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Optimal Layout of Cantilever Trusses

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Abstract. For given allowable stress, Michell (Ref. 1) has investigated the optimal design of a cantilever truss that is to transmit a given load to two given fixed points of support. Disregarding the weight of the connections between the bars, he found that the truss of minimum weight is a truss-like continuum with an infinity of joints, and with bars that are mostly of infinitesimal length. In the present paper, a finite number of joints is enforced by including in the structural weight, which is to be minimized, not only the weight of the bars but also the weight of their connections, which is assumed to be proportional to the number of joints. The concept of two adjoint trusses is introduced, each of which coincides with the Maxwell diagram of the other truss. Two adjoint trusses have the same weight, and an optimal truss is therefore self-adjoint. The optimal configurations of 6-joint and 11-joint cantilever trusses are discussed, and the range of the weight of the typical joint is determined for which the 6-joint truss is optimal.

Key Words. Cantilever truss, self-adjoint truss, optimal configuration, optimal number of joints.

1. Introduction

The problem treated in the following is indicated in Fig. 1: the given horizontal load $2P$ with the given point of application A is to be transmitted to the given fixed points of support B and B′ by a cantilever truss that is symmetric with respect to the vertical through A. The bars of the truss are to be designed for the allowable stress σ_0, and the danger of buckling is to be disregarded. The weight of the truss is supposed to be proportional to

$$W = \sum_i A_i l_i + cn/\sigma_0, \tag{1}$$

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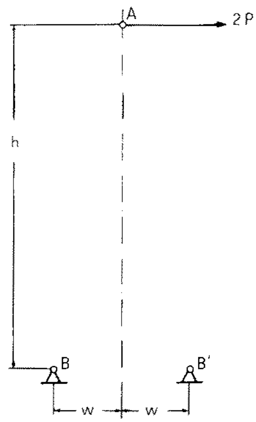

Fig. 1. The data.

where l_i and A_i denote the length and the cross-sectional area of the bar i, c is a known constant, and n is the number of joints. The positions of all joints other than A, B, and B′ are to be chosen to minimize the *objective function*

$$\Omega = \sigma_0 W = \sum_i |s_i| l_i + cn, \tag{2}$$

where S_i denotes the force in bar i.

The term cn in (2) may be viewed as accounting, in a rough manner, for the contribution of gusset plates and rivets to the weight of the truss. For $c = 0$, the optimal structure is a truss-like continuum of the type discussed by Michell (Ref. 1). Structures of this type may be viewed as trusses with an infinity of joints that are, in the main, connected by bars of infinitesimal lengths. They are useful because they furnish a lower bound for the amount of material needed for the bars of a truss, but they are not practical structures. The purpose of the term cn in (2) is to enforce a finite number of joints. Other means of achieveing this have been discussed by Prager (Ref. 2) and Parkes (Ref. 3).

2. Adjoint Cantilever Trusses

Figure 2a shows a cantilever truss with 11 joints. The general arrangement of joints and bars of this truss has been suggested by Michell's truss-like continuum for the considered purpose. Capital letters have been used to label the joints, and lower case letters to label the regions into which the plane of the truss is divided by the bars, the loads, and the reactions.

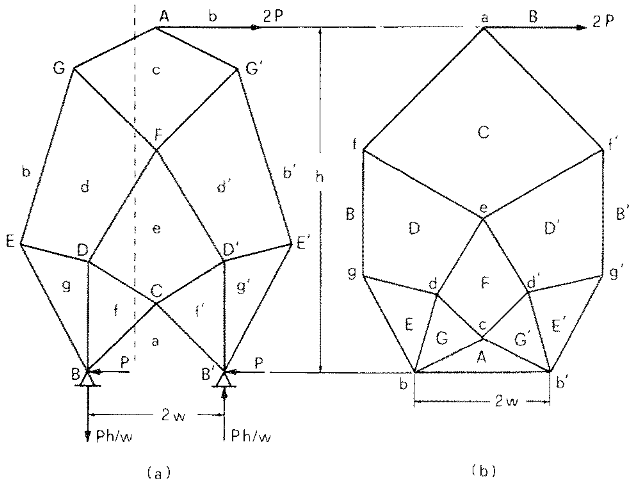

Fig. 2. Adjoint cantilever trusses.

