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A Globally Convergent Method for Nonlinear Programming[1]

S. P. HAN[2]

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Abstract. Recently developed Newton and quasi-Newton methods for nonlinear programming possess only local convergence properties. Adopting the concept of the damped Newton method in unconstrained optimization, we propose a stepsize procedure to maintain the monotone decrease of an exact penalty function. In so doing, the convergence of the method is globalized.

Key Words. Nonlinear programming, global convergence, exact penalty function.

1. Introduction

Consider the nonlinear programming problem

$$\begin{aligned} &\min f(x), \\ &\text{s.t. } g(x) \leq 0, \end{aligned} \tag{1}$$

where $f: R^n \to R$ and $g: R^n \to R^m$. A great deal of attention has been paid to extending Newton and Newton-like methods to solving (1). With the efforts of many authors, this attempt has recently achieved some success. One approach on this line is to generate a sequence $\{x^k\}$ converging to the desired solution by means of solving iteratively the quadratic programming problem

$$\begin{aligned} &\min_x \nabla f(x^k)^T(x-x^k)+\tfrac{1}{2}(x-x^k)^T H_k (x-x^k), \\ &\text{s.t. } g(x^k)+\nabla g(x^k)^T(x-x^k) \leq 0, \end{aligned} \tag{2}$$

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[2] Assistant Professor, Department of Computer Science, Cornell University, Ithaca, New York.

where the $n \times n$ matrix H_k is intended to be an approximation of the Hessian of the Lagrangian

$$L(x, u) = f(x) + u^T g(x).$$

Some results on the convergence and the rate of convergence have been published (Refs. 1–3). However, as in the Newton method for unconstrained optimization, all the results are local. In this work, we show that the direction generated by (2) turns out to be a descent direction of the exact penalty function $\theta_r: R^n \to R$,

$$\theta_r(x) = f(x) + r \sum_{i=1}^{m} g_i(x)_+, \tag{3}$$

where

$$g_i(x)_+ = \max\{0, g_i(x)\}$$

and r is a positive number. Consequently, we introduce a procedure by which stepsizes are determined to maintain the monotone decrease of this function. With this stepsize procedure, the method can be shown to be globally convergent. In this sense, our approach can be viewed as an extension of the damped Newton method to constrained optimization.

For convenience, we shall restrict ourselves to problems with inequality constraints only. The inclusion of equality constraints causes no difficulties, and all of the results go through with minor modification.

We state the method in Section 2 and present global convergence theorems in Section 3. Some comments are given in Section 4.

It is noted that all vectors are column vectors. A row vector is denoted by the superscript T. The notation $\|\cdot\|$ denotes a vector norm and also its induced operator norm.

2. Algorithm

Before the statement of the algorithm, we first define the following quadratic programming problem $Q(x, H)$:

$$\min_p \nabla f(x)^T p + \tfrac{1}{2} p^T H p,$$

$$\text{s.t. } g(x) + \nabla g(x)^T p \le 0,$$

which can be associated with any x in R^n and any $n \times n$ matrix H.

Algorithm. Starting with a point x^0 in R^n, an $n \times n$ matrix H_0, and two positive numbers r and δ, the algorithm proceeds, for $k = 0, 1, \ldots,$

as follows.

Step 1. Having x^k and H_k, find a Kuhn–Tucker point p^k of $Q(x^k, H_k)$.

Step 2. Set

$$x^{k+1} = x^k + \lambda_k p^k$$

for any λ_k in $[0, \delta]$ satisfying

$$\theta_r(x^{k+1}) \leq \min_{0 \leq \lambda \leq \delta} \theta_r(x^k + \lambda p^k) + \epsilon_k,$$

where θ_r is defined in (3) and $\{\epsilon_k\}$ is a sequence of nonnegative numbers satisfying

$$\sum_{i=0}^{\infty} \epsilon_k < \infty.$$

Step 3. Update H_{k+1} by some scheme. □

Remark 2.1. It has been shown by Han (Ref. 3) that, when the sequence $\{H_k\}$ is generated by some well-known quasi-Newton updates, such as the DFP update, the algorithm without the stepsize procedure converges locally with a superlinear rate.

