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Axiomatic Approach in Differential Games[1]

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Abstract. Differential games are usually defined by differential equations. Recently, some work has been done on the possibility of defining such games in a more general, axiomatic way. In this paper, the advantages of this approach are discussed and possible further developments are pointed out.

1. Introduction

An ordinary, autonomous, differential game is essentially determined as follows. Let $t \in [0, \infty)$ denote time and $x \in R^n$ represent the state of a certain system. The evolution $x(t)$ of the system is assumed to be determined by the *kinematic equation*

$$\dot{x} = f(x, u, v) \tag{1}$$

with the initial condition

$$x(0) = x_0 \tag{2}$$

Here u, v are control parameters which are continuously regulated by two players whom, for simplicity, we shall call player u and player v, respectively.

Of particular interest are the following cases:

(a) The kinematic equation is *separable*, that is,

$$\dot{x} = f_1(x, u) + f_2(x, v) \tag{3}$$

(b) The generalized pursuit problem

$$\dot{x}_1 = f_1(x_1, u), \qquad \dot{x}_2 = f_2(x_2, v) \tag{4}$$

with $x = [x_1, x_2]$, x_1, x_2 being vectors of dimension n_1, n_2, with $n_1 + n_2 = n$.

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The game ends when a given end manifold $\mathscr{M} \subset R^{n+1}$ is reached, that is,

$$(T, x(T)) \in \mathscr{M} \tag{5}$$

It is customary to avoid games of infinite duration by including in $\mathscr{M}$ some hyperplane $t = T_0$.

The *payoff* P which player v has to pay to player u at the end of the game is some functional

$$P = P[u, v] = g(x(T)) + \int_0^T h(x, u, v)\, dt \tag{6}$$

In the case of a separable kinematic equation (3) or the generalized pursuit problem (4), we also assume that

$$h(x, u, v) = h_1(x, u) + h_2(x, v) \tag{7}$$

or

$$h(x, u, v) = h_1(x_1, u) + h_2(x_2, v) \tag{8}$$

Player u chooses his control function $u(t)$ so as to maximize P; similarly, player v tries to minimize P. In order to do so, both players know, at each instant t, the general features of the system [Eqs. (1)–(8)], the previous evolution of the system [that is, $x(\tau)$], and also the behavior of the other player for $0 \leqslant \tau < t$. The knowledge of the values of the present ($\tau = t$) is irrelevant in many cases, for example, when all the important information is contained in $x(t)$, which is a continuous function; therefore, the value of $x(t)$ follows from the values of $x(\tau)$ for $\tau < t$. The cases when the knowledge of, say, $v(t)$ is crucial for the determination of $u(t)$ are pathological in the sense that they usually correspond to singularities in the optimal solutions. We will see later how this problem can be avoided, at least theoretically. For a discussion of such singularities and their importance for the game, we refer to the classical book of Isaacs (Ref. 1) and the recent survey paper by Berkovitz (Ref. 2).

Some restrictions have to be imposed on the controls $u(t)$, $v(t)$ in order to be admissible. They have to be measurable for the kinematic equation (1) to make sense. Some boundedness condition is very natural, and one usually assumes $u(t)$, $v(t)$ to be vector-valued functions (of dimensions p, q, respectively) with values in some fixed compact sets U, V, that is,

$$u(t) \in U \subset R^p, \qquad v(t) \in V \subset R^q \tag{9}$$

The admissible values of the state variable may also be restricted to some closed set X, that is,

$$x(t) \in X \subset R^n \tag{10}$$

This last condition creates new and difficult problems, because it means that not all measurable pairs $u(t)$, $v(t)$ satisfying (9) are admissible. In the pursuit case (4), if (10) has the Cartesian product form

$$x_1(t) \in X_1 \subset R^{n_1}, \qquad x_2(t) \in X_2 \subset R^{n_2} \tag{11}$$

it is much simpler to take this condition into account because then a certain $u(t)$ either is or is not admissible, independently of the choice of $v(t)$, and conversely.

For the mathematical problem to be relatively tractable, some assumptions are made on the functions appearing in the basic equations. Typical assumptions are the following:

(a) The function $f(x, u, v)$ is continuous in (x, u, v).

