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Nonzero-Sum Differential Games[1]

A. W. STARR[2] AND Y. C. HO[3]

Abstract. The theory of differential games is extended to the situation where there are N players and where the game is nonzero-sum, i.e., the players wish to minimize different performance criteria. Dropping the usual zero-sum condition adds several interesting new features. It is no longer obvious what should be demanded of a *solution*, and three types of solutions are discussed: *Nash equilibrium*, *minimax*, and *noninferior set of strategies*. For one special case, the linear-quadratic game, all three of these solutions can be obtained by solving sets of ordinary matrix differential equations. To illustrate the differences between zero-sum and nonzero-sum games, the results are applied to a nonzero-sum version of a simple pursuit-evasion problem first considered by Ho, Bryson, and Baron (Ref. 1). *Negotiated* solutions are found to exist which give better results for *both* players than the usual *saddle-point* solution. To illustrate that the theory may find interesting applications in economic analysis, a problem is outlined involving the dividend policies of firms operating in an imperfectly competitive market.

1. Introduction

Since the study of differential games was initiated by Isaacs (Ref. 2) in 1954, many papers on the subject have appeared, mostly dealing with problems of the pursuit-evasion type. The differential games considered in those papers have almost always had the *zero-sum* property, i.e., there is a single performance criterion which one player tries to minimize and the other tries to maximize.

This paper considers a more general class of differential games, where there may be more than two players and where each player tries to minimize

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[2] Graduate Student, Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts.

[3] Gordon McKay Professor of Engineering and Applied Mathematics, Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts.

his individual performance criterion. Each player controls a different set of inputs to a single system, described by a differential equation of arbitrary order. The sum of all the players' criteria is not zero nor is it constant. Dropping the zero-sum hypothesis adds both conceptual and analytic complexity, but in the authors' opinion it extends the utility of the theory of differential games to economic and military applications.

Very little work has been published on this subject, although Case (Ref. 3) extended some of Isaacs' results to the nonzero-sum, N-player case for one special kind of solution. But Case did not explore the implications of dropping the zero-sum hypothesis, nor were any practical applications discussed.

Before introducing our general differential game, we illustrate some of the important conceptual differences between zero-sum games and nonzero-sum games, using simple bimatrix games of the type presented by Luce and Raiffa (Ref. 4).

Game 1: Zero-sum game

		Player 2	
		x	y
Player 1	a	$1, -1$	$0, 0$
	b	$2, -2$	$-2, 2$

Game 2: Zero-sum game

		Player 2	
		x	y
Player 1	a	$-1, 1$	$0, 0$
	b	$2, -2$	$-2, 2$

In Game 1, Player 1 chooses between strategies a and b, while Player 2 simultaneously must choose x or y. The corresponding entries give the costs J_1, J_2 for the two players. For each strategy pair, $J_1 + J_2 = 0$, so the game is zero-sum. (In all games, each player wishes to minimize his own cost and is indifferent to the cost paid by the other player.) Player 2, if he is rational, always plays x, and Player 1, realizing this, plays a. This *saddle-point* solution is apparently the only reasonable one.

Definition. If $J_1(s_1, ..., s_N), ..., J_N(s_1, ..., s_N)$ are cost functions for players $1, ..., N$, then the strategy set $\{s_1^*, ..., s_N^*\}$ is a *Nash equilibrium* strategy set if, for $i = 1, ..., N$,

$$J_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_N^*) \geqslant J_i(s_1^*, ..., s_N^*) \tag{1}$$

where s_i is any admissible strategy for Player i.

In other words, the Nash equilibrium strategy is the optimal strategy for each of the players on the assumption that all of the other players are holding fast to their Nash strategies. In the two-player, zero-sum case, the Nash solution is the familiar saddle-point solution.

In Game 2, also zero-sum, no saddle-point solution exists. But Player 1 can minimize his maximum possible loss by choosing a on the assumption that Player 2 ignores his own cost criterion and attempts to do maximum damage to the criterion of Player 1. By the same reasoning, Player 2 chooses x. Thus (a, x) is a *minimax* solution, but it is not a Nash equilibrium.

It can easily be shown (Ref. 4) that, in a zero-sum game, (i) all Nash equilibria are *equivalent*, i.e., have the same costs, and (ii) if (s_1, s_2) and (s_1^*, s_2^*) are equilibrium pairs, then so are (s_1, s_2^*) and (s_1^*, s_2). Property (ii) is called *interchangeability*.

It is also clear that there can be no mutual interest in a zero-sum game; what is good for one player is harmful to the other. Nor can one player ever gain by disclosing his strategy in advance to his opponent.

Game 3: Dating game

		Player 2	
		x	y
Player 1	a	0, 1	2, 2
	b	2, 2	1, 0

Game 4: Prisoners' dilemma

		Player 2	
		x	y
Player 1	a	2, 2	10, 1
	b	1, 10	5, 5

Game 3, the *dating game*,[4] is nonzero-sum. It has *two* Nash equilibria, (a, x) and (b, y), with different costs. They are not interchangeable, since (a, y) and (b, x) are not equilibria. Note what happens when both players seek to achieve their lowest possible costs. But, if Player 1 announces in advance that he is committed to strategy a, then Player 2 has no choice but to play x. Thus, it is advantageous in some, but not all, nonzero-sum games to disclose one's strategy in advance, i.e., to make the first *move*.

[4] Player 1 (*he*) prefers a football game a to a concert b, while player 2 (*she*) prefers concert y to football x. But each prefers going to the same event ax or by to going to separate events ay or bx.

Game 4 is the classical *prisoners' dilemma*.[5] The only equilibrium solution is (b, y), yet (a, x) gives a better result for both players. The solution (a, x) is vulnerable to *cheating* by one player, while (b, y) is not. This illustrates the *non-optimality* of the Nash equilibrium solution in the nonzero-sum game. There is mutual interest, since both players could gain if cooperation were possible.

These simple examples should convince the reader that there are important phenomena which can arise in nonzero-sum games but not in zero-sum games, and that the Nash equilibrium solution is not the *only* solution of interest.

The most important phenomenon arising from the extension from 2 players to N players is the possibility of *coalitions* among groups of players. Very little can be said unless strict rules governing coalition formation are postulated. One special case, a single coalition of all the players, the so-called *pareto-optimal* solution, is discussed in the following sections.

2. General Differential Games

In the general, nonzero-sum, N-player differential game, the following situation arises: For $i = 1,..., N$, Player i wishes to choose his control u_i to minimize

$$J_i = K_i(x(t_f), t_f) + \int_0^{t_f} L_i(x, u_1, ..., u_N, t)\, dt \tag{2}$$

subject to the constraint

$$\dot{x} = f(x, u_1, ..., u_N, t), \qquad x(t_0) = x_0 \tag{3}$$

where x is a state vector of arbitrary dimension n. There may also be inequality constraints on the state and/or control variables, as well as restrictions on the terminal state. The terminal time t_f may be variable or fixed; here, it is considered fixed for simplicity.

