$, we have

$$\exp[-(q+\beta)u^*(t)]\{\dot{\lambda}_0[t+u^*(t)]-(q+\beta)\lambda_0[t+u^*(t)]\}[1+\dot{u}^*(t)]=\dot{\lambda}_\infty(t)-(q+\beta)\lambda_\infty(t).$$

Applying (19) to the left-hand side and (18) to the right-hand side, we have

$$\exp[-(q+\beta)u^*(t)][-\partial F\{x_0[t+u^*(t)], x_\infty[t+u^*(t)]\}/\partial x_0[t+u^*(t)]] \times [1+\dot{u}^*(t)] = -\partial F[x_0(t), x_\infty(t)]/\partial x_\infty(t),$$

from which (32) follows directly, completing the proof.

An immediate consequence of this theorem is the following corollary.

**Corollary 3.1.** On a converging optimal trajectory,

$$\dot{u}^*(t)\geq -1, \qquad t\geq 0.$$

**Proof.** The proof follows immediately from Theorem 3.7 and from the assumed nonnegative first partial derivatives of the production function.

This result is very important; it guarantees that the trajectory $u$ will intersect any line with a slope of $-1$ at most once, as illustrated in Fig. 1. Trajectories such as that illustrated in Fig. 2 are necessarily nonoptimal. The stock of trainees at time $t$ consists of those new entrants at the time $t-\tau$ with untrainability indices at least equal to $\tau$, but not greater than $u^*(t-\tau)$, if any, who survive until time $t$, integrated over $\tau\geq 0$; see (3). This quantity is simply an integral over all values of $\tau\geq 0$ for which $u^*(t-\tau)$ exceeds $\tau$, i.e., over all values of $\tau$ under the areas in Figs. 1 and 2 indicated by diagonal lines. The importance of Corollary 3.1 is that it guarantees that, if $\tau^*$ is the smallest value of $\tau$ with

$$u^*(t-\tau)=\tau,$$

then

$$u^*(t-\tau)\leq\tau \qquad \text{for all } \tau>\tau^*.$$

An immediate consequence of Corollary 3.1 is that

$$u^*(t+\delta t)\geq u^*(t)-\delta t.$$

This means that if, at any time $t$, all individuals requiring no more than $u^*(t)$ years of training enter training programs, then, for all

$$t<\tau<t+u^*(t),$$

at least, all individuals with untrainability indices less than or equal to $u^*(t)+t-\tau$ [that is, at least all individuals able to complete their training

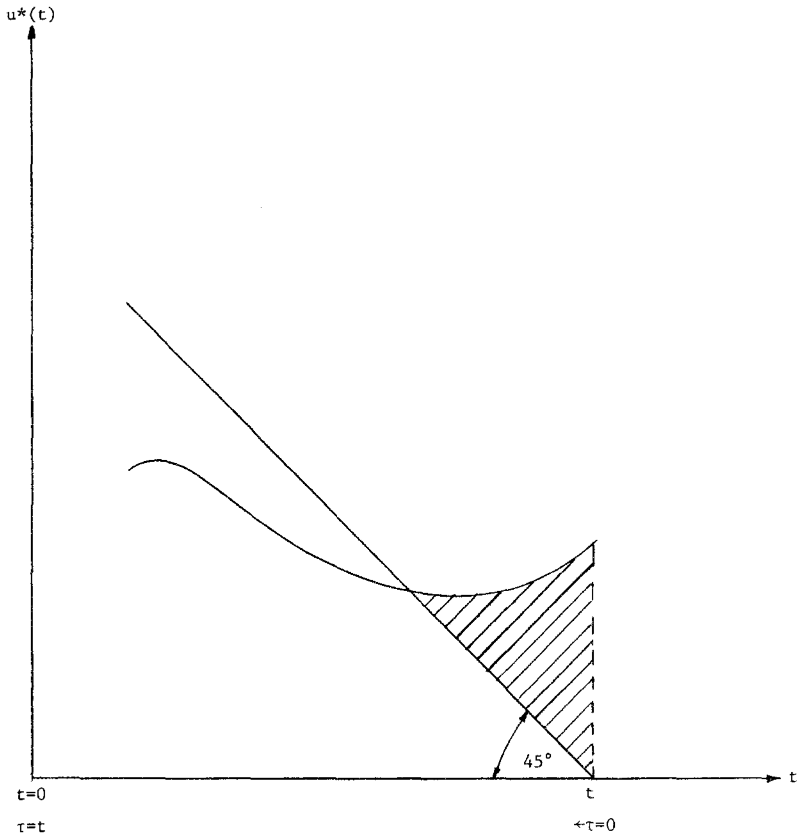


Fig. 1. A possible optimal control scenario.

by time $t+u^*(t)$] will be trained. That is, once one optimal decision has looked forward to a point $t'$, all future optimal decisions must look forward at least that far.[7]

**3.3. Optimal Long-Run Stationary Equilibrium.** In this section, we prove the existence and uniqueness of the stationary optimal control, or long-run stationary equilibrium (Ref. 1, pp. 50–51) and examine the nature of the influence of the various parameters on the stationary optimal control.

