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An Existence Theorem for a Fractional Control Problem[1]

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Abstract. Computational algorithms in mathematical programming have been much in use in the theory of optimal control (see, for example Refs. 1–2). In the present work, we use the algorithm devised by Dinkelback (Ref. 3) for a nonlinear fractional programming problem to prove an existence theorem for a control problem with the cost functional having a fractional form which subsumes the control problem considered by Lee and Marcus (Ref. 4) as a particular case.

1. Introduction

Let a system of differential equations be given, that is,

$$dx^i/dt = f^i(t, x^1,..., x^n, u^1,..., u^m), \qquad i = 1,..., n, \tag{1}$$

where $f^i(t, x^1,..., x^n, u^1,..., u^m) = f^i(t, x, u)$, $i = 1,..., n$, along with their partial derivatives with respect to x^k, $k = 1,..., n$, are real continuous functions on $R^1 \times R^n \times \Omega$, where Ω is a nonempty compact subset of R^m. By the Caratheodary existence theorem (Ref. 5), for each choice of the function $u(t) = (u^1(t),..., u^m(t))$ on $-\infty < t_0 \leqslant t \leqslant t_1 < \infty$ as a measurable vector-valued function taking values in Ω, the differential system

$$\dot{x}^i = f^i(t, x, u(t)), \qquad i = 1,..., n, \tag{2}$$

has a unique absolutely continuous solution $x(t)$ on a subinterval of $t_0 \leqslant t \leqslant t_1$ with the prescribed initial condition $x_0 = x(t_0)$.

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Definition 1.1. A *control* for system (1) with prescribed nonempty compact set $\Omega \subset R^m$ and initial point $x_0 = x(t_0)$ is a measurable vector-valued function $u(t) \subset \Omega$ such that the response $x(t)$ with $x(t_0) = x_0$ is also defined in R^n on $t_0 \leqslant t \leqslant t_1$.

In the optimum control problem considered here, we shall be interested in those controls such that the response $x(t)$ travels from the prescribed initial point $x(t_0) = x_0$ to a given moving target $G(t)$ for each t on a given finite interval $\tau_0 \leqslant t \leqslant \tau_1$, where $G(t) \subset R^n$ is assumed to be a nonempty compact set varying continuously with t. Here, we use the Housdorff metric distance between two nonempty compact subsets of R^n to define continuity (Ref. 4).

Definition 1.2. Given the control problem

(i) $\dot{x}^i = f^i(t, x, u)$, $i = 1,..., n$,

(ii) $\Omega \subset R^m$,

(iii) $x_0 \in R^n$,

(iv) $G(t) \subset R^n$ on $\tau_0 \leqslant t \leqslant \tau_1$,

(v) $C(u)$ the cost functional,

define $\Delta = \Delta\{f^1(t, x, u),..., f^n(t, x, u),\ \Omega, x_0,\ G(t)\}$ as the set of all controls $u(t)$ on various subintervals $t_0 \leqslant t \leqslant t_1$ with $\tau_0 \leqslant t_0 \leqslant t_1 \leqslant \tau_1$ such that $x(t_0) = x_0$ and $x(t_1) \in G(t_1)$. A control $u_*(t)$ in Δ is called optimal in the case where $C(u_*) \leqslant C(u)$ for all $u(t) \in \Delta$.

In Section 2, we state the fractional optimum control problem and establish certain results which will be used to prove the existence theorem, which is given in Section 3.

2. Statement and Analysis of the Problem

The fractional optimum control problem can now be stated below.