The problem as stated is not well-posed. Recall that the simple bimatrix games in the previous section could not be *solved* until one specified what properties the *solution* should have. Similarly, in the differential game, one

[5] Two prisoners awaiting trial are held in separate cells. Each is offered a 50% reduction in his sentence if he divulges evidence about the other's crime. If one refuses, the other can only be convicted of a lesser offense (2 year sentence). A similar situation arises when two superpowers contemplate building antimissile systems.

must demand that the solution have some attribute such as minimax, Nash equilibrium, stability against coalition formation, *pareto-optimality*, and so on. One must also specify what information is available to each player during the course of the game. Here, it is assumed that each player knows the current value of the state vector as well as all system parameters and cost functions, but he does not know the strategies of the rival players. Each player's strategy can then be expressed as a function of time and the current state vector. This does not exclude mixed strategies.

2.1. Nash Equilibrium Solutions. In a game where cooperation among the players is inadmissible or at least difficult to enforce, one is naturally interested in solutions which have the Nash equilibrium property (1). This solution is secure against any attempt by one player to *unilaterally* alter his strategy, since that player can only lose by deviating from his equilibrium control. There may be many equilibrium solutions; one cannot then speak of an equilibrium control for a single player without also stating the corresponding controls for all the other players.

Case (Ref. 3) derived general necessary conditions for a strategy set $\phi = \{\phi_1, ..., \phi_n\}$ to have the Nash property. He used two methods, the value-function approach, analogous to dynamic programming, and the variational approach. He was mainly concerned with problems of the type considered by Isaacs (Ref. 2), where the controls are discontinuous on certain *singular surfaces* in the state space. Since we are not directly concerned with such phenomena, we can state these results much more simply, though perhaps with some loss of rigor and generality.

Consider first the value function approach. Let

$$\phi(x, t) = \{\phi_1(x, t), ..., \phi_N(x, t)\}$$

be any set[6] of control strategies for the N players, resulting in piecewise continuous $u_i(t)$. Then, the *value function* for the ith player is piecewise continuously differentiable and defined as

$$V_i(x_0, t_0, \phi) = K_i(x(t_f), t_f) + \int_{t_0}^{t_f} L_i(x_\phi, \phi, t)\, dt \tag{4}$$

By applying the definition of the Nash property (1) in an obvious extension

[6] Strictly speaking, the set $\phi(x, t)$ must be defined so that the trajectory $x_\phi(t)$ satisfying (3) can be continued from any initial point (x_0, t_0).

of the usual dynamic programming argument, one can show that the value functions V_i are solutions of the partial differential equation

$$\partial V_i/\partial t = -\min_{u_i} H_i(x, t, \Psi_1, ..., \Psi_{i-1}, u_i, \Psi_{i+1}, ..., \Psi_N, \partial V_i/\partial x) \tag{5}$$

where H_i, the Hamiltonian for the ith player, has the functional form[7]

$$H_i(x, t, \phi, \lambda_i^T) = L_i(x, \phi, t) + \lambda_i^T f(x, \phi, t) \tag{6}$$

and where $V_i(x(t_f), t_f) = K_i(x(t_f), t_f)$. This is the generalized Hamilton–Jacobi equation. The equilibrium strategy $\Psi_i(x, t)$ is the control u_i which achieves the minimum in (5). To integrate (5) backward from the terminal manifold, we must be able at each (x, t) to find the *Nash saddle-point* of the vector Hamiltonian $H = [H_1, ..., H_N]$, i.e., to solve an ordinary, continuous, nonzero-sum, N-player game (not a differential game) at every instant t. This is not always possible, but it is possible in an important class of games. A differential game is said to be *normal* (Ref. 3) if (i) it is possible to find a unique Nash equilibrium point Ψ^* for the vector H for all x, λ, t, and (ii) when the equations

$$\partial V_i/\partial t = -H_i[x, t, \Psi^*(x, t, \partial V_1/\partial x, ..., \partial V_N/\partial x), \partial V_i/\partial x] \tag{7}$$

$$\dot{x} = f(x, u_1, ..., u_N, t) \tag{8}$$

$$u_i = \Psi_i^*(x, t, \partial V_1/\partial x, ..., \partial V_N/\partial x) \tag{9}$$

are integrated backward from all the points on the terminal surface, feasible trajectories are obtained.[8] The next section considers a class of games which are normal.

Necessary conditions for a Nash equilibrium can also be obtained by an extension of the variational methods used in optimal control theory. Case (Ref. 3) did this, but his results are apparently valid only for strategies which are functions of time only. In nearly all problems of interest, we require strategies which are functions of the state vector and time. The conditions given below satisfy this requirement. With the Hamiltonians H_i defined in (6),

[7] Note that λ_i here is merely a dummy variable, to be replaced by $\partial V_i/\partial x$ in (5), although it can be interpreted as a costate variable in the variational approach described below.

[8] Note that condition (i) requires that a unique set of controls giving a saddle-point of $H(x, t, u, \lambda)$ can be found as an explicit function of x, t, λ for any λ. This is a *sufficient* (but not necessary) condition for the existence of a Nash trajectory. It is relatively easy to determine if (i) is satisfied, since no differential equation need be solved.

a Nash equilibrium trajectory must satisfy, for $i = 1,\ldots, N$, the following conditions:

$$\dot{x} = f(x, u_1, \ldots, u_N, t), \qquad x(t_0) = x_0 \tag{10}$$

$$\dot{\lambda}_i^T = -(\partial/\partial x)\, H_i(x, t, u_1, \ldots, u_N, \lambda_i) - \sum_{j=1, j\neq i}^{N} \frac{\partial H_i}{\partial u_j}\frac{\partial \Psi_j}{\partial x}(x, t) \tag{11}$$

$$\lambda_i^T(t_f) = [\partial/\partial x(t_f)]\, K_i(x(t_f), t_f) \tag{12}$$

$$u_i = \Psi_i(x, t) \text{ minimizes } H_i(x, t, \Psi_1, \ldots, \Psi_{i-1}, u_i', \Psi_{i+1}, \ldots, \Psi_N, \lambda_i) \tag{13}$$

Note that, for $N = 1$ (optimal control problem), the second term in (11) is absent. The optimal *closed-loop* control $u(x, t)$ can then be obtained by solving for the *open-loop* optimal control $u(t)$ for every initial point (x, t). This method is not valid in the N-player game, however, due to the summation term in (11). In the optimal control problem, these necessary conditions are a set of ordinary differential equations, but in the N-player, nonzero-sum game, they are a set of partial differential equations, generally very difficult to solve.