[7] This important property is most easily seen by noting that

$$(d/dt)[t+u^*(t)]=1+\dot{u}^*(t)\geq 0,$$

by Corollary 3.1.

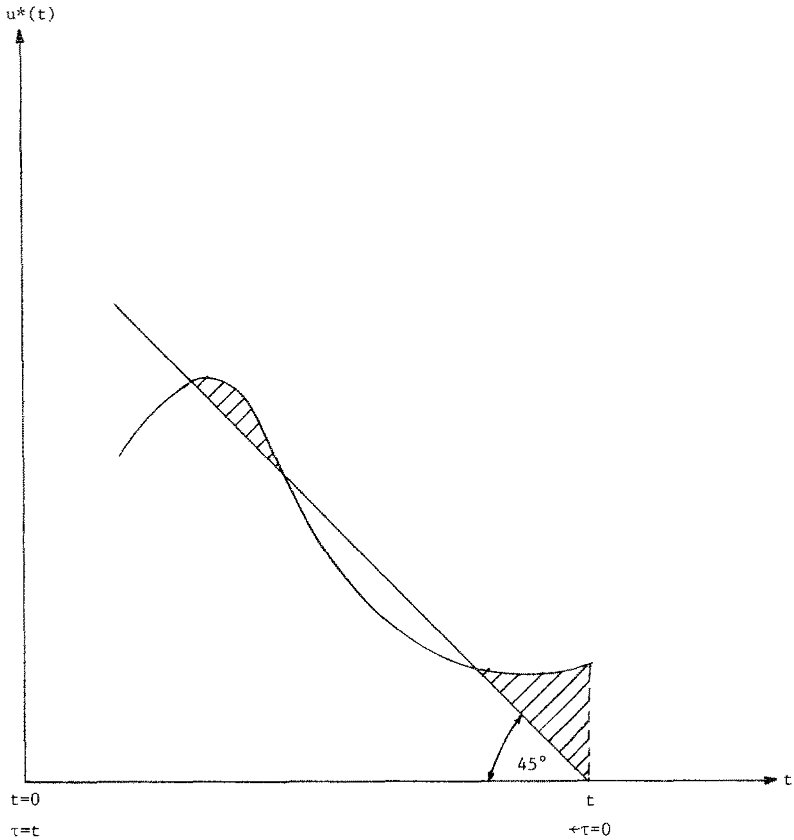


Fig. 2. An impossible optimal control scenario.

**Theorem 3.8.** *Unique Stationary Optimal Control.* For CES production functions with $-1 \leq \sigma < \infty$, there exists a unique stationary optimal control $\bar{u}$ which is the solution to

$$[\hat{H}(\bar{u})/\{1-H(\bar{u})\}]^{1+\sigma}(\exp(\beta\bar{u})/\alpha(\bar{u})] = \delta/(1-\delta), \tag{33}$$

where

$$\hat{H}(\bar{u}) = \int_0^{\bar{u}} \exp(-\beta\xi)h(\xi)\, d\xi. \tag{34}$$

**Proof.** Under stationary conditions,

$$\lambda(t) = \bar{\lambda}.$$

Since $u$ is optimal only if it satisfies the static efficiency condition (23), we have

$$\alpha(\bar{u})\exp(-\beta\bar{u}) = \bar{\lambda}_\infty/\bar{\lambda}_0 = [\partial F(\bar{x}_0, \bar{x}_\infty)/\partial \bar{x}_\infty]/[\partial F(\bar{x}_0, \bar{x}_\infty)/\partial \bar{x}_0]$$
$$= [(1-\delta)/\delta](\bar{x}_0/\bar{x}_\infty)^{1+\sigma}; \tag{35}$$

see (20)–(21) and (12). Equating $\dot{x}_\infty(t)$ and $\dot{x}_0(t)$ in (7) and (9) to zero and solving for the steady-state values of $\dot{x}_\infty$ and $\dot{x}_0$ gives

$$\bar{x}_\infty = 1 - H(\bar{u}), \qquad \bar{x}_0 = \hat{H}(\bar{u}); \tag{36}$$

see (34). Substituting these results into (35) gives (33).