Remark 2.2. The function θ_r is nondifferentiable at some transient surfaces. Since efficient methods for one-dimensional minimization of such nondifferentiable functions are available (e.g., Ref. 4), the algorithm is computationally implementable.

Remark 2.3. From a different viewpoint, the algorithm can also be considered as a descent method for finding a minimum point of the function θ_r. It is noted that, for minimizing nondifferentiable functions like θ_r, the steepest descent method, even with the exact line-search, may fail to work. The cause of the failure is that the sequence generated may jam into a corner. We refer to Ref. 5, p. 75, for an example. However, in our case, the direction generated by solving $Q(x^k, H_k)$, though different from that of the steepest descent, is adequate enough to avoid the jamming situation.

3. Global Convergence

For establishing global convergence theorems, the concept of directional derivative and some of its properties are needed. Recall that a directional derivative of a real-valued function h at a point x in the direction p is defined as

$$D_p h(x) = \lim_{t \to 0+} [(h(x+tp) - h(x))/t].$$

Clearly, if

$$D_p h(x) < 0,$$

then we have

$$h(x+tp) < h(x)$$

for all sufficiently small but nonzero t. The existence of directional derivatives for the function θ_r is ensured by the following lemma. We will not give the proof for this lemma, but refer to Dem'yanov and Malozemov (Ref. 5) for a more general and detailed discussion of this result.

Lemma 3.1. If h_i, $i = 1, \ldots, k$, are continuously differentiable functions from R^n into R and

$$\Phi(x) = \max_i \{h_i(x)\},$$

then, for any direction p, the directional derivative $D_p\Phi(x)$ exists and

$$D_p\Phi(x) = \max_{i \in I(x)} \{\nabla h_i(x)^T p\},$$

where

$$I(x) = \{i : h_i(x) = \Phi(x)\}.$$

Theorem 3.1. Let f and g_i, $i = 1, \ldots, m$, be continuously differentiable at x and H be a positive definite $n \times n$ matrix. If (p, u) is a Kuhn–Tucker pair of $Q(x, H)$ with $p \neq 0$ and

$$\|u\|_\infty \leq r,$$

then

$$D_p\theta_r(x) < 0.$$

Proof. Let

$$I = \{i : g_i(x) > 0\},$$

$$\tilde{I} = \{i : g_i(x) = 0\},$$

$$\hat{I} = \{i : g_i(x) < 0\}.$$

By Lemma 3.1, we have that

$$D_p\theta_r(x)=\nabla f(x)^T p + r\sum_{i\in I}\nabla g_i(x)^T p + r\sum_{i\in \bar{I}}(\nabla g_i(x)^T p)_+.$$

Since (p, u) is a Kuhn–Tucker pair of $Q(x, H)$, we have, for $i=1,\ldots,m$,

$$g_i(x)+\nabla g_i(x)^T p\leq 0,$$

which yields

$$\sum_{i\in \bar{I}}(\nabla g_i(x)^T p)_+=0.$$

Hence, by taking

$$u_i(g_i(x)+\nabla g_i(x)^T p)=0$$

into account, we obtain

$$D_p\theta_r(x)=\nabla f(x)^T p+\sum_{i=1}^m u_i\nabla g_i(x)^T p+\sum_{i=1}^m u_i g_i(x)+r\sum_{i\in I}\nabla g_i(x)^T p.$$

By the Kuhn–Tucker equality

$$\nabla f(x)+\nabla g(x)u+\tfrac{1}{2}(H+H^T)p=0,$$

and by observing that

$$\sum_{i\in \bar{I}\cup\hat{I}} u_i g_i(x)\leq 0,$$

we have

$$D_p\theta_r(x)\leq -\tfrac{1}{2}p^T(H+H^T)p+\sum_{i\in I}(u_i g_i(x)+r\nabla g_i(x)^T p)$$

$$\leq -\tfrac{1}{2}p^T(H+H^T)p+\sum_{i\in I}(u_i-r)g_i(x)<0,$$

since H is positive definite and

$$\|u\|_\infty<r. \qquad \square$$

Before establishing the global convergence theorems, we need a lemma concerning the perturbation of quadratic programs. The proof can be found in Ref. 6.