2.2. Minimax Controls. When a player believes that the other players play Nash equilibrium controls, he should also play the Nash controls. But, if he cannot be sure of how his rivals select their strategies, he may instead choose to minimize his cost against the worst possible set of strategies which they could choose.

Definition. A strategy $\bar{\phi}_i(x, t)$ is the *minimax* strategy for the ith player if, for all admissible $\{\phi_1(x, t), \ldots, \phi_N(x, t)\}$,

$$\max_{\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_N} J_i(\phi_1, \ldots, \bar{\phi}_i, \ldots, \phi_N) \leqslant \max_{\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_N} J_i(\phi_1, \ldots, \phi_i, \ldots, \phi_N) \tag{14}$$

Note that only the ith player's cost function enters into the computation of his minimax control. This is equivalent to finding the equilibrium solution of a two-player, zero-sum differential game, where the opponent of Player i chooses all the controls except the ith and tries to *maximize* J_i. Player i can also calculate his minimax cost $\bar{J}_i$. If he plays $\bar{\phi}_i$, he pays no more than $\bar{J}_i$. He probably pays much less, since the other $N - 1$ players, each with his own cost to minimize, are unlikely to choose the combination of strategies which maximizes J_i (for example, they may play their own minimax controls). Since it fails to take account of the other players' cost criteria and since it is excessively pessimistic, the minimax solution is somewhat unsatisfactory in the nonzero-sum game. In some reasonable, well-behaved games, $\bar{J}_i = \infty$.

Of course, in the two-player, zero-sum game, the Nash solution, if it exists, is also minimax, but this is not true when $N > 2$, nor in a nonzero-sum game.

2.3. Noninferior Controls. It may be of interest to know what could be gained by all players if a *negotiated* solution could be reached and enforced. Clearly, this solution should be selected from the set of strategy N-tuples defined below.

Definition. The strategy N-tuple $\theta = \{\theta_1, ..., \theta_N\}$ belongs to the *noninferior set* if, for any other strategy N-tuple ϕ,

$$\{J_i(\phi) \leqslant J_i(\theta), i = 1, ..., N\} \text{ only if } \{J_i(\phi) = J_i(\theta), i = 1, ..., N\} \tag{15}$$

Solving for the set of noninferior controls is equivalent to solving an optimal control problem with a vector cost criterion. When appropriate convexity conditions are satisfied (Ref. 5), the problem is equivalent to solving an $N - 1$ parameter family of optimal control problems with scalar cost criteria (Refs. 5–7). Each noninferior control N-tuple then minimizes the scalar criterion

$$J = \mu_1 J_1 + \cdots + \mu_N J_N \tag{16}$$

for some $\mu = \{\mu_1, ..., \mu_N\}$ satisfying

$$\sum_{i=1}^{N} \mu_i^2 = 1, \qquad \mu_i \geqslant 0, \qquad i = 1, ..., N \tag{17}$$

Conversely, for any μ satisfying (17) with strict inequalities,[9] the corresponding control N-tuple which minimizes J in (16) belongs to the noninferior set.

The members of the noninferior set are not ordered by the vector criterion. Thus, the *negotiating problem*, equivalent to selecting a vector μ satisfying (17), cannot be solved unless further rules are specified.

3. Linear-Quadratic Games

This section considers a special class of differential games where the system is linear and the cost functions are quadratic functions of the state

[9] Klinger (Ref. 7) has given an example where a solution obtained with one of the $\mu_i = 0$ is not noninferior. Such pathological cases would not occur in the applications of interest to us. Besides, one would probably reject a *negotiated* solution which totally ignored the interests of one player.

vector and controls. Like its counterpart in optimal control theory, the linear-quadratic game is analytically tractable and of some practical interest (Ref. 1).

For the ith player, $i = 1,..., N$, the problem is to choose a control strategy $u_i = \Psi_i(x, t)$ to minimize

$$J_i = \tfrac{1}{2}(x^T S_{if} x)_{t=t_f} + \tfrac{1}{2} \int_{t_0}^{t_f} (x^T Q_i x + \sum_{j=1}^{N} u_j^T R_{ij} u_j)\, dt \tag{18}$$

subject to

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j u_j \tag{19}$$

In some problems, inhomogeneous *forcing* functions appear in the cost criteria or in the state equation. Though they add no great difficulty, they are omitted here to make the presentation simple.

3.1. Nash Equilibrium Controls. If the results of Section 2.1 are applied, the Hamiltonian for the ith player is

$$H_i = \tfrac{1}{2} x^T Q_i x + \tfrac{1}{2} \sum_{j=1}^{N} u_j^T R_{ij} u_j + \lambda_i^T (Ax + \sum_{j=1}^{N} B_j u_j) \tag{20}$$

Using the value-function approach, one sees that the game is normal and the minimizing control for the ith player is

$$u_i^* = -R_{ii}^{-1} B_i^T \lambda_i = -R_{ii}^{-1} B_i^T (\partial V_i / \partial x)^T \tag{21}$$

All the R_{ii} must be positive definite; otherwise, the problem is meaningless. Substituting (21) into the Hamilton–Jacobi equation (7), one obtains a partial differential equation which can easily be solved by separation of variables. Assuming that $V_i(x, t)$ has the form

$$V_i = \tfrac{1}{2} x^T S_i(t) x \qquad \text{so that} \qquad u_i^* = -R_{ii}^{-1} B_i^T S_i x \tag{22}$$

where S_i is symmetric, and taking the symmetric part of the resulting differential equations in the S_j, one obtains

$$\begin{aligned} \dot{S}_i = &- S_i A - A^T S_i - Q_i - \sum_{j=1}^{N} (S_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T S_j - S_i B_j R_{jj}^{-1} B_j^T S_j \\ &- S_j B_j R_{jj}^{-1} B_j^T S_i), \qquad S_i(t_f) = S_{if} \end{aligned} \tag{23}$$

It is easily verified that, for $N = 1$ (optimal control problem), Eq. (22) reduces to the familiar Riccati equation. Of course, Eq. (23) can also be obtained from the necessary conditions (10)–(13). When the set of N coupled matrix equations (23) is integrated backward from the terminal time, the costs incurred by each player are given by (22). One might wish to compare these costs with those incurred when the players use arbitrary linear feedback controls of the form

$$u_i = -K_i(t)x \tag{24}$$

If one starts at (x, t), these costs are

$$J_i = \tfrac{1}{2}x^T P_i x \tag{25}$$

where $P_i(t)$, $i = 1,\dots, N$, satisfy the N uncoupled linear matrix equations

$$\dot{P}_i = -P_i A - A^T P_i - Q_i - \sum_{j=1}^{N} (K_j{}^T R_{ij} K_j - P_i B_j K_j - K_j{}^T B_j{}^T P_i), \qquad P_i(t_f) = S_{if} \tag{26}$$