Figure 2b shows the Maxwell diagram of the bar forces, the scale of force having been chosen to represent the load $2P$ by the segment bb′ of the length $2w$. A bar that separates two regions in the *physical plane* of Fig. 2a transmits an axial force whose intensity is represented in the *force plane* of Fig. 2b by the length of the segment joining the points with the labels of these regions. For example, the force in bar CD, which separates regions e and f in Fig. 2a, has the intensity represented by the length of the segment ef in Fig. 2b.

We shall now show that the point a in Fig. 2b has the distance h from the segment bb′. Indeed, this distance is the sum of the projections of the segments bc, cd, de, ef, and fa on the vertical through a, and represents therefore the sum of the vertical components of the forces in the bars AG, GF, FD, DC, and CB. The vertical equilibrium of the part of the truss that is to the left of the dashed vertical line in Fig. 2a shows that the sum of these vertical components equals Ph/w, and is therefore represented in Fig. 2b by a segment of the length h.

Figure 2b may be regarded as the drawing of a cantilever truss of the height h and the base width $2w$ that transmits the load $2P$ to the given fixed joints b and b′. The regions into which the bars of this truss, the load, and the reactions divide the plane of the truss have been labelled by capital letters in such a manner that the intensity of the force in, say, the bar ef, which separates regions C and D, is represented by the segment CD in Fig. 2a.

Each of Figs. 2a and 2b thus shows a cantilever truss of the required height and base width, while the other figure shows the corresponding bar forces. Accordingly, the sum $\sum_i |S_i| l_i$ has the same value for these *adjoint* *cantilever trusses*, that is, for each truss the total weight of the bars has the same value. A *self-adjoint cantilever truss*, that is, a truss that coincides with its Maxwell diagram, is therefore likely to require a smaller or greater amount of material for its bars than neighboring trusses.

For a truss with the general layout of Fig. 2a to be self-adjoint, each of the following groups of bars must consist of parallel bars:

Group 1: bars BC, DF, and GA;
Group 2: bars CD and FG;
Group 3: bars BD and EG.

3. Optimal Self-Adjoint Cantilever Truss with Six Joints

Figure 3 shows a cantilever truss with six joints. Because the bars BC and DA are parallel, the truss is self-adjoint. With the notations in Fig. 3, the lengths l_i of the bars and the intensities $|S_i|$ of their axial forces are as shown in Table 1.
Accordingly,

$$\sum_i |S_i| l_i = 2(P/w)\{2(\mathrm{BC})(\mathrm{DA}) + (\mathrm{BD})^2 + (\mathrm{CD})^2\}. \tag{3}$$

With the values in Table 1, this yields the following expression for the objective function (2):

$$\Omega = 2(P/w)\{(w/\sin\alpha)^2 + 2wb \cot\alpha + 2b^2 \sin^2\alpha/\sin^2\beta\} + 6c, \tag{4}$$

where

$$b = h - w \cot\alpha. \tag{5}$$

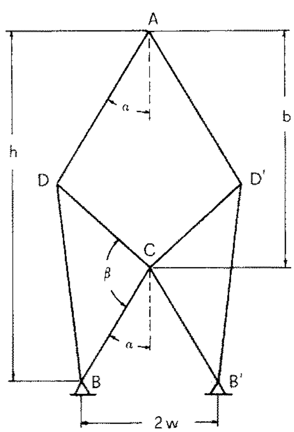

Fig. 3. Self-adjoint cantilever truss with six joints.