Lemma 3.2. Let x' minimize

$$q(x)=\tfrac{1}{2}x^T Hx+b^T x$$

over

$$S = \{x : Ax \le a\},$$

and let $\bar{x}'$ minimize

$$\bar{q}(x) = \tfrac{1}{2} x^T \bar{H} x + \bar{b}^T x$$

over

$$\bar{S} = \{x : \bar{A}x \le \bar{a}\},$$

where A and $\bar{A}$ are $m \times n$ matrices, H and $\bar{H}$ are $n \times n$ matrices, a and $\bar{a}$ are in R^m, and b and $\bar{b}$ are in R^n. If H is positive definite and

$$S^0 = \{x : Ax < a\} \ne \phi,$$

then, for any fixed norm $\|\cdot\|$, there exist positive numbers c and $\bar{\epsilon}$ such that

$$\|x' - \bar{x}'\| \le c\epsilon$$

whenever

$$\epsilon \le \bar{\epsilon}$$

and

$$\epsilon = \max\{\|H - \bar{H}\|, \|A - \bar{A}\|, \|a - \bar{a}\|, \|b - \bar{b}\|\}.$$

Theorem 3.2. Let f and g_i, $i = 1, \ldots, m$, be continuously differentiable, and assume that the following conditions are satisfied:

(i) there exist two positive numbers α and β such that

$$\alpha x^T x \le x^T H_k x \le \beta x^T x$$

for each k and any x in R^n;

(ii) for each k, there exists a Kuhn–Tucker point of $Q(x^k, H_k)$ with a Lagrange multiplier vector bounded by r in ∞-norm.

Then, any sequence $\{x^k\}$ generated from the algorithm either terminates at a Kuhn–Tucker point of (1) or any accumulation point $\bar{x}$ with

$$S^0(\bar{x}) = \{p : g(\bar{x}) + \nabla g(\bar{x})^T p < 0\} \ne \phi$$

is a Kuhn–Tucker point of (1).

Proof. By assumption (ii), we have (p^k, u^k), which is a Kuhn–Tucker pair of $Q(x^k, H_k)$ with

$$\|u^k\|_\infty \le r.$$

If

$$p^k = 0,$$

then (x^k, u^k) satisfies the Kuhn–Tucker conditions of (1), and the sequence terminates at the Kuhn–Tucker point x^k of (1). Suppose that

$$p^k \neq 0$$

for each k. From Theorem 3.1 and the way we choose x^{k+1}, it follows that x^{k+1} exists and

$$\theta_r(x^{k+1}) < \theta_r(x^k) + \epsilon_k.$$

Let $\bar{x}$ be an accumulation point of $\{x^k\}$ with

$$S^0(\bar{x}) \neq \phi.$$

Without loss of generality, we may assume

$$x^k \to \bar{x} \quad \text{and} \quad H_k \to \bar{H}.$$

The existence of $\bar{H}$ follows from assumption (i). Furthermore, $\bar{H}$ is positive definite. It follows from

$$S^0(\bar{x}) \neq \phi$$

and the positive definiteness of $\bar{H}$ that $Q(\bar{x}, \bar{H})$ has a unique Kuhn–Tucker point $\bar{p}$. If

$$\bar{p} = 0,$$

then $\bar{x}$ is a Kuhn–Tucker point of (1) and the theorem follows. Suppose that

$$\bar{p} \neq 0.$$

By Lemma 3.2, we have that

$$p^k \to \bar{p}.$$

Since $\{u^k\}$ is uniformly bounded, there exists an accumulation point u of u^k. From

$$x^k \to x, \qquad p^k \to \bar{p},$$

and the continuity of gradients of f and g, it follows that $\bar{u}$ is a Lagrange multiplier vector of $Q(\bar{x}, \bar{H})$ and

$$\|\bar{u}\|_\infty \leq r.$$

Let $\bar{\lambda} \in [0, \delta]$ be chosen such that

$$\theta_r(\bar{x} + \bar{\lambda}\bar{p}) = \min_{0 \leq \lambda \leq \delta} \theta_r(\bar{x} + \lambda\bar{p}).$$

