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Generalized Complementarity Problem[1]

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Abstract. A general complementarity problem with respect to a convex cone and its polar in a locally convex, vector-topological space is defined. It is observed that, in this general setting, the problem is equivalent to a variational inequality over a convex cone. An existence theorem is established for this general case, from which several of the known results for the finite-dimensional cases follow under weaker assumptions than have been required previously. In particular, it is shown that, if the given map under consideration is strongly copositive with respect to the underlying convex cone, then the complementarity problem has a solution.

1. Introduction

Several problems arising in different fields, such as mathematical programming, game theory, mechanics, and geometry, have the same mathematical form which may be stated as follows: For a given map F from the n-dimensional Euclidean space E^n into itself, find $x \in E^n$ satisfying

$$x \geqslant 0, \qquad F(x) \geqslant 0, \qquad x^T F(x) = 0. \tag{1}$$

Or, equivalently, find a vector x in the nonnegative orthant E_+^n whose image $F(x)$ also lies in E_+^n and such that it is orthogonal to x.

When F is of the form $F(x) = Mx + b$, where M is an $n \times n$ matrix and b is an n vector, the above problem is referred to as the *linear comple-*

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mentarity problem (LCP); otherwise, it is called the *nonlinear complementarity problem* (NCP).

In the past few years, several investigators have been concerned with both the computational and the theoretical (existence and uniqueness) aspects of the above problem. Several important results have been established, many of which appear in Refs. 1–8.

Recently, Habetler and Price (Ref. 9) considered an interesting generalization of the NCP, where the usual nonnegative partial ordering of E^n is replaced by partial orderings generated by a given cone and its polar.

In this paper, which was motivated by the work of Habetler and Price, we consider a further generalization of the NCP. The setting is a locally convex, Hausdorff, vector-topological space over the reals. A generalized complementarity problem relative to the preorderings determined by a convex cone and its polar is defined, and an existence theorem is established, from which the theorems of Habetler and Price follow under weaker assumptions.

2. Generalized Complementarity Problem

Let X be a locally convex, Hausdorff, vector-topological space over the reals, and let Y be any other vector space over the reals (for example, the dual space of X). Suppose that a bilinear form $\langle \cdot, \cdot \rangle$, which maps $X \times Y$ into the reals, is defined. Let K be a closed convex cone[3] in X, and define its polar in Y to be the set

$$K^* = \{y \mid y \in Y, \langle x, y \rangle \geqslant 0 \quad \text{for all } x \in K\}.$$

Obviously, K^* is nonempty, since $0 \in K^*$; and it is easy to see that it is a convex cone in Y. Let $\geqslant^K$ and $\geqslant^{K^*}$ denote the preorderings[4] induced by K in X and by K^* in Y, respectively. That is, for x^1 and x^2 in X, we write $x^1 \geqslant^K x^2$ if, and only if, $x^1 - x^2 \in K$. In particular, $x \geqslant^K 0$ if, and only if, $x \in K$. The preordering $\geqslant^{K^*}$ is defined in a similar fashion. Finally, let F be a map from K into Y. Then, the *generalized complementarity problem* (GCP) is to find $x \in X$ satisfying

$$x \geqslant^K 0, \qquad F(x) \geqslant^{K^*} 0, \qquad \langle x, F(x) \rangle = 0. \tag{2}$$

[3] A set K in a vector space is a cone if, and only if, whenever $x \in K$, then $\lambda x \in K$ for all scalars $\lambda \geqslant 0$.

[4] A preordering is a binary relation which is reflexive and transitive.

Of course, the GCP may be defined independently of the preorderings $\geqslant^K$ and $\geqslant^{K^*}$. Indeed, (2) is equivalent to

$$x \in K, \qquad F(x) \in K^*, \qquad \langle x, F(x) \rangle = 0. \tag{3}$$

If one takes $X = Y = E^n$ and $\langle \cdot, \cdot \rangle$ to be the usual inner product in E^n, then (2) reduces to the complementarity problem considered by Hubetler and Price (Ref. 9). In addition, if one takes $K = E_+^n$, then (2) reduces to (1).

3. An Existence Theorem for the GCP

The following lemma indicates the equivalence between the GCP and a *variational type* problem. Let $C(F, K)$ denote the solution set of (2), that is,

$$C(F, K) = \{x \mid x \in K, F(x) \in K^*, \langle x, F(x) \rangle = 0\}, \tag{4}$$

and let

$$V(F, K) = \{x \mid x \in K, \langle u - x, F(x) \rangle \geqslant 0 \quad \text{for all } u \in K\}. \tag{5}$$

Lemma 3.1. $C(F, K) = V(F, K)$.