As $\bar{u}$ increases from zero to infinity, the left-hand side of (33) increases monotonically from zero to infinity for

$$-1 < \sigma < \infty$$

and from unity to infinity for

$$\sigma = -1.$$

Since

$$\delta/(1-\delta) > 1,$$

i.e., skilled labor is more productive than unskilled labor, there exists a unique $\bar{u}$ satisfying (33). This completes the proof of the theorem.

**Theorem 3.9.** If there is more than one converging Pontryagin path, then they all converge to $\{\bar{u}, \bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}}\}$ defined in (33), (36), (20), and (21). Furthermore, $\boldsymbol{\lambda}(t) > 0, \forall t$ along these paths.

**Proof.** The first part of the theorem follows immediately from Theorem 3.8 and the definition of a Pontryagin path. The second part of the theorem follows from the fact that, if

$$\lambda_i(\tau) \leq 0, \qquad \text{for } i = 0 \text{ or } \infty$$

at some time $\tau$, then

$$\lambda_i(t) < 0, \qquad t \geq \tau,$$

by (19) or (18); therefore, $\boldsymbol{\lambda}(t)$ cannot converge to $\bar{\boldsymbol{\lambda}}$, which is positive. This contradiction with the first part of the theorem completes the proof.

In economic terms, this means that, at each point of time, the present value of the marginal product stream of any worker (or, equivalently, of the earnings stream of any worker) is strictly positive along any converging

Pontryagin path. In fact, it is easily shown that, along any such path,

$$\lambda_i(t)=\int_t^\infty \exp[-(q+\beta)(\tau-t)]\{\partial F[x_0(\tau), x_\infty(\tau)]/\partial x_i(\tau)\}\,d\tau, \qquad i=0, \infty,$$

where

$$\partial F[x_0(\tau), x_\infty(\tau)]/\partial x_i(\tau)$$

is the marginal product of $x_i$ at time $\tau$.

**Theorem 3.10.** The stationary optimal control $\bar{u}$ is an increasing function of $\delta$ and a decreasing function of $q$ and $\beta$.

**Proof.** The left-hand side of (33) is a strictly increasing function of $\bar{u}$. Since the right-hand side of (33) is a strictly increasing function of $\delta$, $0\leq\delta\leq 1$, and since the left-hand side of (33) is a strictly increasing function of $q$ and $\beta$ (specifically, $q+\beta$), the theorem follows immediately.

**Theorem 3.11.** Let $\tilde{u}$ be the solution of the equation

$$\hat{H}(\tilde{u})=1-H(\tilde{u}); \tag{37}$$

then,

$$\text{sign}(\partial\bar{u}/\partial\sigma)=\text{sign}(\tilde{u}-\bar{u}).$$

**Proof.** Replacing $\alpha(\bar{u})$ with $\exp(-q\bar{u})$ in (33), taking natural logarithms, differentiating partially with respect to $\sigma$, and solving for $\partial\bar{u}/\partial\sigma$ gives

$$\partial\bar{u}/\partial\sigma=-\left\{\frac{\log[\hat{H}(\bar{u})/\{1-H(\bar{u})\}]}{q+\beta+(1+\sigma)[(\hat{h}(\bar{u})/\hat{H}(\bar{u}))+(h(\bar{u})/\{1-H(\bar{u})\})]}\right\}, \tag{38}$$

where

$$\hat{h}(\bar{u})=\partial\hat{H}(\bar{u})/\partial\bar{u}.$$

Since the denominator of (38) is positive, the sign of $\partial\bar{u}/\partial\sigma$ is determined by the numerator. Thus, the proof follows immediately from the definition of $\tilde{u}$ in (37).

The golden-rule control $\hat{u}$ is defined as that control satisfying

$$F[\bar{x}_0(\hat{u}), \bar{x}_\infty(\hat{u})]\geq F[\bar{x}_0(\bar{u}), \bar{x}_\infty(\bar{u})], \qquad \forall\bar{u}. \tag{39}$$

**Corollary 3.2.** There exists a unique golden-rule control $\hat{u}$; furthermore, $\hat{u}>\bar{u}$ for $-1\leq\sigma<\infty$.