(b) There is a Lipschitz constant k such that, for every $u \in U$, $v \in V$, $x, \bar{x} \in X$,

$$|f(x, u, v) - f(\bar{x}, u, v)| \leqslant k \, | x - \bar{x} |$$

Here, the symbol $| \cdot |$ denotes the Euclidean norm.

(c) There are constants M, N such that for every $u \in U$, $v \in V$, $x \in X$,

$$|f(x, u, v)| \leqslant M \, | x | + N$$

(d) Some convexity condition is necessary to avoid the appearance of *sliding regimes* or *weak solutions*. Conditions of this type are: (d′) for every $x \in X$, the set $f(x, U, V) = \{f(x, u, v) | \, u \in U, v \in V\}$ is convex; and (d″) for every $x \in X$, $u_0 \in U$, $v_0 \in V$, the sets $f(x, u_0, V) = \{f(x, u_0, v) | \, v \in V\}$ and $f(x, U, v_0) = \{f(x, u, v_0) | \, u \in U\}$ are convex. In the case of a separable function $f(x, u, v)$, (d′) follows immediately from (d″).

Basic for any development of the theory is the definition of a strategy, that is, a general rule of behavior for a player. In most practical cases, a strategy for player u is given by a function $u^*(x)$, with the requirement that $u(t)$ should be chosen at any moment as $u(t) = u^*(x(t))$. A strategy for player v is defined similarly as $v(t) = v^*(x(t))$. This way of prescribing the action of the players is very satisfactory when it works, but unfortunately this is not always the case. Indeed, introducing these expressions into Eq. (1), we obtain the relation

$$\dot{x} = f(x, u^*(x), v^*(x))$$

for which neither the uniqueness nor the existence of solutions is assured (u^* and v^* are not necessarily continuous).

Of course, both players seek *optimal strategies* $u^*(x)$, $v^*(x)$, such that, for all other strategies $u(x)$, $v(x)$, the payoff P satisfies the inequalities

$$P[u, v^*] \leqslant P[u^*, v^*] \leqslant P[u^*, v] \tag{12}$$

Such a couple u^*, v^* constitutes a *saddle point*. The corresponding value of the payoff

$$V^* = P[u^*, v^*] \tag{13}$$

is called the *value* of the game, and it is easy to see that, if it exists, it is unique. The optimal strategy u^* or v^* may not be unique; but, if there are more strategies than one, they are equivalent in the sense that they give the same payoff.

The main difficulty is plain: in order to assert (12) for all possible u, v, the set of all these strategies must constitute a well-defined and known class. Such a class is not always easy to determine.

Recently, Varaiya (Ref. 3) gave another definition of strategy and was successful in applying it to games of the pursuit type. For this purpose, he used notions from the theory of dynamical polysystems.

2. Dynamical Polysystems

Some axiomatic approaches to control and even more general systems were developed recently. An excellent survey was given by Bushaw (Ref. 4). The main idea is to characterize and study systems such as control systems usually defined by a differential equation

$$\dot{x} = f(x, u) \tag{14}$$

where $u(t)$ is a control function chosen arbitrarily from a given class of admissible controls, but seeking a characterization independent of a particular representation like (14). Such *dynamical polysystems* include differential equations

$$\dot{x} = f(x)$$

with nonunique solutions and *contingent equations* defined by a set-valued function $F(x)$ such that

$$\dot{x} \in F(x)$$

or by inequalities

$$\Phi(x, \dot{x}) \geqslant 0$$

For the relation of contingent equations with control systems, we refer to Ref. 5. The advantage of such a theory is the characterization of all properties in an intrinsic way, that is, independently of the analytical representation. A deeper insight is so obtained and, as an example, one can mention the stability theory developed in Ref. 6 and later papers.

The theory of dynamical polysystems in its greatest generality was initiated by Bushaw in Ref. 7, but in such a general setting very few results have been achieved at present.

