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Pontryagin Functions for Multiple Integral Control Problems[1]

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Abstract. An optimal control problem is considered whose performance index is represented by an m-fold multiple integral, the state equations being given by a system of first-order partial differential equations. The concept of a field of optimal control variables with respect to an independent integral is introduced, the significance of these fields being due to the fact that a control pair is optimal whenever it is imbedded in such a field. Since there are m distinct m-fold independent integrals, it is possible to construct m distinct fields of this kind. For each of these, a Pontryagin function is defined, and it is shown that, if an optimal pair is embedded in one of these fields, it satisfies a corresponding Pontryagin maximum principle.

Key Words. Distributed control problems, sufficiency theorems, optimization theorems, calculus of variations.

1. Introduction

It was shown in a previous note (Ref. 1) that, for an m-fold integral optimal control problem, one can establish m distinct sufficiency conditions, which is possible by virtue of the fact that there exist m distinct independent m-fold integrals. In a very recent paper, Klötzler (Ref. 2) described the construction of his so-called field of optimal control variables with respect to an arbitrary independent integral, and he proved that, whenever an admissible control pair is embedded in such a field, this pair is optimal. By means of one of these independent integrals, namely the simplest, Klötzler defined a Pontryagin function, in terms of which he established a maximum principle for multiple integral control problems.

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It is the object of the present note to extend these results in the following sense. Relative to each of the m aforementioned independent integrals, a field of optimal control variables is defined. Such fields are direct analogues of the m distinct geodesic fields in the calculus of variations of m-fold integrals, each of which possesses its own peculiar canonical formalism, including a Hamiltonian function (Ref. 3). The form of the latter clearly suggests the structure of the Pontryagin function to be associated with each of the fields of optimal control variables. Thus, m distinct Pontryagin functions are defined, which in turn give rise to a set of Hamilton–Jacobi equations. It is indicated how solutions of the latter may be used to construct corresponding fields of optimal control variables. A control pair which is embedded in any one such field is optimal, and it is shown that such a pair satisfies a Pontryagin maximum principle. Thus, for a given m-fold integral control problem, one can formulate m distinct maximum principles, of which one turns out to be equivalent to the principle enunciated by Klötzler. The increased analytical complexity of the general development discussed below may, perhaps, be offset by the advantage that the extended theory allows for a choice of several maximum principles instead of being restricted to a particular one.

We shall consider a system described by a state vector with n components $x^j(t^\alpha)$ depending on m independent real variables t^α which vary on a fixed, closed, connected region G in R^m. Unless otherwise specified, Latin indices $j, h, \ldots,$ range from 1 to n, while Greek indices $\alpha, \beta, \ldots,$ range from 1 to m, the summation convention being operative in respect of both sets of indices. It is supposed that the state equation is of the form

$$\partial x^j/\partial t^\alpha = f^j_\alpha(t^\epsilon, x^h(t^\epsilon), u^A(t^\epsilon)), \tag{1}$$

where the given functions f^j_α are of class C^1, while the p control functions $u^A(t^\epsilon)$, $A = 1, \ldots, p$, belong to a set of admissible controls, namely the set of bounded, piecewise continuous functions on G whose values range in a given open set $U \subset R^p$. The range of the state functions is supposed to be an open set $X \subset R^n$, and we shall denote by Ω the set of all admissible pairs $(x^j(t^\epsilon), u^A(t^\epsilon))$ which are related according to (1) and for which the state functions $x^j(t^\alpha)$ satisfy certain boundary conditions on the boundary ∂G of G. These boundary conditions are to be specified presently. It is assumed that Ω is nonempty.