**Proof.** The golden-rule control is the special case of a stationary optimal control for which the discount rate $r=0$; thus, existence and uniqueness follow from Theorem 3.8 for $-1\leq\sigma<\infty$. For the remaining case $\sigma=\infty$, static maximization of the production function gives the unique golden-rule control $\hat{u}$ which satisfies

$$\bar{x}_\infty(\hat{u})/\bar{x}_0(\hat{u})=\pi_0/\pi_\infty,$$

or

$$[1-H(\hat{u})]/\hat{H}(\hat{u})=\pi_0/\pi_\infty. \tag{40}$$

Thus, existence and uniqueness are established for $-1\leq\sigma\leq\infty$. The inequality $\hat{u}>\bar{u}$ follows immediately from Theorem 3.10.

**Theorem 3.12.** For perfectly substitutable labor inputs [the linear production function (12)], the unique, converging optimal control is

$$u^*(t)=\bar{u}=[1/(q+\beta)]\log[\delta/(1-\delta)], \qquad \forall t, \tag{41}$$

for any arbitrary initial conditions.

**Proof.** The adjoint system (18), (19) reduces in this case to

$$\dot{\lambda}_\infty(t)=(q+\beta)\lambda_\infty(t)-(1-\delta),$$
$$\dot{\lambda}_0(t)=(q+\beta)\lambda_0(t)-\delta.$$

The solutions of these differential equations are

$$\lambda_\infty(t)=C_\infty\exp[(q+\beta)t]+(1-\delta)/(q+\beta), \tag{42}$$

$$\lambda_0(t)=C_0\exp[(q+\beta)t]+\delta/(q+\beta), \tag{43}$$

where $C_\infty$ and $C_0$ are constants to be determined by the transversality conditions

$$\lambda_\infty(t)\rightarrow\bar{\lambda}_\infty=(1-\delta)/(q+\beta),$$
$$\lambda_0(t)\rightarrow\bar{\lambda}_0=\delta/(q+\beta);$$

see (12), (20), and (21). Applying these conditions to (42) and (43), we get

$$C_\infty=C_0=0;$$

therefore,

$$\lambda_\infty(t)=\bar{\lambda}_\infty, \qquad \lambda_0(t)=\bar{\lambda}_0, \qquad t\geq 0.$$

Applying this result to the static efficiency condition (23) gives (41). Thus, by Theorem 3.3, $\mathbf{u}^*$ given by (41) is an optimal control. Uniqueness follows from the instability of (42) and (43): any perturbation of $\lambda_i(t)$ from $\bar{\lambda}_i$ will cause $\lambda_i(t)$ to diverge from that point on. This completes the proof.

The economic interpretation of this result is simple. As noted in Section 2, for linear production functions, the marginal productivity of any individual is independent of the level or composition of employment, either at present or in the future. Thus, the critical untrainability level $\bar{u}$ is constant over time. In particular, for the total GNP maximization case, it is exactly that value for which expected discounted lifetime income (marginal product) is independent of whether the individual is trained (see the discussion following Theorem 3.3).

**Theorem 3.13.** *Global Stability of Dynamics.* For any constant control $\bar{u}$ with corresponding steady states $\bar{x}_0$ and $\bar{x}_\infty$ given by (36),

$$x_0(t) \to \bar{x}_0 \qquad \text{and } x_\infty(t) \to \bar{x}_\infty.$$

**Proof.** Replacing $u(\tau)$ with $\bar{u}$ and $t-\tau$ with $\xi$ in (9) yields

$$\begin{aligned}\dot{x}_0(t) &= \beta\left\{c - x_0(t) - \int_0^\infty \exp(-\beta\xi)h[\phi(\xi,\bar{u})]\phi_1(\xi,\bar{u})\,d\xi\right\}\\ &= \beta\left\{c - x_0(t) - \int_{\bar{u}}^\infty \exp(-\beta\xi)h(\xi)\,d\xi\right\}\\ &= \beta\left\{c - x_0(t) - \left[\int_0^\infty \exp(-\beta\xi)h(\xi)\,d\xi - \int_0^{\bar{u}} \exp(-\beta\xi)h(\xi)\,d\xi\right]\right\}\\ &= \beta\{c - x_0(t) - [c - \hat{H}(\bar{u})]\} = -\beta\epsilon_0(t),\end{aligned} \tag{44}$$

where

$$\epsilon_i(t) = x_i(t) - \bar{x}_i, \qquad i = 0, \infty; \tag{45}$$

see (2) and (6), and see (5) and (34). Also, replacing $u(t)$ with $\bar{u}$ in (7), we have

$$x_\infty(t) = \beta[1 - H(\bar{u}) - x_\infty(t)] = -\beta\epsilon_\infty(t); \tag{46}$$

see (36) and (45). Since $0 < \beta < 1$, (44) and (46) imply global stability, completing the proof.

Since

$$\dot{x}_i(t) = \dot{\epsilon}_i(t),$$

see (45), we have the following result.