By Theorem 3.1, we have

$$\theta_r(\bar{x}+\bar{\lambda}\bar{p})<\theta_r(\bar{x}).$$

Set

$$\beta=\theta_r(\bar{x})-\theta_r(\bar{x}+\bar{\lambda}\bar{p}).$$

Since

$$x^k+\bar{\lambda}p^k\to\bar{x}+\bar{\lambda}\bar{p},$$

it follows that, for sufficiently large k, we have

$$\theta_r(x^k+\bar{\lambda}p^k)+\beta/2<\theta_r(\bar{x}). \tag{4}$$

However, by

$$\theta_r(x^{k+1})<\theta_r(x^k)+\epsilon_k \qquad \text{and } \sum_{i=k}^{\infty}\epsilon_i<\beta/2,$$

for sufficiently large k we have

$$\begin{aligned}\theta_r(\bar{x})&<\theta_r(x^{k+1})+\sum_{i=k+1}^{\infty}\epsilon_i\\ &\le\min_{0\le\lambda\le\delta}\theta_r(x^k+\lambda p^k)+\epsilon_k+\sum_{i=k+1}^{\infty}\epsilon_i\\ &<\theta_r(x^k+\bar{\lambda}p^k)+\beta/2,\end{aligned}$$

which contradicts (4). Hence,

$$\bar{p}=0,$$

and $\bar{x}$ is a Kuhn–Tucker point of (1). □

Assumption (ii) of Theorem 3.2 is not as restrictive as it might appear. In the rest of this section, we will give a sufficient condition which ensures the satisfaction of this assumption. First, we introduce the following lemma.

Lemma 3.3. Let f and g_i, $i=1,\dots,m$, be continuously differentiable, and let the following conditions be satisfied:

(i) g_i's are convex;
(ii) $X^0=\{x: g(x)<0\}\neq\phi$;
(iii) for some positive numbers α and β and for any y in R^n,

$$\alpha y^Ty\le y^THy\le\beta y^Ty.$$

Then, for any compact set $U\subset R^n$ and for any vector norm $\|\cdot\|$, there exist $r>0$ such that, if u in R^m is a Lagrange multiplier vector of the quadratic program $Q(x, H)$ with x in U, then

$$\|u\| \leq r.$$

Proof. We can assume that H is symmetric. If not, we can replace it by $\frac{1}{2}(H+H^T)$ without affecting the results.

From assumption (ii) of this lemma, there exists at least one point, say $\hat{x}$, in X^0. Let

$$\eta = \min_i\{-g_i(\hat{x})\}, \tag{5}$$

$$\xi = \max_x\{\|x-\hat{x}\|_2 \colon x \in U\}. \tag{6}$$

We assume further that σ is an upper bound of $\|\nabla f(x)\|_2$ on U and also an upper bound of $\|H\|_2$ and $\|H^{-1}\|_2$.

From the assumptions, it follows that a Kuhn–Tucker point p of $Q(x, H)$ exists and is unique. Let u be a Lagrange multiplier vector of $Q(x, H)$ and

$$\bar{p} = \hat{x} - x.$$

By the convexity of the g_i's, we have that, for $i = 1, \ldots, m$,

$$g_i(x) + \nabla g_i(x)^T\bar{p} \leq g_i(\hat{x}) < 0. \tag{7}$$

Hence, $\bar{p}$ is a feasible point of $Q(x, H)$; and, from the Kuhn–Tucker saddle-point theorem (Ref. 7), it follows that

$$\nabla f(x)^T p + \tfrac{1}{2}p^T H p \leq \nabla f(x)^T\bar{p} + \tfrac{1}{2}\bar{q}^T H\bar{p} + \sum_{j=1}^{m} u_i(g_i(x) + \nabla g_i(x)^T\bar{p}). \tag{8}$$

Thus, by (8), (7), and (5), we have

$$\eta\|u\|_1 \leq \nabla f(x)^T\bar{p} + \tfrac{1}{2}\bar{p}^T H\bar{p} - \nabla f(x)^T p - \tfrac{1}{2}p^T H p. \tag{9}$$