Let $f_0(t^\epsilon, x^j(t^\epsilon), u^A(t^\epsilon))$ denote a class C^1 function by means of which the performance index is defined as an m-fold integral over the region G:

$$I(x^j, u^A) = \int_G f_0(t^\epsilon, x^j(t^\epsilon), u^A(t^\epsilon))\, d(t), \tag{2}$$

where we have used the abbreviation

$$d(t)=dt^1\dots dt^m. \tag{3}$$

The optimal control problem consists of the determination of a pair $(\overset{*}{x}{}^j(t^\epsilon), \overset{*}{u}{}^A(t^\epsilon))\in\Omega$ which affords a minimum to the integral (2). Such a pair will be called *optimal.*

Before proceeding to an investigation of this problem, it is appropriate that we should recall some results obtained previously (Ref. 1). Let us denote by $V^\alpha(t^\epsilon, x^j)$ a set of m class C^2 functions, by means of which we construct the m^2 matrix elements

$$c^\alpha_\beta=\partial V^\alpha/\partial t^\beta+(\partial V^\alpha/\partial x^j)f^j_\beta. \tag{4}$$

These give rise to m distinct functions $\psi_{(r)}(t^\epsilon, x^j, u^A)$, $r=1,\dots,m$, the latter being defined by

$$r!\psi_{(r)}=\delta^{\beta_1\dots\beta_r}_{\alpha_1\dots\alpha_r}c^{\alpha_1}_{\beta_1}\dots c^{\alpha_r}_{\beta_r}, \tag{5}$$

in which $\delta^{\dots}_{\dots}$ denotes the generalized Kronecker delta. The function $\psi_{(r)}$ is, in fact, the sum of all principal $r\times r$ minors of the determinant

$$\Delta=\det(c^\alpha_\beta), \tag{6}$$

so that, in particular

$$\psi_{(1)}=\partial V^\alpha/\partial t^\alpha+(\partial V^\alpha/\partial x^j)f^j_\alpha, \tag{7}$$

while

$$\psi_{(m)}=\Delta. \tag{8}$$

The significance of the functions $\psi_{(r)}$ is due to the fact that each of the m integrals

$$J_{(r)}(x^j, u^A)=\int_G \psi_{(r)}(t^\epsilon, x^j, u^A)\,d(t) \tag{9}$$

is an independent integral, in the sense that it depends solely on the values assumed by $x^j(t^\alpha)$ on the boundary ∂G of G for any given control $u^A(t^\epsilon)$.

This independence property is a direct consequence of a system of m distinct integral formulae. Let us assume that ∂G admits a parametric representation of the form

$$t^\alpha=t^\alpha(\tau^a),\qquad a=1,\dots,m-1,$$

and let us construct the functional determinants

$$\Delta^{\beta_1\dots\beta_{r-1},\alpha_r\dots\alpha_{m-1}}=\partial(V^{\beta_1},\dots,V^{\beta_{r-1}},t^{\alpha_r},\dots,t^{\alpha_{m-1}})/\partial(\tau^1,\dots\dots\dots,\tau^{m-1}). \tag{10}$$

It may then be shown that (Ref. 1, p. 134)

$$r!(m-r)!J_{(r)}(x^j, u^A) = \int_{\partial G} \epsilon_{\beta\beta_1\ldots\beta_{r-1}\alpha_r\ldots\alpha_{m-1}} V^\beta \Delta^{\beta_1\ldots\beta_{r-1}\alpha_r\ldots\alpha_{m-1}} d(\tau), \tag{11}$$

where $\epsilon_{\ldots}$ denotes the m-dimensional permutation symbol. Moreover, let $x^j(t^\alpha)$, $\bar{x}^j(t^\alpha)$ denote two state vectors which coincide on the boundary ∂G of G:

$$x^j(t^\alpha(\tau^a)) = \bar{x}^j(t^\alpha(\tau^a)). \tag{12}$$

It is easily verified that, under these circumstances, the derivatives

$$\partial V^\alpha\{t^\epsilon(\tau^a), x^j(t^\epsilon(\tau^a))\}/\partial\tau^a, \qquad \partial V^\alpha\{t^\epsilon(\tau^a), x^j(t^\epsilon(\tau^a))\}/\partial\tau^a$$

are identical. Thus, the functional determinants (10), which appear in the integrand of (11), assume the same values for all functions $x^j(t^\alpha)$ satisfying the boundary condition (12), and this in turn implies the asserted independence of the integrals (9) by virtue of the integral formulae (11).