Problem (I). Given

(i) a system of differential equations

$$\dot{x}^i = f^i(t, x^1,..., x^n, u^1,..., u^m) = g^i(t, x) + h_j^i(t, x)u^j, \quad i = 1,..., n, \quad j = 1,..., m,$$

with $g^i(t, x)$, $h_j^i(t, x)$, and $\partial g^i(t, x)/\partial x^k$, $\partial h^i_j(t, x)/\partial x^k$, $k = 1,..., n$, continuous on $R^1 \times R^n$,

(ii) a nonempty convex, compact restraint set $\Omega \subset R^m$,

(iii) the initial point $x_0 \in R^n$,

(iv) the continuously moving nonempty compact target set $G(t)$ on the finite interval $\tau_0 \leqslant t \leqslant \tau_1$,

(v) the cost functional

$$C(u) = \int_{t_0}^{t_1} f^{n+1}(t, x(t), u(t))\, dt \Big/ \int_{t_0}^{t_1} f^{n+2}(t, x(t), u(t))\, dt,$$

where

$$f^{n+i}(t, x, u) = g^{n+i}(t, x) + h_j^{n+i}(t, x)u^j,$$

$g^{n+i}(t, x)$ and $h_j^{n+i}(t, x)$, $i = 1, 2$, are continuous on $R^1 \times R^n$, and

$$\int_{t_0}^{t_1} f^{n+2}(t, x(t), u(t))\, dt > 0$$

for all $u(t) \in \Delta$ [the set of controls which is assumed to be such that the responses travel from x_0 to G as above and such that (a) Δ is nonempty and (b) there exists a real bound $B < \infty$ for all responses $x(t)$ corresponding to Δ, that is, $|x(t)| < B$ uniformly for all responses], the problem is to find a control $u_*(t) \in \Delta$ which minimizes $C(u)$ over all $u(t) \in \Delta$.

It can be noted that, if

$$\int_{t_0}^{t_1} f^{n+2}(t, x(t), u(t))\, dt \equiv 1$$

for all $u(t) \in \Delta$, the above problem reduces to one considered by Lee and Marcus (Ref. 4).

We consider another optimum control problem of our interest.

Problem (II)

$$\min_{u(t) \in \Delta} \left[\int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q f^{n+2}(t, x(t), u(t))\}\, dt \right]$$

for $q \in R^1$ under the hypotheses of Problem (I).

Now, Lee and Marcus (Ref. 4) assert that a control $u(t) \in \Delta$ exists which solves Problem (II) for $q \in R^1$. In what follows, we analyze the two problems along the lines of Dinkelback (Ref. 3). Define

$$F(q) = \min_{u(t) \in \Delta} \left[\int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q f^{n+2}(t, x(t), u(t))\}\, dt \right]$$

for $q \in R^1$.

Lemma 2.1. $F(q)$ is concave over R^1.

Proof. Let $(x_\lambda(t), u_\lambda(t))$ solve Problem (II) with

$$q = \lambda q_1 + (1 - \lambda) q_2,$$

where $q_1 \neq q_2$ and $0 \leqslant \lambda \leqslant 1$ and $x_\lambda(t)$ is a solution of the differential system in Problem (I), corresponding to the control $u_\lambda(t)$. Then,

$$\begin{aligned}
&F(\lambda q_1 + (1 - \lambda) q_2) \\
&\quad = \int_{t_0}^{t_1} \{f^{n+1}(t, x_\lambda(t), u_\lambda(t)) - (\lambda q_1 + (1 - \lambda) q_2) f^{n+2}(t, x_\lambda(t), u_\lambda(t))\}\, dt \\
&\quad = \lambda \int_{t_0}^{t_1} \{f^{n+1}(t, x_\lambda(t), u_\lambda(t)) - q_1 f^{n+2}(t, x_\lambda(t), u_\lambda(t))\}\, dt \\
&\qquad + (1 - \lambda) \int_{t_0}^{t_1} \{f^{n+1}(t, x_\lambda(t), u_\lambda(t) - q_2 f^{n+2}(t, x_\lambda(t), u_\lambda(t))\}\, dt \\
&\quad \geqslant \lambda \min_{u(t) \in \Delta} \left[\int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q_1 f^{n+2}(t, x(t), u(t))\}\, dt \right] \\
&\qquad + (1 - \lambda) \min_{u(t) \in \Delta} \left[\int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q_2 f^{n+2}(t, x(t), u(t))\}\, dt \right] \\
&\quad = \lambda F(q_1) + (1 - \lambda) F(q_2).
\end{aligned}$$

Lemma 2.2. $F(q)$ is continuous on R^1.

