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Optimal Design of Rotating Disk for Given Radial Displacement of Edge[1]

J.-M. CHERN[2] AND W. PRAGER[3]

Abstract. Minimum-weight design of axially-symmetric, rotating, elastic disks is usually discussed in terms of uniform strength. In this paper, an alternative constraint is used: under the combined influence of centrifugal forces and a uniform radial traction along the circular edge, the radial displacement at this edge is to have a prescribed value. A necessary and sufficient optimality condition is derived, and its use in the determination of optimal disk profiles is illustrated by examples. It is shown that the resulting disks are far from satisfying the condition of uniform strength although their weights are only very slightly smaller than those of the corresponding disks of uniform strength.

1. Introduction

Design of axially-symmetric, rotating disks for *uniform strength* has been widely discussed in the literature (see, for instance, Ref. 1, p. 65, and Ref. 2). In this context, the term *uniform strength* is usually taken to mean that the radial and circumferential stresses σ_r and σ_θ have the same constant value σ_0 throughout the disk, that is,

$$\sigma_r = \sigma_\theta = \sigma_0 . \tag{1}$$

As will be shown in Section 4, the condition (1) may be derived by assuming that the disk is made of an elastic-plastic material with the yield condition of von Mises and demanding that the yield limit should be simultaneously reached throughout the disk.

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[2] Research Associate, Division of Engineering, Brown University, Providence, Rhode Island.

[3] University Professor of Engineering and Applied Mathematics, Brown University, Providence, Rhode Island.

In the present paper, an alternative constraint is used: under the combined influence of centrifugal forces and a uniform radial traction T along the edge $r = a$ of the disk, the radial displacement $u(a)$ at the edge is to have the given value u_0, that is,

$$u(a) = u_0 . \tag{2}$$

A necessary and sufficient optimality condition for the constraint (2) is established in Section 2. This is used in Section 3 to determine the profile of the optimal disk and the associated radial and circumferential stresses. While these stresses are far from satisfying (1), the optimal disks under the constraints (1)–(2) are shown in Section 4 to have practically the same volume but different profiles.

2. Optimality Condition

Consider a thin elastic disk of arbitrary planform that is subject to given body forces per unit volume and given edge tractions along the part s_T of its edge, while the remainder s_U of the edge is rigidly supported. The disk is to have a given *elastic compliance*, which will be defined as the virtual work of a suitably chosen set of fictitious edge tractions on the actual displacements of the disk.

With respect to rectangular Cartesian coordinates x_i, $i = 1, 2$, in the median plane of the disk, let $h(x_1, x_2)$ be the variable thickness of the disk, and denote the given body force per unit volume by f_i and the given traction per unit length of the edge by T_i. Furthermore, let u_i be the displacements in the median plane of the disk, and denote the associated strains and elastic stresses by ϵ_{ij} and $\sigma_{ij} = c_{ijkl}\epsilon_{kl}$, where lowercase subscripts have the range 1,2, repeated subscripts imply summation over this range, and the quantities c_{ijkl} are elastic constants.

To define the elastic compliance C that is relevant for a considered problem, introduce appropriate fictitious edge tractions $\bar{T}_i$ on s_T and set

$$C = \int \bar{T}_i u_i \, ds_T , \tag{3}$$

where the integration is extended over the part s_T of the edge. (For example, under conditions of axial symmetry, a fictitious uniform radial traction of unit intensity along the circular edge of the disk furnishes the product of the circumference of the disk and the radial edge displacement as the value of the compliance C.) Displacements, strains, and stresses caused in the elastic disk by the sole action of the edge tractions $\bar{T}_i$ will be denoted by $\bar{u}_i$, $\bar{\epsilon}_{ij}$, and $\bar{\sigma}_{ij} = c_{ijkl}\bar{\epsilon}_{kl}$.

The design objective considered in the following is minimization of the volume

$$V = \int h \, dA \tag{4}$$

of the disk subject to the constraint

$$C = \int \bar{T}_i u_i \, ds_T = C_0 , \tag{5}$$

where dA is the area element of the planform and C_0 is a given positive constant.