Proof. It is obvious that $C(F, K) \subset V(F, K)$. To show that $V(F, K) \subset C(F, K)$ also, let $x \in V(F, K)$. Taking $u = 0$ in (5), we have $\langle x, F(x) \rangle \leqslant 0$. Also, by taking $u = \lambda x$ with $\lambda > 1$, we see that $\langle x, F(x) \rangle \geqslant 0$. Therefore, $\langle x, F(x) \rangle = 0$. It remains to be shown that $F(x) \in K^*$. Assume the contrary, $F(x) \notin K^*$. Then, there exists $u^0 \in K$ such that $\langle u^0, F(x) \rangle < 0$. But, from (5), we have

$$0 > \langle u^0, F(x) \rangle \geqslant \langle x, F(x) \rangle = 0,$$

a contradiction. This completes the proof.

The next lemma is a generalization of similar results given in Ref. 4 and Refs. 10–11. Let D be a nonempty compact and convex set in X, and let ψ be a function which maps DXY into the reals.

Lemma 3.2. If, for every fixed $y \in Y$, $\psi(\cdot, y)$ is quasiconvex on D, and the function $(u, v) \to \psi(u, F(v))$ is continuous on $D \times D$, then there exists $\bar{x} \in D$ such that

$$\psi(\bar{x}, F(\bar{x})) \leqslant \psi(x, F(\bar{x})) \qquad \text{for all} \quad x \in D. \tag{6}$$

Proof. For every $x \in D$, let

$$\Lambda(x) = \{u \mid u \in D, \psi(u, F(x)) = \min_{v \in D} \psi(v, F(x))\}. \tag{7}$$

This defines a point-to-set map from D into its subsets. It is a routine matter to check that the set D and the map Λ satisfy the conditions of Fan's generalization of Kakutani's fixed-point theorem (Ref. 12). Indeed, for every $x \in D$, the set $\Lambda(x)$ is a nonempty convex subset of D. This follows from the compactness of D, the continuity and quasiconvexity in v of $\psi(v, F(x))$. Also, the map Λ is upper semicontinuous on D or, equivalently, its graphical representation is closed in $D \times D$, which follows from the continuity of $\psi(u, F(v))$ on $D \times D$. Therefore, there exists $\bar{x} \in D$ such that $\bar{x} \in \Lambda(\bar{x})$; and, from (7), we see that $\bar{x}$ satisfies (6). This completes the proof.

Several interesting results follow from Lemma 3.2, when the function ψ takes special forms. In particular, we have the following corollary.

Corollary 3.1. Let D be a nonempty compact and convex set in X, and let F map D into Y in such a way that the function $(u, v) \to \langle u, F(v)\rangle$ is continuous on $D \times D$. Then, there exists $\bar{x} \in D$ such that

$$\langle x - \bar{x}, F(\bar{x})\rangle \geqslant 0 \qquad \text{for all} \quad x \in D.$$

If we take X to be a normed space, say a Banach space, and let F be a map from D into X, then Fan's Theorem 2 of Ref. 13 follows from Lemma 3.2.

Corollary 3.2 (Fan). If $F : D \to X$ is continuous on the nonempty compact and convex set $D \subset X$, then there exists $\bar{x} \in D$ such that

$$\| \bar{x} - F(\bar{x})\| \leqslant \| x - F(\bar{x})\| \qquad \text{for all} \quad x \in D.$$

Now we give our main existence theorem. Let X, Y, $\langle\ ,\ \rangle$, K, F be as in Section 2.

Theorem 3.1. The GCP, as given by (2), has a solution if (i) the function $(u, v) \to \langle u, F(v)\rangle$ is continuous on $K \times K$, and (ii) there exists a nonempty compact and convex subset D in K with the property that, for every $x \in K - D$, there exists $z \in D$ such that

$$\langle x - z, F(x)\rangle > 0. \tag{8}$$

Proof. For every $u \in K$, let

$$D_u = \{x \mid x \in D, \langle u - x, F(x)\rangle \geqslant 0\}. \tag{9}$$

From the continuity assumption, it follows that D_u is a closed subset of D for every $u \in K$. We also assert that the intersection of any finite number of the D_u's is nonempty, i.e., for arbitrary $u^1,..., u^m \in K$, we have $\bigcap_{i=1}^{m} D_{u^i} \neq \varnothing$. To see this, let $\hat{D}$ be the convex closure of $D \cup \{u^1,..., u^m\}$. Since $\hat{D}$ is a nonempty compact and convex subset in K, it follows from Corollary 3.1 that there exists $\hat{x} \in \hat{D}$ such that

$$\langle x - \hat{x}, F(\hat{x})\rangle \geqslant 0 \qquad \text{for all} \quad x \in \hat{D}. \tag{10}$$

In particular,

$$\langle u^i - \hat{x}, F(\hat{x})\rangle \geqslant 0, \qquad i = 1,..., m. \tag{11}$$

It remains to be shown that $\hat{x} \in D$. But, if $\hat{x} \notin D$, then it follows from (8) that there exists $z \in D$ such that $\langle z - \hat{x}, F(\hat{x})\rangle < 0$, which contradicts (10). Now, from the finite intersection property of compact sets, we have that $\bigcap_{u \in K} D_u \neq \varnothing$, which implies that there exists $\bar{x} \in D$ such that $\langle x - \bar{x}, F(\bar{x})\rangle \geqslant 0$ for all $x \in K$. From Lemma 3.1, it follows that $\bar{x}$ solves the GCP.