Now, consider the dual problem of $Q(x, H)$

$$\max_{v \in R^m} -(\nabla f(x) + \nabla g(x)v)^T H^{-1}(\nabla f(x) + \nabla g(x)v) + v^T g(x),$$

$$\text{s.t. } v \geq 0.$$

Since $v = 0$ is dual feasible, by Dorn's duality theorem (Ref. 7) we have

$$\nabla f(x)^T p + \tfrac{1}{2}p^T H p \geq -\nabla f(x)^T H^{-1}\nabla f(x). \tag{10}$$

From (9) and (10), it follows that

$$\begin{aligned}\|u\|_1 &\le (1/\eta)(\nabla f(x)^T\bar{p}+\tfrac{1}{2}\bar{p}^T H\bar{p}+\nabla f(x)^T H^{-1}\nabla f(x))\\ &\le (1/\eta)(\sigma\xi+\tfrac{1}{2}\sigma\xi^2+\sigma^3),\end{aligned}$$

which, by the equivalence of norms, implies the desired result. □

We also need the following lemma on the compactness of some level sets.

Lemma 3.4. If

$$X=\{x: g(x)\le 0\}$$

is compact and the g_i's are lower semicontinuous and convex, then

$$X_c=\left\{x: \sum_{i=1}^{m} g_i(x)_+\le c\right\}$$

is compact for any finite real number c.

Proof. Define

$$\Phi(x)=\sum_{i=1}^{m} g_i(x)_+.$$

By the lower semicontinuity and convexity of the g_i's, the function Φ is closed and convex. Since $X_0=X$ is compact, it follows from Ref. 8, Lemma 4.1.14, p. 139, that X_c is compact for any finite c. □

A global convergence theorem is given below.

Theorem 3.3. Let f and g_i, $i=1,\ldots,m$, be continuously differentiable, and let the following conditions hold:

(i) f is bounded below;
(ii) g_i's are convex;
(iii) the set

$$X=\{x: g(x)\le 0\}$$

is compact and

$$X^0=\{x: g(x)<0\}\neq\phi;$$

(iv) there exist positive numbers α and β such that, for each k and for any x in R^n,

$$\alpha x^T x\le x^T H_k x\le \beta x^T x.$$

Then, for any starting point x^0, there exists a positive number $\bar{r}$ such that, if

$$r \geq \max\{\bar{r}, 1\},$$

then any sequence $\{x^k\}$ generated from the algorithm either terminates at a Kuhn–Tucker point of (1) or any accumulation point of this sequence is a Kuhn–Tucker point of (1).

Proof. It is evident that such a sequence exists. By (ii) and (iii), we also have that, for any x in R^n, the set

$$S^0(x) = \{p: g(x) + \nabla g(x)^T p < 0\} \neq \phi.$$

Therefore, we need only to prove that assumption (ii) of Theorem 3.2 holds.

Let x^0 be a given starting point, and let f be bounded below by $-\sigma$. Define

$$c = f(x^0) + \sum_{i=1}^{m} g_i(x^0)_+ + \sigma + \sum_{i=0}^{\infty} \epsilon_i.$$

By Lemma 3.4, the set

$$X_c = \left\{x: \sum_{i=1}^{m} g_i(x)_+ \leq c\right\}$$

is compact. Hence, it follows from Lemma 3.3 that there exists an $\bar{r} > 0$ such that, if