Proof. See Ref. 3 and Ref. 6, page 326.

Lemma 2.3. $F(q)$ is strictly monotonically decreasing, i.e.,

$$q_1, q_2 \in R^1 \quad \text{and} \quad q_1 < q_2 \Rightarrow F(q_2) < F(q_1).$$

Proof. Let $q_1 < q_2$ be given, and let $(x_1(t), u_1(t))$ solve Problem (II) with $q = q_1$. Then, since

$$\int_{t_0}^{t_1} f^{n+2}(t, x_1(t), u_1(t))\, dt > 0,$$

we have

$$\begin{aligned}
F(q_1) &= \int_{t_0}^{t_1} \{f^{n+1}(t, x_1(t), u_1(t)) - q_1 f^{n+2}(t, x_1(t), u_1(t))\}\, dt \\
&> \int_{t_0}^{t_1} \{f^{n+1}(t, x_1(t), u_1(t)) - q_2 f^{n+2}(t, x_1(t), u_1(t))\}\, dt \\
&\geqslant \min_{u(t) \in \Delta} \left[\int_{t_1}^{t_2} \{f^{n+1}(t, x(t), u(t)) - q_2 f^{n+2}(t, x(t), u(t))\}\, dt \right] = F(q_2).
\end{aligned}$$

Lemma 2.4. $F(q) = 0$ has a unique solution in R^1.

Proof. Lemmas 2.2 and 2.3 and the fact that $\lim_{q\to-\infty} F(q) = +\infty$ and $\lim_{q\to+\infty} F(q) = -\infty$ imply the result.

Lemma 2.5. Let $u(t) \in \Delta$ and

$$q = \int_{t_0}^{t_1} f^{n+1}(t, x(t), u(t))\, dt \Big/ \int_{t_0}^{t_1} f^{n+2}(t, x(t), u(t))\, dt.$$

Then, $F(q) \leqslant 0$.

Proof

$$F(q) = \min_{u(t)\in\Delta} \left[\int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q f^{n+2}(t, x(t), u(t))\}\, dt \right]$$

$$\leqslant \int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q f^{n+2}(t, x(t), u(t))\}\, dt = 0.$$

3. Existence Theorem

We are now in a position to prove the main result of this paper. We shall first prove a theorem ensuring the existence of an optimal control for Problem (I) and then develop an algorithm to reach it. Denote the value of

$$\min_{u(t)\in\Delta} \int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q_* f^{n+2}(t, x(t), u(t))\}\, dt$$

by $F(q_*, x_*(t), u_*(t)) = F(q_*)$, where $(x_*(t), u_*(t))$ minimizes the above.

Theorem 3.1. If $F(q_*) = 0$, then

$$q_* = \min_{u(t)\in\Delta} \left[\int_{t_0}^{t_1} f^{n+1}(t, x(t), u(t))\, dt \Big/ \int_{t_0}^{t_1} f^{n+2}(t, x(t), u(t))\, dt \right]$$

$$= \int_{t_0}^{t_1} f^{n+1}(t, x_*(t), u_*(t))\, dt \Big/ \int_{t_0}^{t_1} f^{n+2}(t, x_*(t), u_*(t))\, dt,$$

and $u_*(t)$ solves Problem (I).

Proof. Surely, there is a unique solution to $F(q) = 0$, say q_* (Lemma 2.4). Then, the existence of an optimal control is guaranteed by (Ref. 4), which solves Problem (II) with $q = q_*$. Then,

$$0 = F(q_*) = \int_{t_0}^{t_1} \{f^{n+1}(t, x_*(t), u_*(t)) - q_* f^{n+2}(t, x_*(t), u_*(t))\}\, dt$$

$$\leqslant \int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q_* f^{n+2}(t, x(t), u(t))\}\, dt$$

for all $u(t) \in \Delta$, so that

$$q_* = \int_{t_0}^{t_1} f^{n+1}(t, x_*(t), u_*(t))\, dt \Big/ \int_{t_0}^{t_1} f^{n+2}(t, x_*(t), u_*(t))\, dt$$

$$\leqslant \int_{t_0}^{t_1} f^{n+1}(t, x(t), u(t)\, dt \Big/ \int_{t_0}^{t_1} f^{n+1}(t)\, x(t), u(t))\, dt$$

for all $u(t) \in \Delta$, and $u_*(t)$ solves Problem (I) with $C(u^*) = q_*$.