In many applications, it seems natural to have $R_{ij} = 0$ for $i \neq j$, so that the ith player's cost function does not contain the other players' controls. But, with this choice of the $R_{ij} (i \neq j)$, there is no choice of the remaining parameters R_{ii}, S_{if}, Q_i, which can make the game zero-sum. (Choosing $R_{ii} = 0$ would permit infinite controls.) In fact, with $N = 2$, the game is zero-sum only if

$$R_{12} = -R_{22}, \qquad R_{21} = -R_{11}, \qquad S_{2f} = -S_{1f}, \qquad Q_2 = -Q_1 \tag{27}$$

Substituting (27) into (23), one obtains two equations which are both satisfied by the choice $S_1 = -S_2 = S$. The result is a single matrix Riccati equation

$$\dot{S} = -SA - A^T S - Q_1 + S(B_1 R_{11}^{-1} B_1{}^T - B_2 R_{22}^{-1} B_2{}^T)S, \qquad S(t_f) = S_{1f} \tag{28}$$

and the equilibrium controls (which are also saddle-point controls, because the game is zero-sum) are

$$u_1^* = -R_{11}^{-1} B_1{}^T S x, \qquad u_2^* = R_{22}^{-1} B_2{}^T S x \tag{29}$$

in agreement with the results of Ho, Bryson, and Baron (Ref. 1).

3.2. Minimax Controls. Finding the minimax control for the ith player is equivalent to solving a two-player, zero-sum game, where the opponent of the ith player chooses all but the ith control and tries to maximize J_i. Applying the results of Section 3.1, we see that the minimax control for the ith player is

$$\bar{u}_i = -R_{ii}^{-1} B_i{}^T \bar{S}_i x \tag{30}$$

where

$$\dot{\bar{S}}_i = -\bar{S}_i A - A^T \bar{S}_i - Q_i + \bar{S}_i \sum_{j=1}^{N} B_j R_{ij}^{-1} B_j{}^T \bar{S}_i\,, \qquad \bar{S}_i(t_f) = S_{if} \tag{31}$$

provided that

$$R_{ii} > 0, \qquad R_{ij} < 0, \qquad j \neq i, \qquad j = 1,..., N \tag{32}$$

If conditions (32) are not satisfied, the ith minimax control may fail to exist, so that the ith minimax cost is infinite. Note that a minimax control might exist for some players and fail to exist for others. A case of interest is $R_{ij} = 0$ for all $i, j \neq i$. In this special case, the minimax control is either identically zero or does not exist.

3.3. Noninferior Controls. It was concluded in Section 2.3 that the set of noninferior controls could be obtained by solving an $(N - 1)$-parameter family of optimal control problems. In the linear-quadratic game, this is very easy. The noninferior controls are

$$\hat{u}_i(\mu) = -\left[\sum_{j=1}^{N} \mu_j R_{ji}\right]^{-1} B_i{}^T \hat{S}(\mu)\, x \tag{33}$$

where

$$\dot{\hat{S}}(\mu) = -\hat{S}A - A^T\hat{S} - \sum_{j=1}^{N} Q_j + \hat{S} \sum_{i=1}^{N} B_i \left[\sum_{j=1}^{N} \mu_j R_{ji}\right]^{-1} B_i{}^T \hat{S}$$

$$\hat{S}(\mu, t_f) = \sum_{i=1}^{N} \mu_i S_{if} \tag{34}$$

where

$$\sum_{i=1}^{N} \mu_i = 1, \qquad \mu_i \geqslant 0, \qquad i = 1,..., N$$

There is no reason to expect the Nash equilibrium control to belong to the noninferior set. Hence, one expects to find the *prisoners' dilemma* situation in most linear-quadratic nonzero-sum games.

In the next section, the above results are applied in an example which, though very simple, displays many of the interesting differences between zero-sum and nonzero-sum differential games.

4. Example: A Simple Pursuit-Evasion Problem

In 1965, Ho, Bryson, and Baron (Ref. 1) presented a simple pursuit-evasion problem which could be formulated and solved as a two-player, zero-sum differential game. In this section, that problem is generalized to a nonzero-sum game, allowing the players to have different cost criteria.

The problem concerns the lateral maneuvers of an interceptor (pursuer) and a target (evader) in space, approaching each other on a nominal collision course. If one ignores external forces, the relative position vector r of the pursuer with respect to the evader obeys the kinematic equations

$$\dot{r} = v, \qquad \dot{v} = a_p - a_e \tag{35}$$

where the accelerations a_p and a_e are controlled by the pursuer and evader, respectively. The cost criteria are

$$\begin{aligned} J_p &= \tfrac{1}{2}\sigma_p^2 r_f^T r_f + \tfrac{1}{2}\int_0^T (a_p^T a_p/c_p + a_e^T a_e/c_{pe})\, dt \\ J_e &= -\tfrac{1}{2}\sigma_e^2 r_f^T r_f + \tfrac{1}{2}\int_0^T (a_e^T a_e/c_e + a_p^T a_p/c_{ep})\, dt \end{aligned} \tag{36}$$

where $r_f = r(T)$ and the final time T is fixed. [Following (27), we can obtain[10] the zero-sum game of Ho, Bryson, and Baron by taking $\sigma_e = \sigma_p$, $c_{pe} = -c_e$, and $c_{ep} = -c_p$.]

The Nash equilibrium controls for this game are found using (23), where, for convenience, $S_1 = P$, $S_2 = E$. Specifically,

$$\begin{aligned} a_p &= -c_p[0 \quad I]\, P \begin{bmatrix} r \\ v \end{bmatrix}, \qquad a_e = c_e[0 \quad I]\, E \begin{bmatrix} r \\ v \end{bmatrix} \\ \dot{P} &= -P\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} P + c_p P \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} P + c_e P \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} E + c_e E \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} P \\ &\quad - (c_e^2/c_{pe})\, E \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} E \\ \dot{E} &= -E\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} E + c_e E \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} E + c_p E \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} P + c_p P \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} E \\ &\quad - (c_p^2/c_{ep})\, P \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} P \\ P(T) &= \sigma_p^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad E(T) = -\sigma_e^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{37}$$

[10] In the original problem, the quadratic terms in the controls arose when energy constraints were adjoined to the cost criterion which was just a quadratic function of terminal miss distance. The authors do not mean to imply that the problem as originally stated should have been treated as a nonzero-sum game.