$$x \in X_c$$

and

$$\alpha y^T y \leq y^T H y \leq \beta y^T y$$

for any y in R^n, then a Lagrange multiplier vector u of $Q(x, H)$ exists and

$$\|u\|_\infty < \bar{r}.$$

Therefore, it is only necessary to show that

$$x^k \in X_c$$

for each k. It is clear that

$$x^0 \in X_c.$$

Assume that

$$x^k \in X_c$$

and

$$\theta_r(x^k) \leq \theta_r(x^0) + \sum_{i=0}^{k-1} \epsilon_i.$$

Then,

$$\|u^k\|_\infty \leq \bar{r};$$

and, by Theorem 3.1 and the choice of x^{k+1} in the algorithm, we have

$$\begin{aligned} f(x^{k+1}) + r \sum_{i=1}^{m} g_i(x^{k+1})_+ &\leq f(x^k) + r \sum_{i=1}^{m} g_i(x^k)_+ + \epsilon_k \\ &\leq f(x^0) + r \sum_{i=1}^{m} g_i(x^0)_+ + \sum_{i=0}^{k} \epsilon_i. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{m} g_i(x^{k+1})_+ &\leq (1/r)(f(x^0) - f(x^{k+1})) + \sum_{i=1}^{m} g_i(x^0)_+ + (1/r) \sum_{i=0}^{k} \epsilon_i \\ &\leq f(x^0) + \sigma + \sum_{i=1}^{m} g_i(x^0)_+ + \sum_{i=0}^{\infty} \epsilon_i = c. \end{aligned}$$

Therefore,

$$x^{k+1} \in X_c,$$

and the proof is complete.

If we further assume f to be strictly convex, then (1) has an unique Kuhn–Tucker point which is actually its optimal solution. Therefore, we have the following result.

Corollary 3.1. Let all the assumptions of Theorem 3.3 hold. Furthermore, if f is strictly convex, then any sequence $\{x^k\}$ generated from the algorithm converges to the optimal solution of (1).

4. Conclusions

We conclude this paper with the following comments.

(i) It has been shown by Han (Ref. 3) that the algorithm converges locally with a superlinear rate when the Davidson–Fletcher–Powell update is used to generate the matrices $\{H_k\}$ and the stepsizes λ_k are set to be one. For this reason, if the DFP update is used, we suggest that the stepsize procedure be discarded to achieve a superlinear rate when the points have moved close to the solution and the matrices have become a good approximation to the Hessian of the Lagrangian.

(ii) A different way to generate the direction p^k is to solve the dual problem of $Q(x^k, H_k)$:

$$\min \tfrac{1}{2}(\nabla f(x^k)+\nabla g(x^k)u)^T H_k^{-1}(\nabla f(x^k)+\nabla g(x^k)u)-g(x)^T u,$$

$$\text{s.t. } u \geq 0,$$

and, with u^k as its solution, set

$$p^k = -H_k^{-1}(\nabla f(x^k)+\nabla g(x^k)u^k).$$

A study of the local convergence of this algorithm without stepsize procedure is contained in Ref. 9. We note that all the analysis in this paper can be carried through for this case and the global convergence theorems are also valid.

(iii) An approximate line-search procedure (like Armijo's search and Goldstein's search) is desirable. Since the function θ_r is nondifferentiable, these procedures do not work in our case. It is of some practical value to develop a workable one.

References

1. ROBINSON, S. M., *Perturbed Kuhn–Tucker Points and Rates of Convergence for a Class of Nonlinear-Programming Algorithms*, Mathematical Programming, Vol. 7, pp. 1–16, 1974.
2. GARCIA-PALOMARES, U. M., and MANGASARIAN, O. L., *Superlinearly Convergent Quasi-Newton Algorithms for Nonlinearly Constrained Optimization Problems*, Mathematical Programming, Vol. 11, pp. 1–13, 1976.
3. HAN, S. P., *Superlinearly Convergent Variable Metric Algorithms for General Nonlinear Programming Problems*, Mathematical Programming, Vol. 11, pp. 263–282, 1976.
4. KOWALIK, J., and OSBORNE, M. R., *Methods for Unconstrained Optimization Problems*, American Elsevier, New York, New York, 1968.
5. DEM'YANOV, V. F., and MALOZEMOV, V. N., *Introduction to Minimax*, John Wiley and Sons, New York, New York, 1974.
6. DANIEL, J. M., *Stability of the Solution of Definite Quadratic Programs*, Mathematical Programming, Vol. 5, pp. 41–53, 1973.
7. MANGASARIAN, O. L., *Nonlinear Programming*, McGraw-Hill Book Company, New York, New York, 1969.
8. STOER, J., and WITZGALL, C., *Convexity and Optimization in Finite Dimensions, I*, Springer-Verlag, Berlin, Germany, 1970.
9. HAN, S. P., *Dual Variable Metric Methods for Constrained Optimization Problems*, SIAM Journal on Control and Optimization, Vol. 15, No. 4, 1977.