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On the Observability of Linear, Time-Invariant Systems with Unknown Inputs[1]

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Abstract. This paper considers the observation of linear, time-invariant dynamical systems in the general case in which some of the input functions are unknown. By arguments based on the concepts of controlled and conditioned invariance, a convenient expression for the observability subspace is found which includes the well-known expression for the case in which the input functions are given.

1. Introduction

Observability has been presented and subsequently developed as a dual of controllability (Refs. 1–5). A plant is usually said to be completely observable if, from the knowledge of input and output functions in a finite time interval, it is possible to deduce the state trajectory in the same time interval. In practice, many cases occur in which some of the input variables are not accessible, so that we can conveniently distinguish the inputs in two classes; control inputs and disturbances. When the system equations are known, it may be possible, even in the presence of disturbances, to deduce the state trajectory from the knowledge of control inputs and system outputs in a finite time interval. The purpose of this paper is to give necessary and sufficient conditions for complete or partial observability when some inputs are completely unknown.

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We deal with a linear, purely dynamical, time-invariant system described by the equations

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in R^n$ is the *state vector*, $u_1 \in R^m$ is the *control vector*, $u_2 \in R^l$ is the *disturbance vector*, $y \in R^s$ is the *output vector*, and A, B_1, B_2, C are real, constant matrices of proper sizes. We call $\mathscr{F}_1 = \mathscr{R}(B_1)$ the subspace of control actions and $\mathscr{F}_2 = \mathscr{R}(B_2)$ the subspace of disturbance actions.

It is well known that, in the particular case where $B_1 \neq 0$, $B_2 = 0$, from the observation of input and output functions in a finite interval of time, it is possible to recognize the orthogonal projection of the state on the least subspace which is invariant under A^T and contains $\mathscr{R}(C^T)$. The word *least* is justified because the intersection of two invariants is an invariant. This subspace is sometimes called *observability subspace* and its orthogonal complement *unobservability subspace.*

In this particular case, when the input functions are completely known, the observation of the system (1)–(2) reduces to the observation of the corresponding autonomous system; that is, since

$$y(t) = C\Phi(t, 0)\, x_0 + C \int_0^t \Phi(t, \tau)\, B_1 u_1(\tau)\, d\tau \tag{3}$$

where $\Phi(t, \tau)$ is the state-transition matrix, it is possible to determine by a simple subtraction the output functions of the corresponding autonomous system, namely, the zero-input output functions.

By similar reasoning, the general case in which a part of the input is known and a part is unknown can be reduced to the case of completely unknown input. Thus, it is sufficient to consider only this last case. In the next section, we state a theorem that provides the observability subspace as the least conditioned invariant under the matrix A^T, with respect to the subspace $\mathscr{F}_2^{\perp}$, containing $\mathscr{R}(C^T)$, and which includes the previous results, corresponding to $B_2 = 0$.

2. Observability Subspace for Systems with Unknown Inputs

First, we recall some definitions and results given in a previous paper (Ref. 6) which provide a background for the analysis presented here. Consider an $n \times n$ matrix A and a subspace $\mathscr{F} \subseteq R^n$. We use the following definitions and properties: (a) an $(A, \mathscr{F})$-controlled invariant is a subspace $\hat{\mathscr{J}}$ such that $A\hat{\mathscr{J}} \subseteq \hat{\mathscr{J}} + \mathscr{F}$; (b) an $(A, \mathscr{F})$-conditioned invariant is a subspace $\check{\mathscr{J}}$ such that $A(\check{\mathscr{J}} \cap \mathscr{F}) \subseteq \check{\mathscr{J}}$; (c) the orthogonal complement of an $(A, \mathscr{F})$-controlled invariant is an $(A^T, \mathscr{F}^\perp)$-conditioned invariant; and (d) the least $(A, \mathscr{F})$-conditioned invariant containing a given subspace $\mathscr{X}$ is the subspace

$$\check{\mathscr{J}}_m = \mathscr{Y}_{n-1} \tag{4}$$

where $\mathscr{Y}_{n-1}$ is defined by the recursive relationship

$$\mathscr{Y}_0 = \mathscr{X}, \qquad \mathscr{Y}_i = \mathscr{X} + A(\mathscr{Y}_{i-1} \cap \mathscr{F}), \qquad i = 1, \ldots, n-1 \tag{5}$$

Now, consider the system

$$\dot{x} = Ax + Bu \tag{6}$$

$$y = Cx \tag{7}$$

where u is a completely unknown input vector. We state the following theorem:

Theorem 2.1. The observability subspace of system (6)–(7) with unknown inputs (i.e., the subspace of maximal dimension where the orthogonal projection of the state can be recognized solely from the knowledge of the output in a finite interval of time) is the least $(A^T, \mathscr{F}^\perp)$-conditioned invariant containing $\mathscr{R}(C^T)$.

By the duality property (c) mentioned above, it follows from Theorem 2.1 that the state can be recognized within a vector on the greatest $(A, \mathscr{F})$-controlled invariant contained in $\mathscr{N}(C)$. This is intuitively obvious, because the greatest subspace of $\mathscr{N}(C)$ which could contain a finite arc of the trajectory in the state space is exactly the greatest $(A, \mathscr{F})$-controlled invariant contained in $\mathscr{N}(C)$.

3. Proof of Theorem 2.1.

First, we prove that the least $(A^T, \mathscr{F}^\perp)$-conditioned invariant containing $\mathscr{R}(C^T)$, which can be defined exactly by means of statement (d) of the previous section, is a subspace where the orthogonal projection of the state can be recognized from a record of the output functions of a finite length. Later, we verify that it is the greatest subspace where the system can be observed.