Defining $\tau = T - t$ as the *time-to-go*, we can easily verify that the solutions to these equations are

$$P(t) = (p(\tau)/c_p) \begin{bmatrix} I & \tau I \\ \tau I & \tau^2 I \end{bmatrix}, \qquad E(t) = (e(\tau)/c_e) \begin{bmatrix} I & \tau I \\ \tau I & \tau^2 I \end{bmatrix} \tag{38}$$

where the scalar functions $p(\tau)$ and $e(\tau)$ are solutions of

$$\begin{gathered} dp/d\tau = -\tau^2(p^2 + 2pe - ae^2), \qquad p(0) = c_p \sigma_p{}^2 \\ de/d\tau = -\tau^2(e^2 + 2pe - bp^2), \qquad e(0) = -c_e \sigma_e{}^2 \\ a = c_p/c_{pe}\,, \qquad b = c_e/c_{ep} \end{gathered} \tag{39}$$

By introducing the following nonlinear scaling of the time-to-go τ

$$t' = \tfrac{1}{3}\tau^3 \tag{40}$$

we reduce these equations to

$$dp/dt' = -p^2 - 2pe + ae^2, \qquad de/dt' = -e^2 - 2pe + bp^2 \tag{41}$$

Besides being in a more convenient form for computation, Eqs. (41) have another significance. In the simpler pursuit-evasion game with *velocity control*

$$\dot{r} = v_p - v_e \tag{42}$$

with a_p and a_e in the cost criteria (27) replaced by v_p and v_e, respectively, the Nash equilibrium solution is

$$v_p = -p(t')\, Ir, \qquad v_e = e(t')\, Ir \tag{43}$$

with associated costs[11]

$$J_p = (1/2c_p)\, pr^2, \qquad J_e = (1/2c_e)\, er^2 \tag{44}$$

where $t' = T - t$ is the time-to-go. Thus, one can simultaneously solve the velocity-control problem and the acceleration-control problem, the solution to the latter being obtained from that of the former by using (38) and the inverse of (40).

Since a performance criterion may always be multiplied by a positive constant, there is no loss of generality in taking $\sigma_p{}^2 = \sigma_e{}^2 = 1$. Furthermore,

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[11] The players measure their costs associated with their own controls in different units, each unit chosen to give unit weighting to the terminal miss. One must be careful in interpreting these costs; a direct comparison is not always meaningful.

(41) is left unchanged if p and e are multiplied by α while t' is divided by α ($\alpha > 0$). Thus, we need only consider the case where

$$-p(0)\, e(0) = c_p c_e \sigma_p^2 \sigma_e^2 = 1 \tag{45}$$

since any other case can be obtained by a linear scaling on t'.

The costs incurred can also be computed assuming that the players use arbitrary linear feedback controls

$$\textit{velocity control:} \qquad v_p = -k_p(t')\, r, \qquad v_e = k_e(t')\, r \tag{46}$$

In this problem, we have

$$J_p(r, t) = [p^*(t')/2c_p]\, r^T r, \qquad J_e(r, t) = [e^*(t')/2c_e]\, r^T r \tag{47}$$

where the scalar functions $p^*(t')$, $e^*(t')$ are solutions of the following linear equations obtained from (26):

$$\begin{aligned} dp^*/dt' &= k_p^2 + ak_e^2 - 2p^*(k_e + k_p), \qquad p^*(0) = c_p \sigma_p^2 \\ de^*/dt' &= k_e^2 + bk_p^2 - 2e^*(k_p + k_e), \qquad e^*(0) = -c_e \sigma_e^2 \end{aligned} \tag{48}$$

The corresponding costs for the acceleration-control problem can be obtained by the same transformations as in the case of the Nash equilibrium, using (40), (38), (22), provided that the feedback controls are restricted to have the form

$$a_p = -k_p(\tau)\, \tau[I\ \tau I] \begin{bmatrix} r \\ v \end{bmatrix}, \qquad a_e = k_e(\tau)\, \tau[I\ \tau I] \begin{bmatrix} r \\ v \end{bmatrix} \tag{49}$$

This is equivalent to feeding back the predicted terminal miss, assuming that no further controls are used. If the feedback does not have this form, (26) does not reduce to a scalar equation.

For the velocity-control problem, Eqs. (41) have been solved on an analog computer. The results can also be used to understand the acceleration-control problem, since the transformations (40) and (38) do not affect the qualitative behavior of the solutions. All curves are plotted as a function of time-to-go (the *clock time* runs from right to left) with the normalizations described above [see (45)].

The solutions to (41) for the zero-sum case are shown in Fig. 1. To obtain the zero-sum case, one must take $c_{pe} = -c_e$ and $c_{ep} = -c_p$. In addition, by (45), $c_p c_e = 1$. This leaves only one free parameter, which we

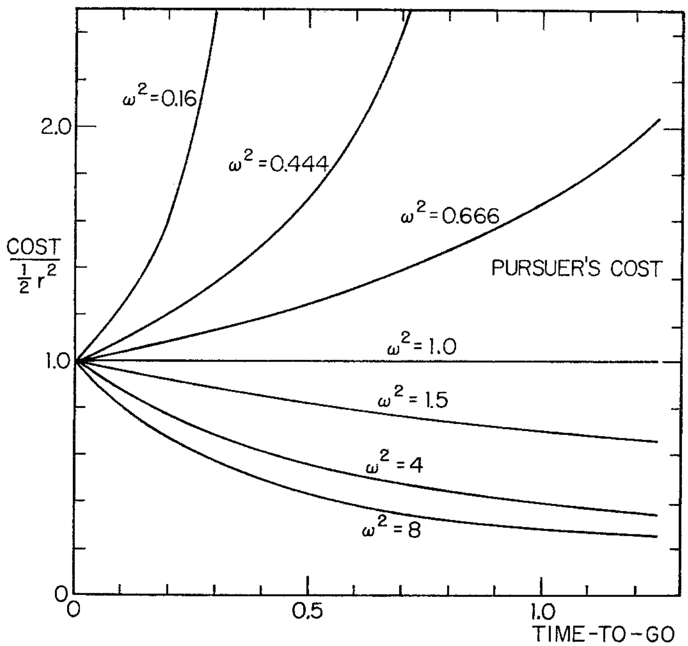

Fig. 1. Equilibrium costs for the zero-sum case.

take to be $\omega^2 = c_p/c_e$. Then, $a = 1/b = -\omega^2$, $c_p = \omega$, and $c_e = 1/\omega$. If ω is large, the pursuer places a smaller weight on his control *energy* (relative to his weighting on terminal miss) than does the evader. Since the costs starting at (r, t) are [from (44)]

$$J_p = [p(t')/2\omega]\, r^2, \qquad J_e = [e(t')/2]\, \omega r^2 \tag{50}$$

the quantities of interest (the costs, apart from the factor $\frac{1}{2}r^2$) are p/ω and $e\omega$, and these are the quantities plotted in all figures.