**Corollary 3.3.** *Exponential Damping of Perturbations.* Any perturbation of equilibrium dies out exponentially at a rate $\beta$.

**3.4. Competitive Market Dynamics.** In this section, we describe the dynamics under competitive market conditions. We also compare the market solution with the optimal solution and discuss various governmental policies for inducing the market to behave optimally. We assume static expectations:

$$w_i^E(t+\tau)=w_i(t), \qquad i=0,\infty, \qquad 0\leq\tau<\infty, \tag{47}$$

where $w_i^E(t+\tau)$ is the value of $w_i(t+\tau)$ that each individual expects as of time $t$.

If a new entrant to the labor force at time $t$ decides not to be trained, then the present value of his lifetime earnings is

$$W_\infty(t)=\int_0^\infty \exp[-(r+\gamma)\tau]w_\infty^E(t+\tau)\,d\tau. \tag{48}$$

If an individual with untrainability index $u$ decides to get trained, the present value of his expected earnings stream is

$$\begin{aligned}W_0(t)&=\int_u^\infty \exp[-(r+\gamma)\tau]w_0^E(t+\tau)\,d\tau\\&=\exp[-(r+\gamma)u]\int_0^\infty \exp[-(r+\gamma)\tau]w_0^E(t+u+\tau)\,d\tau.\end{aligned} \tag{49}$$

If new entrants to the labor force make the training decision so as to maximize the present value of expected lifetime earnings, then, at each instant $t$, there will be a critical value $u^m(t)$ satisfying

$$W_0(t)=W_\infty(t), \tag{50}$$

such that all individuals with untrainability indices less than $u^m(t)$ will choose to be trained, while those with untrainability indices in excess of $u(t)$ will choose not to be trained [those with indices identical to $u^m(t)$ are indifferent]. Now, if individuals knew all future wage rates [i.e., $w_\infty(t+\tau)$ and $w_0(t+\tau)$ for all $\tau>0$], then the market control would be optimal; see (23). However, the market does not know future wage rates. Applying the static expectations assumption (47) to (48)–(49) gives

$$W_\infty(t)=w_\infty(t)/(r+\gamma)$$

$$W_0(t)=\exp[-(r+\gamma)u]w_0(t)/(r+\gamma);$$

applying these results to (50) gives

$$\exp[-(r+\gamma)u^m(t)]=w_\infty(t)/w_0(t), \tag{51}$$

where $u^m(t)$ is the market control at time $t$. Thus, we can state the following theorem.

**Theorem 3.14.** Under dynamic conditions, the market (with static expectations) is in general not optimal.

The one case for which the market solution is always optimal is the linear production function (12); this is summarized in the following theorem.

**Theorem 3.15.** With the linear production function (12), the free market solution maximizes the present value of total GNP.

**Proof.** Since $w_0(\tau)=\delta$ and $w_\infty(t)=1-\delta$ for all $t\geq 0$, the market control (51) becomes

$$u^m(t)=[1/(r+\gamma)]\log[\delta/(1-\delta)],$$

which is identical to the optimal control; see (41).

Since the linear production function has been discussed earlier, additional elaboration is not required. We do note, however, that the market will not maximize the present value of *per capita* GNP unless individuals use the discount rate $r+n$, rather than $r$, or act as if the death rate is really $\beta$, not $\gamma$.

**Theorem 3.16.** *Stationary Optimality of Market Solution.* Under stationary conditions, a control is optimal (it maximizes the discounted total GNP) iff it is the market solution.

**Proof.** From (51), under stationary conditions,

$$\exp[-(r+\gamma)\bar{u}^m]=\bar{w}_\infty/\bar{w}_0=\bar{F}_\infty/\bar{F}_0=\bar{\lambda}_\infty/\bar{\lambda}_0,$$

by marginal productivity conditions and by (20) and (21). This is the condition for a stationary optimal control; see (35).

**3.4.1. Tax Subsidy Scheme.** Let $T_0(t)$ be the hourly tax rate for skilled workers and $T_\infty(t)$ be the hourly tax rate for unskilled labor. Then, from (51),

$$\exp[-(r+\gamma)u^m(t)]=[w_\infty(t)-T_\infty(t)]/[w_0(t)-T_0(t)].$$

By reference to (23), it is seen immediately that the market will behave optimally iff

$$\lambda_\infty(t)/\lambda_0[t+u^*(t)]=[w_\infty(t)-T_\infty(t)]/[w_0(t)-T_0(t)]. \tag{52}$$