We start from

$$y(t) = Cx(t), \qquad t \in [0, T] \tag{8}$$

which, in order to emphasize the iterative character of the argument, can be written as

$$q_0(t) = Y_0 x(t), \qquad t \in [0, T] \tag{9}$$

where $q_0(t) = y(t)$ is a vector of known functions of the time and $Y_0 = C$ is a known constant matrix. Using solely the vector equation (9), that is, pseudo-inverting the matrix Y_0, we can obtain the orthogonal projection of the vector $x(t)$ on the range of the transpose of the coefficient matrix. We denote this subspace by the symbol $\mathscr{Y}_0 : \mathscr{Y}_0 = \mathscr{R}(Y_0^T) = \mathscr{R}(C^T)$.

In general, more knowledge of the state can be gained by using also the differential equations (6); in fact, taking the first derivatives of (9) and using (6), we have

$$\dot{q}_0(t) = Y_0 A x(t) + Y_0 B u(t) \quad \text{a.e. in } [0, T] \tag{10}$$

Since the input vector function $u(t)$ is unknown, in order to deduce some information on the state from Eq. (10), we must employ its projection on the subspace $\mathscr{R}(Y_0 B)^\perp = \mathscr{N}(B^T Y_0^T)$. Letting P_1 denote the projecting matrix on this subspace,[4] we obtain

$$P_1 \dot{q}_0(t) = P_1 Y_0 A x(t) \tag{11}$$

Note that we take a twofold advantage of this projection: we drop the unknown input and we obtain a vector equation, both sides of which are again differentiable a.e. in $[0, T]$. In more compact notation, we can write Eqs. (9) and (11) together as

$$q_1(t) = Y_1 x(t), \qquad t \in [0, T] \tag{12}$$

where

$$q_1 = \begin{bmatrix} q_0 \\ P_1 \dot{q}_0 \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} Y_0 \\ P_1 Y_0 A \end{bmatrix} \tag{13}$$

In order to deduce information about the state, it is convenient to employ Eqs. (12) instead of (9), because $\mathscr{Y}_1 = \mathscr{R}(Y_1^T) \supseteq \mathscr{Y}_0$. In fact, since

$$\mathscr{R}(P_1) = \mathscr{R}(P_1^T) = \{y : B^T Y_0^T y = 0\} \tag{14}$$

is the locus of the vectors $y \in R^s$ which are mapped by Y_0^T into $\mathscr{F}^\perp = \mathscr{N}(B^T)$, the set $\mathscr{R}(Y_0^T P_1^T) = Y_0^T \mathscr{R}(P_1^T)$ is equal to $\mathscr{R}(Y_0^T) \cap \mathscr{N}(B^T) = \mathscr{Y}_0 \cap \mathscr{F}^\perp$, so

[4] The projecting matrix on a subspace $\mathscr{X} = \mathscr{R}(X)$, where it is assumed that the column vectors of X are linearly independent, is expressed by $P = X(X^T X)^{-1} X^T$. Also, P is symmetric.

that the range of the transpose of the coefficient matrix of (11) is $A^T(\mathscr{Y}_0 \cap \mathscr{F}^\perp)$; hence,

$$\mathscr{Y}_1 = \mathscr{Y}_0 + A^T(\mathscr{Y}_0 \cap \mathscr{F}^\perp) \tag{15}$$

and, therefore, $\mathscr{Y}_1 \supseteq \mathscr{Y}_0$.

We can now start from Eqs. (12) and, by means of the same procedure, derive the equations

$$q_2(t) = Y_2 x(t), \qquad t \in [0, T] \tag{16}$$

which make it possible to determine the projection of the state on the subspace

$$\mathscr{Y}_2 = \mathscr{R}(Y_2^T) = \mathscr{Y}_0 + A^T(\mathscr{Y}_1 \cap \mathscr{F}^\perp) \tag{17}$$

Relationship (17) can be proved by the same arguments as those used in the proof of (14), noting that $A^T(\mathscr{Y}_1 \cap \mathscr{F}^\perp) \supseteq A^T(\mathscr{Y}_0 \cap \mathscr{F}^\perp)$, so that we can have $\mathscr{Y}_0$ instead of $\mathscr{Y}_1$ in the right side of (17).

Iterating $n - 1$ times, we finally obtain

$$q_{n-1}(t) = Y_{n-1} x(t), \qquad t \in [0, T] \tag{18}$$

where $q_{n-1}(t)$ is a known function of the output and of the derivatives of its projections on proper subspaces, on which they are differentiable, and Y_{n-1} is a known matrix such that

$$\mathscr{Y}_{n-1} = \mathscr{R}(Y_{n-1}^T) = \mathscr{Y}_0 + A^T(\mathscr{Y}_{n-2} \cap \mathscr{F}^\perp) \tag{19}$$

Thus, the range of the transpose of Y_{n-1}, where it is always possible to recognize the projection of the state function, is the least $(A^T, \mathscr{F}^\perp)$-conditioned invariant subspace containing $\mathscr{Y}_0 = \mathscr{R}(C^T)$.

It is still to be proved that it is not possible to find any greater subspace where the projection of the state can be observed: if such a subspace were to exist, a trajectory on the greatest $(A, \mathscr{F})$-controlled invariant contained in $\mathscr{N}(C)$ would be partially observable. This is clearly a contradiction, because such a trajectory does not affect the output.

4. Conclusion

In this paper, we have presented an application of the properties of controlled and conditioned invariance, namely, the solution of the problem of observing a system with unknown inputs. It is remarkable how these generalizations of the concept of invariance, particularly suitable for the analysis of control systems, make it possible to state the main result and give its proof in a very simple way.

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