In the zero-sum case, (32) can be solved analytically. The resulting costs, shown in Fig. 1 for $\sigma_1 = \sigma_2 = 1$, are

$$2J_1/r^2 = -2J_2/r^2 = 1/[1 + (\omega - 1/\omega)\, t'] \tag{51}$$

Note that, when $\omega < 1$, the cost becomes infinite when the time-to-go is $\omega/(1 - \omega^2)$. For larger t', no solution exists. The miss distance is then infinite; the evader escapes. This occurs because the pursuer places too large a price on his control energy. For $\omega > 1$, finite solutions always exist; the pursuer can afford the chase.

In the general, nonzero-sum case, the presence of three free parameters makes it difficult to present the results clearly. However, there is one case of special interest, the case $a = b = 0$, so that *neither player has the other player's control in his cost function.* In some pursuit-evasion applications, this may be more plausible than the usual zero-sum assumption. With this choice and the time-scaling (45), only one free parameter remains, the parameter

$$\omega^2 = c_p/c_e \tag{52}$$

which is the ratio of the price of evader's energy to the price of pursuer's energy, where the *prices* are measured in units which give unit weighting by each player on the terminal miss.[12] Some members of this one-parameter family of Nash equilibrium cost curves are plotted in Fig. 2. Since this game is nonzero-sum, cost curves for both pursuer and evader are given. Note that ω has the same interpretation as in the zero-sum case, so that Fig. 2 can be compared with Fig. 1.

When $\omega \gg 1$, there is very little that the evader can do to avoid capture, and the cost curves closely resemble the corresponding ones in the zero-sum case. Multiplying the pursuer's cost by ω and the evader's cost by $1/\omega$ gives the feedback gains. Thus, when the evader's control is relatively expensive, he uses very little control.

When $\omega = 1$, the Nash equilibrium behavior is very different from the corresponding zero-sum case, where the costs and feedback gains were constant functions of the time-to-go. As ω decreases further, the difference becomes more striking. Infinite costs are not obtained for any $\omega > 0$. The reader is cautioned not to interpret these curves as *trajectories*. Along any trajectory with $a = b = 0$, the costs for each player increase monotonically with the time-to-go.

In Fig. 3, the Nash equilibrium cost curves for $a = b = 0$ and $\omega = 1$ are compared with the costs with the same parameters when nonequilibrium feedback gains are used (in this case, multiples ρ_p and ρ_e of the equilibrium gains for pursuer and evader, respectively). Figure 3a demonstrates the Nash property for *unilateral* deviations from equilibrium by the evader. With $\rho_p = 1$, $\rho_e > 1$, both players incur higher costs for any t'; while, for $\rho_p = 1$, $\rho_e < 1$, the evader still pays more but the pursuer pays less. Figure 3b demonstrates the Nash property for deviations by the pursuer.

Figure 3c shows that, with $\rho_p = 0.8$, $\rho_e = 0.71$, lower costs are incurred

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[12] These prices are measured in *different* units, each chosen to make the corresponding *prices* on terminal miss unity.

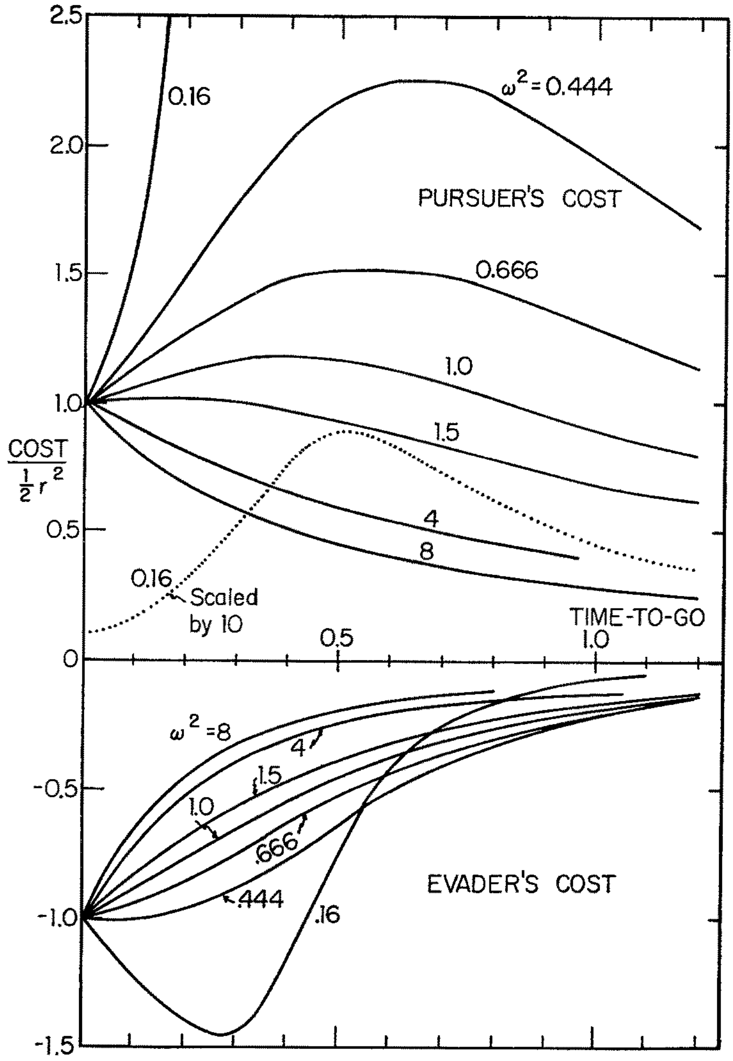

Fig. 2. Nash equilibrium costs for the nonzero-sum case with $a = b = 0$.

by *both* players. But this solution does not have the Nash property; the evader can make a small improvement in his cost by changing to $\rho_e = 1.2$, at great additional cost to the pursuer who still plays $\rho_p = 0.8$. A similar effort by the pursuer to improve his cost against $\rho_e = 0.71$ leads to a higher cost for the evader. Thus, we are confronted with an example of the *prisoners' dilemma*