We impose the condition that the tax rate be purely redistributive:

$$-T_0(t)x_0(t)=T_\infty(t)x_\infty(t). \tag{53}$$

Solving (52) and (53) for $T_0(t)$ and $T_\infty(t)$ gives

$$T_0(t)=x_\infty(t)\left\{\frac{\lambda_\infty(t)w_0(t)-\lambda_0[t+u^*(t)]w_\infty(t)}{\lambda_\infty(t)x_\infty(t)+\lambda_0[t+u^*(t)]x_0(t)}\right\}$$

and

$$T_\infty(t)=-x_0(t)\left\{\frac{\lambda_\infty(t)w_0(t)-\lambda_0[t+u^*(t)]w_\infty(t)}{\lambda_\infty(t)x_\infty(t)+\lambda_0[t+u^*(t)]x_0(t)}\right\}.$$

Since this tax-subsidy scheme simply adjusts market wages so that the net market rate is equivalent to the appropriate $\lambda$ in the optimal control solution, the net rates are necessarily nonnegative. This analysis leads to the following theorem.

**Theorem 3.17.** There exists a stable tax-subsidy system which makes the optimal control policy controllable[8] (i.e., which induces the market to behave optimally under dynamic conditions).

**3.4.2. Monetary Policy.** It may be possible to find a time-varying interest rate which induces the market to behave optimally. Let $q(t)$ be the time-varying short-term interest rate, assumed known for all $t+\tau$, $\tau\geq 0$, to each individual entering the labor force at time $t$, with $q(t)\to r$. Then, the market efficiency condition becomes

$$\exp\left\{-\gamma u^m(t)-\int_t^{t+u^m(t)} q(\tau)\,d\tau\right\}=w_\infty(t)/w_0(t).$$

The problem, then, is to find a trajectory $\mathbf{q}$, with $q(t)\to r$, that satisfies this equation for

$$u^m(t)=u^*(t);$$

i.e.,

$$\exp\left\{-\int_t^{t+u^*(t)} q(\tau)\,d\tau-\gamma u^*(t)\right\}=w_\infty(t)/w_0(t)$$

$$=\frac{\partial F[x_0(t),x_\infty(t)]/\partial x_\infty(t)}{\partial F\{x_0[t+u^*(t)],x_\infty[t+u^*(t)]\}/\partial x_0[t+u^*(t)]}$$

[8] A policy is said to be controllable by a given set of instruments if there exist values of the instruments, varying over time in general, which cause the private and governmental sectors together to realize the policy. If the values of the instruments converge to finite values, then the policy is said to be controllable with stable instruments (see Ref. 1, pp. 120–121).

or

$$\int_t^{t+u^*(t)} q(\tau)\, d\tau = -\gamma u^*(t) - \left( \frac{\partial F[x_0(t), x_\infty(t)]/\partial x_\infty(t)}{\partial F\{x_0[t+u^*(t)], x_\infty[t+u^*(t)]\}/\partial x_0[t+u^*(t)]} \right) = b(t).$$

Let

$$a(t) = t + u^*(t);$$

then,

$$\int_t^{a(t)} q(\tau)\, d\tau = b(t).$$

Differentiating with respect to $t$, we have

$$q[a(t)]\dot{a}(t) - q(t) = \dot{b}(t),$$

or

$$q[a(t)] = [q(t) + \dot{b}(t)]/\dot{a}(t). \tag{54}$$

Note that

$$a(t) \geq t,$$

since

$$u^*(t) \geq 0;$$

also,

$$\dot{a}(t) \geq 0$$

by Corollary 3.1 (and the equality can occur only when $\sigma = \infty$).

**3.5. Nonsubstitutable Inputs.** When $\sigma = \infty$ in the production function (11), the production function is not differentiable along the ray

$$\pi_0 x_0(t) = \pi_\infty x_\infty(t); \tag{55}$$

see (13); in this case,

$$\partial F(t)/\partial x_i(t)^- = \pi_i,$$

while

$$\partial F(t)/\partial x_i(t)^+ = 0,$$

$i = 0, \infty$. Consequently, the Hamiltonian also is not differentiable when

(55) holds. This difficulty is extremely relevant, since, if the stationary optimal control exists, it is on this ray. We present some results for this case below.

**Theorem 3.18.** The following relation holds.

$$\lim_{\sigma \to \infty} \bar{u}_\sigma = \bar{\bar{u}},$$

where $\bar{\bar{u}}$ satisfies

$$\pi_0 \hat{H}(\bar{\bar{u}}) = \pi_\infty [1 - H(\bar{\bar{u}})],$$

and where the subscript $\sigma$ has been added to $\bar{u}$ to show its dependence on $\sigma$.