The well-known method of equivalent integrals in the calculus of variations (in the sense of Carathéodory) now suggests the introduction of equivalent performance indices in terms of equivalent integrands

$$g_{(r)}(t^\epsilon, x^j, u^A) = f_0(t^\epsilon, x^j, u^A) - \psi_{(r)}(t^\epsilon, x^j, u^A). \tag{13}$$

Let $v^A(t^\epsilon, x^j)$ denote p functions defined on a region B in the domain R^{m+n} of the variables (t^ϵ, x^j), such that $G \times X \subset B$. Adopting the terminology introduced by Klötzler (Ref. 2), we shall call the set $\{t^\epsilon, x^j, v^A(t^\epsilon, x^j): (t^\epsilon, x^j) \in B\}$ a *field* $F_{(r)}(B)$ *of optimal control variables with respect to* $\psi_{(r)}$ whenever the following conditions are satisfied:

$$g_{(r)}(t^\epsilon, x^j, u^A) \geq 0, \qquad \forall u^A \in U, \tag{14}$$

with

$$\min_{u^A \in U} g_{(r)}(t^\epsilon, x^j, u^A) = g_{(r)}(t^\epsilon, x^j, v^A(t^\epsilon, x^j)) = 0, \qquad \forall (t^\epsilon, x^j) \in B. \tag{15}$$

Moreover, a pair $(\overset{*}{x}{}^j, \overset{*}{u}{}^A)$ is said to be *embedded in the field* $F_{(r)}(B)$ if $(\overset{*}{x}{}^j, \overset{*}{u}{}^A)$ satisfy the state equation (1) with $\overset{*}{u}{}^A = v^A(t^\epsilon, \overset{*}{x}{}^j)$, while

$$J_{(r)}(\overset{*}{x}{}^j, \overset{*}{u}{}^A) \leq J_{(r)}(x^j, u^A) \tag{16}$$

for all admissible pairs (x^j, u^A). It should be emphasized that, by virtue of the independence property of the integrals $J_{(r)}$, *the conditions* (16) *are essentially boundary conditions*. Thus, if $(\overset{*}{x}{}^j, \overset{*}{u}{}^A)$ is embedded in $F_{(r)}(B)$, it follows from (1), (13), (9), (14), (15), and (16) that

$$I(x^j, u^A) - I(\overset{*}{x}{}^j, \overset{*}{u}{}^A) = \int_G g_{(r)}(t^\epsilon, x^j, u^A)\, d(t) - \int_G g_{(r)}(t^\epsilon, \overset{*}{x}{}^j, \overset{*}{u}{}^A)\, d(t)$$

$$- J_{(r)}(\overset{*}{x}{}^j, \overset{*}{u}{}^A) + J_{(r)}(x^j, u^A) \geq 0. \tag{17}$$

It therefore follows that, whenever a pair $(\overset{*}{x}{}^j, \overset{*}{u}{}^A)$ is embedded in the field $F_{(r)}(B)$, *this pair is optimal.* This conclusion is due to Klötzler (Ref. 2) for unspecified independent integrals; for the independent integrals (9) it is equivalent to a sufficiency criterion stated previously (Ref. 1). When $r = 1$, the latter reduces to a sufficiency theorem stated by Butkovsky (Ref. 4), and for $m = 1$ it is a special case (for fixed domain of integration G) of a sufficiency condition formulated by Leitmann (Ref. 5).

2. Pontryagin Functions

It is well known that each of the m distinct functions $\psi_{(r)}$, as defined by (5), gives rise to a definite field theory of multiple integral problems in the calculus of variations. Moreover, it has been shown (Ref. 3, pp. 406–413) that each of these m field theories requires its own peculiar canonical formalism, which is based on a set of m canonical momentum vectors, each of which possesses $n + m$ components. In order to define the m Pontryagin functions for the optimal control problem under consideration here, we shall follow an analogous procedure and begin by introducing $m^2 + nm$ parameters, to be denoted by λ^α_β, λ^α_j. By means of the latter, we define the m^2 matrix elements

$$p^\alpha_\beta = \lambda^\alpha_\beta + \lambda^\alpha_j f^j_\beta, \tag{18}$$

in terms of which m functions

$$\pi_{(r)}(t^\epsilon, x^j, u^A, \lambda^\alpha_\beta, \lambda^\alpha_j), \qquad r = 1, \ldots, m,$$

are constructed according to

$$r!\,\pi_{(r)} = \delta^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_r} p^{\beta_1}_{\alpha_1} p^{\beta_2}_{\alpha_2} \cdots p^{\beta_r}_{\alpha_r}. \tag{19}$$