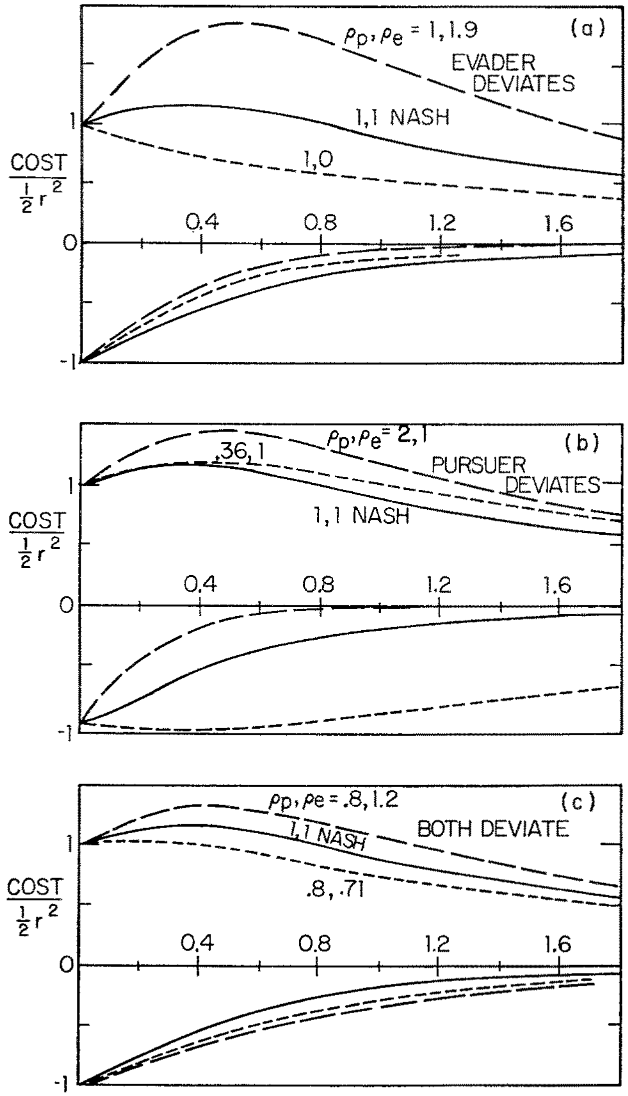

Fig. 3. Effect of deviations from the Nash equilibrium controls on costs for nonzero-sum case with $a = b = 0$ and $\omega = 1$. Controls are multiples ρ_p and ρ_e of Nash equilibrium controls.

situation described in Section 1. The same situation arises for other values of ω. Thus, even in this very simple differential game, it is evident that the Nash equilibrium is in no sense *optimal* and that it would be in the mutual interest of the two players to agree upon some other solution, if such an agreement could be enforced.

For the case considered here ($a = b = 0$), the *minimax* controls cannot be obtained from (31), because (32) is not satisfied. Inspection of the cost functions for the problem readily shows that $a_e = 0$ is a minimax control for the evader (with minimax cost $J_e = 0$) but the pursuer has no minimax control.

The *noninferior controls* for the velocity control problem[13] can be obtained by applying the results of Section 3.3. The family of noninferior controls is easily found to be

$$v_p = -sc_p\alpha r, \qquad v_e = sc_e\beta r \tag{53}$$

where

$$\begin{aligned} s(\mu, t) &= s_f/(1 + \gamma s_f t'), \qquad t' = t_f - t \\ s_f(\mu) &= \mu\sigma_p^2 - (1-\mu)\,\sigma_e^2, \qquad \gamma = \alpha c_p + \beta c_e \\ \alpha &= 1/[\mu + (1-\mu)\,c_e/c_{pe}], \qquad \beta = 1/[1 - \mu + \mu c_e/c_{pe}] \\ 0 &\leqslant \mu \leqslant 1 \end{aligned} \tag{54}$$

Note that, when $s_f < 0$, that is, when $\mu < \sigma_e^2/(\sigma_p^2 + \sigma_e^2)$, the noninferior controls become infinite at some finite t'. With $a = b = 0$, $\gamma = 1/\mu(1-\mu)$, and there is some range $\mu^* < \mu \leqslant 1$ such that the noninferior controls are finite for all t' (one must put a sufficiently large weight μ on the pursuer's cost). When a set of noninferior controls is used, the costs incurred by the two players can be obtained by solving (26). The resulting costs are

$$\hat{J}_p = \tfrac{1}{2}\hat{p}(t)\,r^2, \qquad \hat{J}_e = \tfrac{1}{2}\hat{e}(t)\,r^2 \tag{55}$$

where

$$\hat{p}(t) = \xi s_f^2 t'/(1 + \gamma s_f t') + \sigma_p^2/(1 + \gamma s_f t')^2$$

$$\hat{e}(t) = \eta s_f^2 t'/(1 + \gamma s_f t') - \sigma_e^2/(1 + \gamma s_f t')^2$$

and

$$\xi = c_p\alpha^2 + c_e^2\beta^2/c_{pe}$$

$$\eta = c_e\beta^2 + c_p^2\alpha^2/c_{ep}$$

[13] The corresponding results for the acceleration control problem can again be obtained from the results for the velocity control problem via (40) and (38).

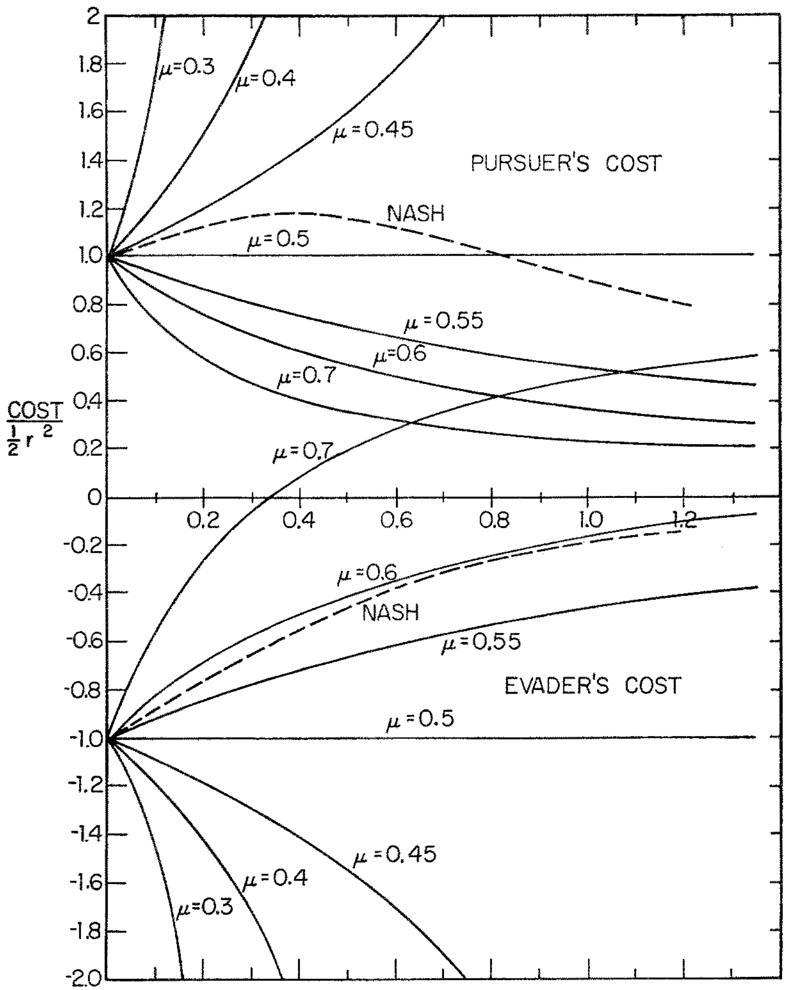

Fig. 4. Set of noninferior costs, with weighting μ on pursuer's criterion, for the case $a = b = 0$ and $\omega = 1$.