**Proof.** Modifying (33) to take account of $\pi_0 \neq \pi_\infty$ gives

$$[\hat{H}(\bar{u})/\{1 - H(\bar{u})\}]^{1+\sigma}[\exp(\beta\bar{u})/\alpha(\bar{u})] = \delta\pi_0^{-\sigma}/(1-\delta)\pi_\infty^{-\sigma}.$$

Multiplying both sides by $(\pi_0/\pi_\infty)^{1+\sigma}$ and raising both sides to the $(1+\sigma)^{-1}$ power gives

$$\hat{H}(\bar{u})\pi_0 \exp[\bar{u}(q+\beta)/(1+\sigma)]/[1 - H(\bar{u})]\pi_\infty = [\delta\pi_0/(1-\delta)\pi_\infty]^{1/(1+\sigma)}.$$

Taking the limit as $\sigma \to \infty$ gives

$$\hat{H}(\bar{\bar{u}})\pi_0/[1 - H(\bar{\bar{u}})]\pi_\infty = 1;$$

this completes the proof.

**Theorem 3.19.** For the case of nonsubstitutable labor inputs ($\sigma = \infty$), $\bar{\bar{u}}$ is a stationary optimal control.

**Proof.** Let $\mathbf{x}(u)$ denote the steady state corresponding to a constant control $u$. Let the functional $J_\sigma[\mathbf{u}, \mathbf{x}(u)]$ be the value of the objective function (14) if $\mathbf{x}(u)$ is the initial condition and $\mathbf{u}$ is any control trajectory. By telescoping, we have

$$\begin{aligned} J_\sigma[\mathbf{u}, \mathbf{x}(\bar{\bar{u}})] - J_\infty[\bar{\bar{\mathbf{u}}}, \mathbf{x}(\bar{\bar{u}})] &= J_\sigma[\mathbf{u}, \mathbf{x}(\bar{\bar{u}})] - J_\sigma[\mathbf{u}, \mathbf{x}(\bar{u}_\sigma)] + J_\sigma[\mathbf{u}, \mathbf{x}(\bar{u}_\sigma)] \\ &\quad - J_\sigma[\bar{\mathbf{u}}_\sigma, \mathbf{x}(\bar{u}_\sigma)] + J_\sigma[\bar{\mathbf{u}}_\sigma, \mathbf{x}(\bar{u}_\sigma)] - J_\infty[\bar{\bar{\mathbf{u}}}, \mathbf{x}(\bar{\bar{u}})]. \end{aligned}$$

Now, we examine the right-hand side as $\sigma$ becomes large. The first difference is small because of the continuity of $J_\sigma$ in $\mathbf{x}(u)$ and because

$$\lim_{\sigma \to \infty} \mathbf{x}(\bar{u}_\sigma) = \mathbf{x}(\bar{\bar{u}}).$$

The second difference is negative by the optimality of $\bar{u}_\sigma$ from Theorem 3.8. The third difference is small because the functional $J_\sigma$ converges uniformly to the functional $J_\infty$ and its arguments

$$\bar{\mathbf{u}}_\sigma \to \bar{\bar{\mathbf{u}}} \quad \text{and} \quad \mathbf{x}(\bar{u}_\sigma) \to \mathbf{x}(\bar{\bar{u}}).$$

Hence,

$$J_\sigma[\mathbf{u}, \mathbf{x}(\bar{\bar{u}})] - J_\infty[\bar{\bar{\mathbf{u}}}, \mathbf{x}(\bar{\bar{u}})] \leq 0, \qquad \forall \mathbf{u},$$

implying that $\bar{\bar{u}}$ is a stationary optimal control with the initial condition $\mathbf{x}(\bar{\bar{u}})$.

## 4. Extensions

**4.1. Horizontal Heterogeneity.** The model of *vertical* heterogeneity of labor (i.e., skilled and unskilled workers) examined herein extends directly to *horizontal heterogeneity* (*equal* but nontransferable skills, e.g., plumbers and electricians). For example, suppose that we allow two types of skilled labor $x_0^{(1)}$ and $x_0^{(2)}$, with output per worker given by

$$f(t) = \{\delta_1[x_0^{(1)}(t)]^{-\sigma} + \delta_2[x_0^{(2)}(t)]^{-\sigma} + (1-\delta_1-\delta_2)[x_\infty(t)]^{-\sigma}\}^{-1/\sigma}.$$