Aiming at much less generality, working in locally compact, complete metric spaces (in practice, finite-dimensional Euclidean spaces) and with relatively strict assumptions, Barbashin (Ref. 8) and the author (Ref. 6) developed a rather satisfactory theory having sufficient results to be practically applicable, but nevertheless including all *classical* control systems, differential equations without uniqueness, and so on. In this approach, the following set of axioms is assumed concerning the *attainable set* $F(t, t_0, x_0)$, that is, the set of points x which can be reached at time t starting at x_0, t_0. The axioms are:

(I) $F(t, t_0, x_0)$ is a closed nonempty set defined for every $x \in X$, $t \geqslant t_0$.

(II) $F(t_0, t_0, x_0) = \{x_0\}$ for every x_0, t_0.

(III) For $t_0 \leqslant t_1 \leqslant t_2$,

$$F(t_2, t_0, x_0) = \bigcup_{x_1 \in F(t_1, t_0, x_0)} F(t_2, t_1, x_1)$$

(IV) Given $x_1 \in X$, $t_0 \leqslant t_1$, there exists an $x_0 \in X$ such that $x_1 \in F(t_1, t_0, x_0)$.

(V) $F(t, t_0, x_0)$ is continuous in t.

(VI) $F(t, t_0, x_0)$ is upper semicontinuous in (t_0, x_0) uniformly in any finite t-interval.

From these axioms, many properties are derived which show that these systems behave as one expects or wishes them to behave. The most important consequence of the axioms is the existence of *trajectories*. These are defined as functions $x = \varphi(t)$, $t_0 \leqslant t \leqslant t_1$, with the property that, if $t_0 \leqslant t_a \leqslant t_b \leqslant t_1$, then $\psi(t_b)$ is attainable from $\varphi(t_a)$ in the corresponding time interval. This shows that systems defined by axioms (I)–(VI) can be determined by the trajectories.

3. Applications to Differential Games

The similarity between the control equation (14) and the kinematic equation (1) of a differential game is obvious. Equation (14) can indeed be imagined as a differential game with only one player. One therefore expects to be able to develop the theory of differential games starting from the knowledge of the attainable sets, that is, the sets of points x reachable from some x_0 by a suitable choice of $u(t)$ and $v(t)$. But here the situation is much more difficult than in the case of a single control.

To realize this, consider a deterministic control system (14) and any corresponding optimization problem. It is completely equivalent for the single player to determine his best course of action $u(t)$ for the whole game at the beginning or step by step as the one-player game proceeds. More precisely, the player can predict completely the future course of the game as a function of his own action. The fact that this is not true for the two-player game (1) is the essential feature of game theory.

A general theory of differential games from the approach of the attainable sets has not been attempted to the author's knowledge. In a recent paper, Kirillova (Ref. 9) did some work in this direction for the pursuit problem, but she did not use a concept of strategy in the sense of Varaiya. For that reason, her results, even if applicable to some special cases, do not really correspond to the theory of differential games.

A kind of mixed approach, utilizing some features of the axiomatic systems, was initiated by Varaiya for the pursuit game (Ref. 3). He considers two players with positions $x(t)$, $y(t)$. The initial conditions x_0, y_0 are given. The payoff is the time of capture, that is, the time when $x(T) = y(T)$. But instead of equations of type (4), the kinematics of the players are assumed to be given by the *attainability functions*

$$x(t) \in F(t, t_0, x_0), \qquad y(t) \in G(t, t_0, y_0) \tag{15}$$

which satisfy axioms (I)–(VI) of the previous section. Note that x, the maximizing player, is the evader, and y is the pursuer. Under these conditions, Varaiya proves the existence of an optimal pursuit strategy, provided, of course, that the pursuer is able to catch the evader at all.

The most interesting point in this approach is the definition of a strategy. Comparing the kinematics of Eq. (4) with (15), we see that, in the latter, the control functions $u(t)$, $v(t)$ have disappeared. This can be achieved in (4), if we assume that both players select $x(t)$ and $y(t)$ directly among the admissible trajectories, without going through $u(t)$ and $v(t)$. In this way, we let the trajectories $x(t)$ and $y(t)$ themselves play the role of control functions. The great advantage of this is that the set of admissible trajectories for each player is a compact set in the topology of uniform convergence in some basic interval $[0, T_0]$.