For the case $a = b = 0$, these parameters are $\alpha = 1/\mu$, $\beta = 1/(1 - \mu)$, $\gamma = \omega/\mu + 1/\omega(1 - \mu)$, $\xi = \omega/\mu^2$, and $\eta = 1/\omega(1 - \mu)^2$ where ω is defined in (52). For the particular value $\omega = 1$ (the price ratio is the same for both players) these noninferior costs $\hat{p}(t')$ and $\hat{e}(t')$ are plotted in Fig. 4 for several values of the weighting parameter μ. Note that, as the weighting on the pursuer's cost function μ increases, the results become more favorable to the pursuer. The Nash equilibrium costs for the same ω are shown for comparison. Presumably, neither player would accept a *negotiated* solution giving him a higher cost than the Nash solution. This would restrict the *negotiable range* to approximately $0.5 < \mu < 0.6$. The exact limits of this range depend on time-to-go. However, in some games, threat situations may exist (Ref. 4) which force one player to accept a noninferior solution which is worse for him than the Nash solution (recall that the Nash solution does not have the minimax property). Of course, the present mathematical formulation of the problem gives us no help in selecting a member of the noninferior set. Further rules are needed to define a unique, *fair* noninferior solution. The lack of a set of such rules acceptable to all parties is a major obstacle in practical negotiations.

5. Economic Example

The authors believe that the theory of nonzero-sum, N-player differential games may find some interesting applications in economic analysis, for example, where there is a mutual interest among competing firms. A simple model is outlined below to illustrate this idea.

Consider the dividend policies for N firms, each manufacturing a single product. The products are substitutable but not identical. This means that an increase in the price of the ith product results in lower, but not zero, sales of the ith product and increased sales for all the other products. In this model, the amount produced by each firm is a function only of the firm's capital, and everything produced is sold at whatever price the market offers. These *market clearing prices* are in turn determined by the amounts of *all* the N products currently offered for sale. A firm can generate new capital only from its own profits (no borrowing allowed). Given the appropriate production, demand, and production cost functions, one can obtain a vector function giving the net profit flow for each firm as a function of the vector of capital levels of all the firms.

The task of the management of the ith firm is to choose the continuous dividend rate u_i to maximize the *shareholder's utility function*

$$J_i = \int_{t_0}^{t_f} u_i \exp[-\alpha(t - t_0)]\, dt + x_i(t_f) \exp[-\alpha(t_f - t_0)] \tag{56}$$

subject to

$$\dot{x}_i = f_i(x_1, \ldots, x_N) - u_i \tag{57}$$

$$u_i \geqslant 0, \qquad x_i \geqslant \bar{x}_i \tag{58}$$

where x_i is the capital level of the ith firm,

$$f_i = F_i(x_i)P_i(F_1(x_1),..., F_N(x_N)) - C_i(x_i)$$

the net profit function, F_i the production function, P_i the market-clearing price function, C_i the production cost function, and α the interest rate. Clearly, this is a nonlinear, nonzero-sum differential game. Even with very simple F_i and P_i, the inequality constraints make it difficult to analyze. Nevertheless, if interest is great enough, it might be possible to extend some of the computational methods of optimal control theory to obtain Nash equilibria and other relevant solutions.

6. Conclusion

A general class of differential games is introduced where N players seek to minimize different cost criteria. Their interests are not diametrically opposed. Dropping the usual zero-sum hypothesis leads to several interesting phenomena which are not present in zero-sum games. Several *solutions* with different features are proposed. For one solution, the Nash equilibrium, which is secure against unilateral deviations by any player, the appropriate controls can be obtained by solving a set of coupled nonlinear partial differential equations, provided that the *Nash saddle-point* of a vector Hamiltonian can be found. For the linear-quadratic case, the Nash controls and costs can be obtained in terms of the solutions of a set of coupled matrix differential equations resembling (but more complicated than) the matrix Riccati equations which arise in optimal control theory. Solutions with other properties are also discussed, for both the general nonlinear game and the linear-quadratic case. The minimax solution, where each player minimizes his maximum possible cost, can be found by solving a two-player, zero-sum differential game. Finding the set of noninferior (or pareto-optimal) solutions, from which any negotiated solution can be chosen, involves solving an $(N - 1)$-parameter family of optimal control problems.

These results are applied to a nonzero-sum version of a simple pursuit-evasion problem. Even this very simple differential game displayed the *prisoners' dilemma* phenomenon, showing that there is more to a differential game than just finding the Nash equilibrium. Finally, an economic example is outlined involving the dividend policies of firms operating in an imperfectly competitive market.

References

1. Ho, Y. C., Bryson, A. E., and Baron, S., *Differential Games and Optimal Pursuit-Evasion Strategies*, IEEE Transactions on Automatic Control, Vol. AC-10, No. 4, 1965.
2. Isaacs, R., *Differential Games*, John Wiley and Sons, New York, 1965.
3. Case, J. H., *Equilibrium Points of N-Person Differential Games*, University of Michigan, Department of Industrial Engineering, TR No. 1967-1, 1967.
4. Luce, R. D., and Raiffa, H., *Games and Decisions*, John Wiley and Sons, New York, 1957.
5. Dacunha, N. O., and Polak, E., *Constrained Minimization Under Vector-Valued Criteria in Finite-Dimensional Spaces*, University of California at Berkeley, Electronics Research Laboratory, Memorandum No. ERL-M188, 1966.
6. Zadeh, L. A., *Optimality and Non-Scalar-Valued Performance Criteria*, IEEE Transactions on Automatic Control, Vol. AC-8, No. 1, 1963.
7. Klinger, A., *Vector-Valued Performance Criteria*, IEEE Transactions on Automatic Control, Vol. AC-9, No. 1, 1964.

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