4. Finite-Dimensional Case

Although the results of this section are valid in any finite-dimensional Hilbert space, we shall restrict ourselves to the n-dimensional Euclidean space E^n. Let K be a nonempty, closed, convex cone in E^n, and let K^* be its polar cone; that is, $K^* = \{y \mid y \in E^n, x^Ty \geqslant 0 \text{ for all } x \in K\}$, which is also a closed convex cone. Let $\geqslant^K$ and $\geqslant^{K^*}$ be the preorderings induced by K and K^* in E^n, respectively. Finally, let F be a map from K into E^n. Then, the GCP in E^n is to find $x \in E^n$ satisfying

$$x \geqslant^K 0, \qquad F(x) \geqslant^{K^*} 0, \qquad x^TF(x) = 0. \tag{12}$$

This is the problem considered by Habetler and Price (Ref. 9). They established the existence of a solution to (12) for the following classes of maps.

Definition 4.1. A map F from K into E^n is *strongly* K-*monotone* if there exists a scalar $k > 0$ such that, for all x and y in K satisfying $x \geqslant^K y$, we have

$$(x - y)^T(F(x) - F(y)) \geqslant k \mid x - y \mid^2, \tag{13}$$

where $| \cdot |$ denotes the Euclidean norm.

Definition 4.2. A map F from K into E^n is *strongly* K-*copositive* if it is differentiable in K, and there exists a scalar $k > 0$ such that, for all $y \in K$ and $x \in K$, we have

$$x^T J(y) x \geqslant k \mid x \mid^2, \tag{14}$$

where $J(y)$ is the Jacobian matrix of F at y.

Theorem 4.1 (Habetler–Price). The GCP, as given by (12), has a solution if (i) K is solid[5] and $-F(0) \in \mathrm{Int}(K^*)$, and (ii) F is either continuous and strongly K-monotone or strongly K-copositive on K.

Theorem 4.2 (Habetler–Price). The GCP, as given by (12), has a solution if (i) K is solid and subpolar,[6] and (ii) F is either continuous and strongly K-monotone or strongly K-copositive on K.

The following class of maps, not necessarily differentiable, which is considered in Ref. 5 and defined on E_+^n, is easily seen to include both above classes.

Definition 4.3. A map F from K into E^n is *strongly* K-*copositive* if there exists a scalar $k > 0$ such that, for all $x \in K$ (or equivalently $x \geqslant^K 0$), we have

$$x(F(x) - F(0)) \geqslant k \mid x \mid^2. \tag{15}$$

Remark. It is obvious that every strongly K-monotone map is strongly K-copositive. It is also easy to show that every differentiable map satisfying (14) also satisfies (13). To see this, let x be fixed in K, and define the function $\phi : [0, 1] \to$ reals by $\phi(t) = x^T F(tx)$. Differentiating, we get $\phi'(t) = x^T J(tx)x$. From (14), we have $\phi'(t) \geqslant k \mid x \mid^2$. Therefore,

$$x^T(F(x) - F(0)) = \phi(1) - \phi(0) = \int_0^1 \phi'(t)\, dt \geqslant \int_0^1 k \mid x \mid^2 dt = k \mid x \mid^2.$$

The following theorem generalizes the results of Habetler and Price.

Theorem 4.3. The GCP, as given by (12), has a solution if F is continuous and strongly K-copositive on K.

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[5] A cone K is solid if it has a nonempty interior.

[6] A cone K is subpolar if $K \subset K^*$.

Proof. If $F(0) = 0$, then $x = 0$ is a solution to (12). Therefore, assume $F(0) \neq 0$. Since F is strongly K-copositive, there exists a scalar $k > 0$ such that

$$x^T F(x) \geqslant x^T F(0) + k \mid x \mid^2 \qquad \text{for all} \quad x \in K. \tag{16}$$

Let $\rho = \mid F(0) \mid / k$ and $D = \{x \mid x \in K, \mid x \mid \leqslant \rho\}$. It is obvious that D is a nonempty compact and convex subset of K; and, for every $x \in K - D$, we have

$$k \mid x \mid^2 > \mid x \mid \mid F(0) \mid. \tag{17}$$

Now, from (16)–(17) and Schwartz's inequality, we have

$$x^T F(x) > x^T F(0) + \mid x \mid \mid F(0) \mid \geqslant 0 \qquad \text{for all} \quad x \in K - D, \tag{18}$$

and thus condition (8) of Theorem 3.1 is satisfied uniformly with $z = 0$. Therefore, the GCP has a solution.

We conclude by observing that no special assumptions are required for the cone K or the vector $F(0)$ in the statement of Theorem 4.3.

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