A strategy for the pursuer y is then defined as a mapping

$$\beta : \mathscr{X} \to \mathscr{Y} \tag{16}$$

where $\mathscr{X}$ and $\mathscr{Y}$ stand for the sets of admissible trajectories $\{x(t)\}$ and $\{y(t)\}$. In order to be a strategy, such a mapping has to satisfy the additional condition that its determination does not depend on the future. Precisely,

$$\begin{aligned} x_1(\tau) &= x_2(\tau), \qquad 0 \leqslant \tau \leqslant t \\ \beta[x_1] &= y_1, \qquad \beta[x_2] = y_2 \end{aligned} \tag{17}$$

imply that

$$y_1(t) = y_2(t)$$

We may say that $y(t)$ depends, for each t, only on the past values of $x(t)$. The value of $x(t)$ corresponding to the present is accounted for by the continuity of $x(t)$ [see the remark in the introduction, after formula (8)]. A similar definition is given for an x-strategy $\alpha[y]$.

The drawback of Varaiya's results is that, in general, there is no optimal evasion strategy (and, therefore, no true saddle point). The situation is the following.

A pursuit strategy $\beta[x]$ is called *feasible* if it guarantees that the pursuer catches the evader within the allowed time interval $[0, T_0]$. We assume that such strategies exist, because otherwise the evader can avoid capture and there is no optimization problem.

Denote by $T[x, \beta]$ the time of capture, which depends on the behavior $x(t)$ of the evader and the strategy β of the pursuer. The game is said to have a *good solution* if there is a pursuit strategy β^* such that

$$\sup_{x(t)} T[x, \beta^*] = T^* = \inf_{\beta} \sup_{x(t)} T[x, \beta] \tag{18}$$

where $x(t)$ ranges over all the admissible x-trajectories and β over all the feasible pursuit strategies. In other words, using the strategy β^*, the pursuer is sure to catch the evader in a time $t \leqslant T^*$, and this value T^* is the best such value. Under the assumed conditions, Varaiya proves that the game has a good solution. It should be noted that, in this way, the evader does not choose $x(t)$ according to some strategy $\alpha[y]$.

4. Further Development of the Generalized Pursuit Game

The asymmetry of the solution mentioned in the preceding section was overcome recently, to some extent, by Varaiya and Lin. Here, we can only give the main ideas and refer the reader, for the precise assumptions and all other details, to the original paper (Ref. 10). The game considered in Ref. 10 is less general than the one considered in the previous section, in the sense that it is defined by differential equations of the type (4), but more general, in the sense that the payoff is of the type (6), (8).

The strategies $\alpha[y]$, $\beta[x]$ are defined as before. Given such a pair of strategies, the corresponding solution of the game should be a pair of trajectories $x(t)$, $y(t)$ satisfying

$$x(t) = \alpha[y(t)], \qquad y(t) = \beta[x(t)] \tag{19}$$

Unfortunately, neither the existence nor the uniqueness of such a solution can be assured in general, as examples show.

A pair $(x_0(t), y_0(t))$ is defined to be an *outcome* of the pair (α, β) if there are two sequences of admissible trajectories $x_n \to x_0$, $y_n \to y_0$, $n = 1, 2,...,$ such that

$$\lim x_n = \lim \alpha[y_n] = x_0, \qquad \lim y_n = \lim \beta[x_n] = y_0 \tag{20}$$

Let $O(\alpha, \beta) = \{(x, y) | (x, y) \text{ is an outcome of } (\alpha, \beta)\}$. Let P denote the payoff of the game. Under some given conditions, $O(\alpha, \beta)$ is nonempty and compact, and the following values can be defined:

$$\begin{aligned} P^+(\beta) &= \sup_{\alpha} \max_{(x,y)\in O(\alpha,\beta)} P[x, y] \\ P_-(\alpha) &= \inf_{\beta} \min_{(x,y)\in O(\alpha,\beta)} P[x, y] \\ V^+ &= \min_{\beta} P^+(\beta) \\ V_- &= \max_{\alpha} P_-(\alpha) \end{aligned} \tag{21}$$

Note that the inequality $V^+ \geqslant V_-$ always holds.