By virtue of the symmetry of the generalized Kronecker delta in the index pairs

$$\binom{\alpha_1}{\beta_1}, \binom{\alpha_2}{\beta_2}, \ldots, \binom{\alpha_r}{\beta_r},$$

we have

$$r![\partial\pi_{(r)}/\partial p^\epsilon_\gamma] = r\delta^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_r} \delta^{\beta_1}_\epsilon \delta^\gamma_{\alpha_1} p^{\beta_2}_{\alpha_2} \cdots p^{\beta_r}_{\alpha_r} = r\delta^{\gamma\alpha_2\ldots\alpha_r}_{\epsilon\beta_2\ldots\beta_r} p^{\beta_2}_{\alpha_2} \cdots p^{\beta_r}_{\alpha_r}.$$

This suggests that we write

$$(r-1)!\overset{(r)}{P}{}^{\alpha}_{\beta}=\delta^{\alpha\alpha_2\ldots\alpha_r}_{\beta\beta_2\ldots\beta_r}p^{\beta_2}_{\alpha_2}\ldots p^{\beta_r}_{\alpha_r}, \tag{20}$$

so that

$$\partial\pi_{(r)}/\partial p^{\epsilon}_{\gamma}=\overset{(r)}{P}{}^{\gamma}_{\epsilon}. \tag{21}$$

In passing, we note that, for $r=m$, the quantities $\overset{(m)}{P}{}^{\alpha}_{\beta}$ are the cofactors of p^{β}_{α} in the determinant of the m^2 elements (18), while when $r=1$, we have

$$\overset{(1)}{P}{}^{\alpha}_{\beta}=\delta^{\alpha}_{\beta}.$$

It also follows directly from (18) that

$$\partial p^{\epsilon}_{\gamma}/\partial\lambda^{\alpha}_{\beta}=\delta^{\epsilon}_{\alpha}\delta^{\beta}_{\gamma},\qquad \partial p^{\epsilon}_{\gamma}/\partial\lambda^{\alpha}_{j}=\delta^{\epsilon}_{\alpha}f^{j}_{\gamma},\qquad \partial p^{\epsilon}_{\gamma}/\partial x^{j}=\lambda^{\epsilon}_{h}(\partial f^{h}_{\gamma}/\partial x^{j}). \tag{22}$$

With the aid of (21), it is therefore inferred that

$$\partial\pi_{(r)}/\partial\lambda^{\alpha}_{\beta}=[\partial\pi_{(r)}/\partial p^{\epsilon}_{\gamma}](\partial p^{\epsilon}_{\gamma}/\partial\lambda^{\alpha}_{\beta})=\overset{(r)}{P}{}^{\gamma}_{\epsilon}\delta^{\epsilon}_{\alpha}\delta^{\beta}_{\gamma}=\overset{(r)}{P}{}^{\beta}_{\alpha}, \tag{23}$$

$$\partial\pi_{(r)}/\partial\lambda^{\alpha}_{j}=[\partial\pi_{(r)}/\partial p^{\epsilon}_{\gamma}](\partial p^{\epsilon}_{\gamma}/\partial\lambda^{\alpha}_{j})=\overset{(r)}{P}{}^{\gamma}_{\epsilon}\delta^{\epsilon}_{\alpha}f^{j}_{\gamma}=\overset{(r)}{P}{}^{\beta}_{\alpha}f^{j}_{\beta}, \tag{24}$$

$$\partial\pi_{(r)}/\partial x^{j}=[\partial\pi_{(r)}/\partial p^{\epsilon}_{\gamma}](\partial p^{\epsilon}_{\gamma}/\partial x^{j})=\overset{(r)}{P}{}^{\gamma}_{\epsilon}\lambda^{\epsilon}_{h}(\partial f^{h}_{\gamma}/\partial x^{j})=\overset{(r)}{P}{}^{\beta}_{\alpha}\lambda^{\alpha}_{h}(\partial f^{h}_{\beta}/\partial x^{j}). \tag{25}$$