Algorithm. Suppose that $u_*(t) \in \Delta$ is an optimal control solving Problem (I). Keeping in view Theorem 3.1, we see that Problem (I) can be formulated as follows. Find a control $u_m(t)$ such that $| q(u_*) - q(u_m)| < \epsilon$ for any given $\epsilon > 0$. Since $F(q)$ is continuous, we have a second formulation: find $u_l(t)$ and

$$q_l = \int_{t_0}^{t_1} f^{n+1}(t, x_l(t), u_l(t))\, dt \Big/ \int_{t_0}^{t_1} f^{n+2}(t, x_l(t), u_l(t))\, dt = c(u_l)$$

such that $|F(q_*) - F(q_l)| = |F(q_l)| < \delta$ for any given $\delta > 0$. The algorithm can be started by choosing any $u_1(t) \in \Delta$ with $C(u_1) = q_1$, so that $F(q_1) \leqslant 0$, which is possible in the light of Lemma 2.5. Then, let $k = 2$, and proceed as follows.

(A) By means of any known method (say Ref 4), find a control $u_k(t)$ that solves Problem (II) with $q = q_k$ such that

$$F(q_k) = \min_{u(t) \in \Delta} \left[\int_{t_0}^{t_1} \{f^{n+1}(t, x(t), u(t)) - q_k f^{n+2}(t, x(t), u(t))\}\, dt \right].$$

(A_1) If $|F(q_k)| < \delta$, stop. If $F(q_k) < 0$, then $u_k(t) = u_l(t)$. If $F(q_k) = 0$, then $u_k(t) = u_*(t)$.

(A_2) If $|F(q_k)| \geqslant \delta$, evaluate $q_{k+1} = C(u_k)$ and go to (A) replacing q_k by q_{k+1}.

Proof of Convergence. (a) First, we show that $q_{k+1} < q_k$ for all $k = 1, 2, \ldots$ with $|F(q_k)| \geqslant \delta$. Lemma 2.5 implies that $F(q_k) < 0$. By definition,

$$\int_{t_0}^{t_1} f^{n+1}(t, x_k(t), u_k(t))\, dt = q_{k+1} \int_{t_0}^{t_1} f^{n+2}(t, x_k(t), u_k(t))\, dt.$$

Hence,

$$\begin{aligned} F(q_k) &= \int_{t_0}^{t_1} \{f^{n+1}(t, x_k(t), u_k(t)) - q_k f^{n+2}(t, x_k(t), u_k(t))\}\, dt \\ &= (q_{k+1} - q_k) \int_{t_0}^{t_1} f^{n+2}(t, x_k(t), u_k(t))\, dt < 0. \end{aligned}$$

Therefore, $q_{k+1} < q_k$, since

$$\int_{t_0}^{t_1} f^{n+2}(t, x_k(t), u_k(t))\, dt > 0.$$

(b) Our assertion is that $\lim_{k\to\infty} q_k = q(u_*) = q_*$. If this is not true, we must have $\lim_{k\to\infty} q_k = q > q_*$. In that case, we have a sequence $u_k(t)$, with q_k such that $\lim_{k\to\infty} F(q_k) = F(q) = 0$ [see step (A_1) of the algorithm]. Since $F(q)$ is strictly monotonic decreasing (Lemma 2.3), we obtain

$$0 = F(q) < F(q_*) = 0,$$

which is a contradiction. Hence, it follows that $\lim_{k\to\infty} f(q_k) = F(q_*)$; and then, by Lemma 2.2, we have $\lim_{k\to\infty} q_k = q_*$.

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