Under the assumptions made, the following *saddle point theorem* can be proved: There exist strategies α^*, β^* such that, for all admissible strategies α, β,

$$\max_{(x,y)\in O(\alpha,\beta^*)} P[x, y] \leqslant \max_{(x,y)\in O(\alpha^*,\beta^*)} P[x, y] = \min_{(x,y)\in O(\alpha^*,\beta^*)} P[x, y] \leqslant \min_{(x,y)\in O(\alpha^*,\beta)} P[x, y]$$

Furthermore, $P[x, y] = \text{Const} = V_F$ for all $(x, y) \in O(\alpha^*, \beta^*)$.

Therefore, (α^*, β^*) has the properties of a saddle point. The corresponding value of the game is V_F, where the subindex F stands for *fair game*, which is the name Varaiya and Lin give to the optimization problem so defined.

5. Differential Games from the Viewpoint of Functional Analysis

Let us consider again the differential game given in the general form of Eqs. (1), (2), (5), (6). The evolution $x(t)$ of the game is determined by the two functions $u(t)$, $v(t)$ chosen by the players from the corresponding sets of admissible controls $\{u\} = \mathscr{U}$, $\{v\} = \mathscr{V}$. Therefore, there is a functional relationship of the form $x = x[u, v]$, and a similar one for the payoff $P = P[u, v]$. The choice of $u(t)$, $v(t)$ by the players is made according to the desire to maximize and minimize P and with the information pattern already explained.

It is natural to define a strategy for players u and v by

$$\alpha : \mathscr{V} \to \mathscr{U}, \qquad \beta : \mathscr{U} \to \mathscr{V}$$

satisfying the Varaiya condition. We observe that

$$\begin{gathered} v_1, v_2 \in \mathscr{V}, \qquad \alpha[v_1] = u_1, \qquad \alpha[v_2] = u_2 \\ v_1(\tau) = v_2(\tau) \quad \text{for} \quad 0 \leqslant \tau \leqslant t \end{gathered} \tag{22}$$

imply that

$$u_1(t) = u_2(t)$$

and, similarly, for $\beta[u]$.

From here, we can reproduce many results mentioned above, provided the sets $\mathscr{U}$, $\mathscr{V}$, with a suitably chosen topology, are compact. This was done in Ref. 11, but the drawback is that, in general, $\mathscr{U}$ and $\mathscr{V}$ are not compact. Nevertheless, one can sometimes avoid this difficulty with a special device.

In the cases considered by Varaiya and Lin (Refs. 3, 10), the control functions were simply replaced by the admissible trajectories x, y, which form compact sets. For example, if Eq. (1) is linear, that is,

$$\dot{x} = Ax + Bu + Cv$$

if $|u| \leqslant M$, $|v| \leqslant N$ are the constraints, and if the payoff is terminal, that is,

$$P = g(x(T_0))$$

with fixed T_0 and continuous $g(\cdot)$, then one can adopt for $\mathcal{U}$, $\mathcal{V}$ the weak topology which makes them compact (see Ref. 12, pages 292 and 430). With this topology, P is continuous in u, v and the theory can easily be developed as before.

It is probable that this kind of approach can be extended to more general games. At least for the separable case (3), (7), we already know from Pontryagin (Ref. 13) that this case is reasonably tractable by analytical methods and the interaction between the two controls is not too involved.

Finally, we mention that one of the main tools used by Varaiya and Lin in the proof of the above results is the comparison of the given game with some *upper* and *lower* games (or *majorant* and *minorant* games). These are obtained from the given game by introducing a delay σ into the transmission of information to one or the other player; obviously, this gives an advantage to his opponent.

This method, used earlier by Fleming (Ref. 14), is really a particular case of the much more general method of attack of these problems based on comparison of different games. In many cases, it should be possible to construct, for a given game, comparison *lower games* and *upper games* by changing some feature of the original game. Devices of this sort have been used in many particular cases (see, for example, many concrete games in Ref. 1), but there is no general theory. In this sense of comparison criteria, the use of all kinds of possible *Liapunov functions* also comes to mind.

The fact that so little has been done in these areas shows that the whole subject is in its early stages of development and that the axiomatic or functional analytic approach is likely to yield powerful methods to deal with differential games.

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