The structure of the Hamiltonian functions associated with the m distinct field theories in the calculus of variations (Ref. 3) now suggests that *the rth Pontryagin function of our optimal control problem* be defined as follows:

$$H^{*}_{(r)}(t^{\epsilon},x^{j},u^{A},\lambda^{\alpha}_{\beta},\lambda^{\alpha}_{j})=-f_0(t^{\epsilon},x^{j},u^{A})+\pi_{(r)}(t^{\epsilon},x^{j},u^{A},\lambda^{\alpha}_{\beta},\lambda^{\alpha}_{j}), \tag{26}$$

where it should be emphasized that, to each value of $r=1,\ldots,m$, there corresponds one such function. For future reference, we note already at this state that, because of (22)–(25), the derivatives of these functions are given by

$$\partial H^{*}_{(r)}/\partial\lambda^{\alpha}_{\beta}=\overset{(r)}{P}{}^{\beta}_{\alpha}, \tag{27}$$

$$\partial H^{*}_{(r)}/\partial\lambda^{\alpha}_{j}=\overset{(r)}{P}{}^{\beta}_{\alpha}f^{j}_{\beta}, \tag{28}$$

$$\partial H^{*}_{(r)}/\partial x^{j}=-\partial f_0/\partial x^{j}+\overset{(r)}{P}{}^{\beta}_{\alpha}\lambda^{\alpha}_{h}\partial f^{h}/\partial x^{j}. \tag{29}$$

Let us now suppose that we are given a field $F_{(r)}(B)$ of optimal control variables as defined in the previous section. Because of the similar structures of the functions (5), (19), and the explicit forms of (18) and (26), it follows that the condition (15) for such a field may be expressed as

$$\max_{u^A \in U} H^*_{(r)}(t^\epsilon, x^j, u^A, \partial V^\alpha/\partial t^\beta, \partial V^\alpha/\partial x^j) = 0. \tag{30}$$

For instance, when $r = 1$, we have from (19), (18), and (26) that

$$H^*_{(1)} = -f_0 + \lambda^\alpha_\alpha + \lambda^\alpha_j f^j_\alpha. \tag{31}$$

In this case, the criterion (30) can be written in the form

$$\max_{u^A \in U} K^*(t^\epsilon, x^j, u^A, \partial V^\alpha/\partial x^j) + \partial V^\alpha/\partial t^\alpha = 0, \tag{32}$$

where

$$K^*(t^\epsilon, x^j, u^A, \lambda^\alpha_j) = -f_0 + \lambda^\alpha_j f^j_\alpha \tag{33}$$

is the Pontryagin function as defined by Klötzler (Ref. 2, p. 8), while (32) represents Klötzler's generalization of the so-called Bellman equation of the single integral case ($m = 1$).

Proceeding with the case for which the integer r is arbitrary, let us assume that there exists a Hamiltonian function $H_{(r)}(t^\epsilon, x^j, \lambda^\alpha_\beta, \lambda^\alpha_j)$ defined by

$$H_{(r)}(t^\epsilon, x^j, \lambda^\alpha_\beta, \lambda^\alpha_j) = \max_{u^A \in U} H^*_{(r)}(t^\epsilon, x^j, u^A, \lambda^\alpha_\beta, \lambda^\alpha_j), \tag{34}$$

in which case the condition (30) can be written in the form of Hamilton–Jacobi equation:

$$H_{(r)}(t^\epsilon, x^j, \partial V^\alpha/\partial t^\beta, \partial V^\alpha/\partial x^j) = 0. \tag{35}$$

Conversely, let $U^A(t^\epsilon, x^j, \lambda^\alpha_\beta, \lambda^\alpha_j)$ denote some u^A for which the maximum in (34) is attained for given $(t^\epsilon, x^j, \lambda^\alpha_\beta, \lambda^\alpha_j)$, such functions U^A not necessarily being unique, i.e.,

$$H_{(r)}(t^\epsilon, x^j, \lambda^\alpha_\beta, \lambda^\alpha_j) = H^*_{(r)}(t^\epsilon, x^j, U^A, \lambda^\alpha_\beta, \lambda^\alpha_j). \tag{36}$$