Let $v(t)$ be the fraction of the flow of new trainees [i.e., of $u(t)L(t)$] being trained for Skill Class 1. Then, the optimal control $v^*(t)$ is the bang–bang control

$$v^*(t) = 0 \qquad \text{if } x_0^{(2)}[t+u^*(t)]/x_0^{(1)}[t+u^*(t)] < \bar{x}_0^{(2)}/\bar{x}_0^{(1)},$$

$$v^*(t) = 1 \qquad \text{if } x_0^{(2)}[t+u^*(t)]/x_0^{(1)}[t+u^*(t)] > \bar{x}_0^{(2)}/\bar{x}_0^{(1)},$$

with the singular control

$$v^*(t) = \bar{v} = [1 + (\delta_1/\delta_2)^{1+\sigma}]^{-1}$$

once

$$x_0^{(2)}[t+u^*(t)]/x_0^{(1)}[t+u^*(t)] = \bar{x}_0^{(2)}/\bar{x}_0^{(1)}.$$

In other words, horizontal imbalances *within* any skill level are corrected in minimum time, since, by assumption, any worker being trained to a particular skill level is equally capable of being trained for any occupation at that skill level.

All of the results reported herein hold for this extended model.

**4.2. Capital.** When capital is introduced into the model as an argument in the linear homogeneous production function, all of the results reported above hold with obvious modifications. The additional state equation is

$$\dot{k}(t) = s(t)F[x_0(t), x_\infty(t), k(t)] - \mu k(t),$$

where $k(t)$ is the capital–labor ratio, $s(t)$ is the savings rate, $\mu$ is the constant depreciation rate. The optimal control $s^*(t)$ is a bang–bang control with the singular control

$$\bar{s} = \mu \bar{k} / F(\bar{x}_0, \bar{x}_\infty, \bar{k}),$$

which is the familar result for Ramsey models (see Ref. 1, Chapter 3).

## References


1. ARROW, K. J., and KURZ, M., *Public Investment, The Rate of Return and Optimal Fiscal Policy*, The John Hopkins Press, Baltimore, Maryland, 1970.
2. DOBELL, A. R., and HO, Y. C., *An Optimal Unemployment Rate*, Quarterly Journal of Economics, Vol. 81, pp. 675–683, 1967.
3. DOBELL, A. R., and HO, Y. C., *Optimal Investment Policy: An Example of a Control Problem in Economic Theory*, IEEE Transactions on Automatic Control, Vol. AC-12, No. 1, 1967.
4. BLACKBURN, J. O., *An Optimal Unemployment Rate: Comment*, Quarterly Journal of Economics, Vol. 83, pp. 518–520, 1969.
5. DOBELL, A. R., *Reply*, Quarterly Journal of Economics, Vol. 83, pp. 521–523, 1969.
6. TU, P. N. V., *Optimal Educational Investment Program in an Economic Planning Model*, Canadian Journal of Economics, Vol. 2, pp. 52–64, 1969.
7. BUDELIS, J. J., *Optimal Economic Growth with Explicit Consideration of Human Capital Development: Optimal Paths for Some Differential–Difference Equations*, Harvard University, PhD Thesis, 1970.
8. LELE, M. M., JACOBSON, D. H., and MCCABE, J. L., *Qualitative Application of a Result in Control Theory to Problems of Economic Growth*, Harvard University, Division of Engineering and Applied Physics, Technical Report No. 162, 1969.
9. LELE, M. M., *Optimal Control in a Bounded State Space with Economic Application*, Harvard University, PhD Thesis, 1970.
10. MCGUIRE, T. W., and SETHI, S. P., *Optimal and Market Control in a Dynamic Economic System with Endogenous Heterogeneous Labor*, Proceedings of the IFORS/IFAC International Conference, Coventry, England, 1973.
11. ARROW, K. J., et al., *Capital–Labor Substitution and Economic Efficiency*, Review of Economics and Statistics, Vol. 43, pp. 225–250, 1961.
12. SETHI, S. P., *A Useful Transformation of Hamiltonians Occurring in Optimal Control Problems in Economic Analyses*, Journal of Management Science and Applied Cybernetics, Vol. 2, No. 3, 1973.

13. BANKS, H. T., and JACOBS, M. Q., *The Optimization of Trajectories of Linear Functional Differential Equations*, SIAM Journal of Control, Vol. 8, pp. 461–488, 1970.
14. BATE, R. R., *Optimal Control of Systems with Transport Lags*, Advances in Control Systems, Vol. 7, Edited by C. T. Leondes, Academic Press, New York, New York, 1969.
15. SETHI, S. P., *Sufficient Conditions for the Optimal Control of a Class of Systems with Continuous Lags*, Journal of Optimization Theory and Applications, Vol. 13, No. 5, 1974.