Table 1

l_i	$\lvert S_i \rvert$
$BC = w/\sin \alpha$	$(P/w)(DA)$
$CD = b \sin \alpha/\sin \beta$	$(P/w)(CD)$
$DA = b \sin(\alpha+\beta)/\sin \beta$	$(P/w)(BC)$
$BD = \{(BC)^2 + (CD)^2 - 2(BC)(CD) \cos \beta\}^{1/2}$	$(P/w)(BD)$

The condition

$$\partial\Omega/\partial\beta = 0$$

furnishes

$$\beta = \pi/2;$$

and, with the value of β, the condition

$$\partial\Omega/\partial\alpha = 0$$

furnishes, in view of

$$\partial b/\partial\alpha = w/\sin^2 \alpha,$$

$$b = w \cot 2\alpha/\sin^2 \alpha. \tag{6}$$

Thus,

$$\cot \gamma = (CD)/(BC) = (b/w) \sin^2 \alpha = \cot 2\alpha. \tag{7}$$

The optimal layout of the truss in Fig. 3 thus corresponds to

$$\beta = \pi/2, \qquad \gamma = 2\alpha. \tag{8}$$

Because the triangle BCD has a right angle at C, the angle 2α at D is acute, and $d^2\Omega/d\alpha^2$ is found to be positive. The layout characterized by (8) thus requires a minimum of structural material for the bars of the truss.

For $h = 5w$, for instance, one finds

$$\alpha = 27.885°, \qquad \Omega = 41.117Pw + 6c. \tag{9}$$

Figure 4a shows this optimal truss.

4. Optimal Self-Adjoint Cantilever Truss with Eleven Joints

Figure 4b shows an optimal cantilever truss with eleven joints, which is self-adjoint. It can be shown that optimality requires that all angles marked

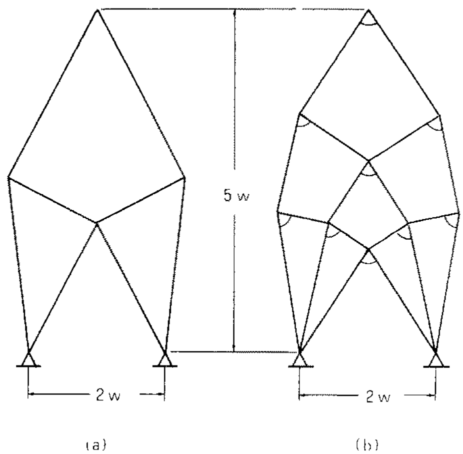

Fig. 4. Optimal cantilever trusses with six and eleven joints.

with a circular arc must have the same value 2α, while the adjacent angles must be right. These conditions furnish the value of α for a given value of h/w. For $h/w = 5$, for instance, one finds

$$\alpha = 34.069°, \qquad \Omega = 36.409Pw + 11c. \tag{10}$$

Figure 4b shows this truss.

5. Optimal Number of Joints

A cantilever truss of the considered kind has at least three joints, namely the joints A, B, and B′ in Fig. 1. For $h/w = 5$, this truss, which is also self-adjoint, has

$$\alpha = 11.310°, \qquad \Omega = 51.999Pw + 3c. \tag{11}$$

Comparison of (9), (10), and (11) shows that the six-joint truss of Fig. 4a is optimal for

$$0.942 < c/(Pw) < 3.627. \tag{12}$$

As c drops below the lower bound in (12), the eleven-joint truss of Fig. 4b becomes preferable; and, as c grows beyond the upper bound in (12), the three-joint truss is optimal.

References

1. MICHELL, A. G. M., *The Limits of Economy in Frame-Structures*, Philosophical Magazine, Series 6, Vol. 8, pp. 589–597, 1904.
2. PRAGER, W., *A Note on Discretized Michell Structures*, Computer Methods in Applied Mechanics and Engineering, Vol. 3, pp. 349–355, 1974.
3. PARKES, E. W., *Joints in Optimum Frameworks*, International Journal of Solids and Structures, Vol. 11, pp. 1017–1022, 1975.