Then a solution V^α of (35) which is of class C^2 in the region B gives rise to a field $F_{(r)}(B)$ with

$$v^A(t^\epsilon, x^j) = U^A(t^\epsilon, x^j, \partial V^\alpha/\partial t^\beta, \partial V^\alpha/\partial x^j). \tag{37}$$

This conclusion follows directly from the fact that the condition (15) for $F_{(r)}(B)$ is satisfied as a result of (34) and (35). The pair $(\overset{*}{x}{}^j, \overset{*}{u}{}^A) \in \Omega$ with $\overset{*}{u}{}^A = v^A(t^\epsilon, x^j)$ as given by (37) is embedded in this field $F_{(r)}(B)$, and, from the concluding remark of the previous section, it follows that *this pair is optimal.*

3. Maximum Principles

Let us suppose that we are given a field $F_{(r)}(B)$ of optimal control variables constructed by mean of class C^2 solutions $V^\alpha(t^\epsilon, x^h)$ of the partial differential equation (35) according to the method outlined above. It is now asserted that, *for an optimal pair embedded in* $F_{(r)}(B)$, *a Pontryagin maximum principle is valid in the following form*: There exist $m(m+n)$ functions $\lambda^\alpha_\beta(t^\epsilon, x^h)$, $\lambda^\alpha_j(t^\epsilon, x^h)$ such that

$$\max_{u^A \in U} H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, u^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h) = H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, \overset{*}{u}{}^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h), \tag{38}$$

$$\overset{(r)}{P}{}^\beta_\alpha(\partial \overset{*}{x}{}^j/\partial t^\beta) = \partial H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, \overset{*}{u}{}^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h)/\partial \lambda^\alpha_j, \tag{39}$$

$$\overset{(r)}{P}{}^\beta_\alpha(d\lambda^\alpha_j/dt^\beta) = -\partial H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, \overset{*}{u}{}^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h)/\partial x^j, \tag{40}$$

$$\overset{(r)}{P}{}^\beta_\alpha(d\lambda^\alpha_\gamma/dt^\beta) = -\partial H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, \overset{*}{u}{}^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h)/\partial t^\gamma, \tag{41}$$

where

$$d/dt^\beta = \partial/\partial t^\beta + (\partial/\partial x^h) f^h_\beta. \tag{42}$$

It will be shown that these relations may be established as a consequence of the identifications

$$\lambda^\alpha_\beta = \partial V^\alpha/\partial t^\beta, \qquad \lambda^\alpha_j = \partial V^\alpha/\partial x^j. \tag{43}$$

The assertion (38) follows directly from the construction of the field $F_{(r)}(B)$, while (39) is an immediate consequence of (28) and (1), the latter being included in the embedding condition. As regards (40) and (41), we observe that, since the V^α represent class C^2 solutions of (35) by hypothesis, we have

$$\partial H_{(r)}/\partial x^j + [\partial H_{(r)}/\partial \lambda^\alpha_\beta](\partial^2 V^\alpha/\partial t^\beta\, \partial x^j) + [\partial H_{(r)}/\partial \lambda^\alpha_h](\partial^2 V^\alpha/\partial x^h\, \partial x^j) = 0, \tag{44}$$

together with

$$\partial H_{(r)}/\partial t^\gamma + [\partial H_{(r)}/\partial \lambda^\alpha_\beta](\partial^2 V^\alpha/\partial t^\beta\, \partial t^\gamma) + [\partial H_{(r)}/\partial \lambda^\alpha_h](\partial^2 V^\alpha/\partial x^h\, \partial t^\gamma) = 0. \tag{45}$$

But, from the definition (35), taken in conjunction with (36), it follows that the partial derivatives of $H_{(r)}(t^\epsilon, x^h, \lambda^\alpha_\beta, \lambda^\alpha_h)$ coincide with the corresponding derivatives of $H^*_{(r)}(t^\epsilon, x^h, u^A, \lambda^\alpha_\beta, \lambda^\alpha_h)$ whenever

$$u^A = U^A(t^\epsilon, x^h, \lambda^\alpha_\beta, \lambda^\alpha_h),$$

since for these values all terms involving $\partial H^*_{(r)}/\partial u^A$ vanish. But this condition on u^A is satisfied by construction, since the optimal pair $(\overset{*}{x}{}^j, \overset{*}{u}{}^A)$ is supposed to be embedded in the field $F_{(r)}(B)$ as determined by (37). With the aid of (27) and (28), the equations (44) and (45) may therefore be written in the form

$$\partial H^*_{(r)}/\partial x^j + \overset{(r)}{P}{}^{\beta}_{\alpha}[\partial^2 V^\alpha/\partial t^\beta \partial x^j + (\partial^2 V^\alpha/\partial x^h \partial x^j) f^h_{\beta}] = 0,$$

and

$$\partial H^*_{(r)}/\partial t^\gamma + \overset{(r)}{P}{}^{\beta}_{\alpha}[\partial^2 V^\alpha/\partial t^\beta \partial t^\gamma + (\partial^2 V^\alpha/\partial x^h \partial t^\gamma) f^h_{\beta}] = 0,$$

from which (40) and (41) follow directly in terms of (43).

It should be observed that relations (39)–(42) can be written in a slightly different form. When λ^α_β, λ^α_j are given by (43), the definition (18) reduces to

$$p^\alpha_\beta = dV^\alpha(t^\epsilon, \overset{*}{x}{}^j)/dt^\beta \tag{46}$$

for the pair $(\overset{*}{x}{}^j, \overset{*}{u}{}^A)$, so that, for the latter,

$$dp^\alpha_\beta/dt^\gamma = dp^\alpha_\gamma/dt^\beta.$$

However, it follows directly from the definition (20) that

$$(r-1)!\, d\overset{(r)}{P}{}^\alpha_\beta/dt^\alpha = r\delta^{\alpha\alpha_2\cdots\alpha_r}_{\beta\beta_2\cdots\beta_r}(dp^{\beta_2}_{\alpha_2}/dt^\alpha)p^{\beta_3}_{\alpha_3}\cdots p^{\beta_r}_{\alpha_r},$$

which vanishes identically as a result of the skew-symmetry of the Kronecker delta in the superscripts (α, α_2) and the symmetry of $dp^{\beta_2}_{\alpha_2}/dt^\alpha$ in these indices as implied by (46). Hence,

$$d\overset{(r)}{P}{}^\alpha_\beta/dt^\alpha = 0 \tag{47}$$

for the optimal pair $(\overset{*}{x}{}^j, \overset{*}{u}{}^A)$, and accordingly the relations (39)–(41) are respectively equivalent to

$$d(\overset{(r)}{P}{}^\beta_\alpha \overset{*}{x}{}^j)/dt^\beta = \partial H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, \overset{*}{u}{}^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h)/\partial\lambda^\alpha_j, \tag{48}$$

$$d(\overset{(r)}{P}{}^\beta_\alpha \lambda^\alpha_j)/dt^\beta = -\partial H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, \overset{*}{u}{}^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h)/\partial x^j, \tag{49}$$

$$d(\overset{(r)}{P}{}^\beta_\alpha \lambda^\alpha_\gamma)/dt^\beta = -\partial H^*_{(r)}(t^\epsilon, \overset{*}{x}{}^h, \overset{*}{u}{}^A, \lambda^\epsilon_\beta, \lambda^\epsilon_h)/\partial t^\gamma. \tag{50}$$

In conclusion, it should be emphasized that, for a given optimal control problem, there are m distinct maximum principles of the type enunciated above, namely one for each value of the integer r which appears in the relations (38)–(41). The most appropriate choice of r must obviously be dictated by the specific problem under consideration. When one puts $r = 1$ in (38)–(41), the resulting relations may be expressed in terms of the function K^* as defined by (33), and one thus obtains the maximum principle established by Klötzler (Ref. 2). It is observed by the latter that the formulation of his principle depends crucially on the embedding hypothesis: since it is by no means clear whether all optimal pairs can be thus embedded, one cannot deduce the necessity of the principle along these lines. These observations naturally apply with equal relevance to each of the m principles formulated above.

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