The principle of stationary mutual potential energy (Ref. 3) is used to derive a sufficient condition for optimality. If u_i^*, $\bar{u}_i^*$ are two kinematically admissible displacement fields and ϵ_{ij}^*, $\bar{\epsilon}_{ij}^*$ the corresponding strain fields, the *mutual potential energy* of the design $h(x_1, x_2)$ for those displacement fields and the loads f_i, T_i, and $\bar{T}_i$ is defined as

$$U[u^*, \bar{u}^*; h] = \tfrac{1}{2}\left[\int hc_{ijkl}\epsilon_{ij}^*\bar{\epsilon}_{kl}^* \, dA - \int hf_i\bar{u}_i^* \, dA - \int T_i\bar{u}_i^* \, ds_T - \int \bar{T}_i u_i^* \, ds_T\right]. \tag{6}$$

The *principle of stationary mutual potential energy* states that the functional (6) is stationary in the neighborhood of $u^* = u$, $\bar{u}^* = \bar{u}$, where u and $\bar{u}$ respectively are the actual displacements caused by the loads f_i, T_i and by the loads $\bar{T}_i$. The proof of this principle exactly follows that given in Ref. 3 for beams and need not be reformulated here. It follows from the definitions (5)–(6) and the principle of virtual work that

$$U[u, \bar{u}; h] = -\tfrac{1}{2}\int hc_{ijkl}\epsilon_{ij}\bar{\epsilon}_{kl} \, dA = -\tfrac{1}{2}\int \bar{T}_i u_i \, ds_T . \tag{7}$$

Assume now that the design h is optimal for the constraint (5), and let $h^* = h + \delta h$ be a neighboring design with a compliance C^* satisfying

$$C^* \leqslant C_0 . \tag{8}$$

Any design of this kind will be called *feasible*. If $u^* = u + \delta u$, $\bar{u}^* = \bar{u} + \delta\bar{u}$ are the displacement fields of the design h^* under the action of the loads f_i, T_i, and $\bar{T}_i$, respectively, it follows from (5), (7)–(8) that

$$U[u^*, \bar{u}^*; h^*] \geqslant U[u, \bar{u}; h]. \tag{9}$$

On the other hand, because the fields u, $\bar{u}$ are kinematically admissible for the design h^* and neighboring to the actual fields u^*, $\bar{u}^*$ of this design, it follows from the principle of stationary mutual potential energy that

$$U[u^*, \bar{u}^*; h^*] = U[u, \bar{u}; h^*]. \tag{10}$$

Substitution of (9) into (10) and use of the definition (6) yields

$$\int (c_{ijkl}\epsilon_{ij}\bar{\epsilon}_{kl} - f_i\bar{u}_i)\, \delta h \, dA \geqslant 0, \tag{11}$$

where $\delta h = h^* - h$. The equality sign in (11) only applies to the feasible designs with $C^* = C_0$, which will be called *bounding feasible designs*. According to (4) and (11), the condition

$$c_{ijkl}\epsilon_{ij}\bar{\epsilon}_{kl} - f_i\bar{u}_i = k^2, \tag{12}$$

where k^2 is a positive constant, is *sufficient* for the volume of bounding feasible designs to be stationary in the neighborhood of the design h. That this condition is also *necessary* is seen when the determination of the optimal design for given compliance is treated as a problem in calculus of variations.

3. Optimal Design of Rotating Disk for Given Edge Displacement

To illustrate the use of the optimality condition (12), consider an axially-symmetric, elastic disk which carries the radial traction T per unit length of the edge $r = a$ and rotates at the constant angular velocity ω about the axis of symmetry. The disk is to be designed for minimum volume subject to the constraint that the radial displacement at the edge $r = a$ should have the prescribed value u_0. The radial displacement u, the strains ϵ_r, ϵ_θ, and the stresses σ_r, σ_θ produced by the centrifugal forces $F = fh$ and the edge traction T are all functions of r only, and $f = \rho\omega^2 r$, where ρ is the density of the disk material.

In view of the constraint on the edge displacement, we introduce a radial traction $\bar{T}$ of unit intensity at the edge $r = a$, so that the prescribed value of the compliance C is $C = 2\pi a u_0$. The fields of radial displacement and radial and circumferential strains and stresses produced by the traction $\bar{T}$ will be denoted by $\bar{u}$, $\bar{\epsilon}_r$, $\bar{\epsilon}_\theta$ and $\bar{\sigma}_r$, $\bar{\sigma}_\theta$.

If Young's modulus and Poisson's ratio are denoted by E and μ, the strains are related to the displacements and stresses by

$$\begin{aligned} \epsilon_r &= du/dr = (1/E)(\sigma_r - \mu\sigma_\theta), \qquad \epsilon_\theta = u/r = (1/E)(\sigma_\theta - \mu\sigma_r), \\ \bar{\epsilon}_r &= d\bar{u}/dr = (1/E)(\bar{\sigma}_r - \mu\bar{\sigma}_\theta), \qquad \bar{\epsilon}_\theta = \bar{u}/r = (1/E)(\bar{\sigma}_\theta - \mu\bar{\sigma}_r) \end{aligned} \tag{13}$$

and must satisfy the compatibility conditions

$$(d/dr)(r\epsilon_\theta) = \epsilon_r, \qquad (d/dr)(r\bar{\epsilon}_\theta) = \bar{\epsilon}_r. \tag{14}$$

Assuming the thickness h of the disk to be known, we may first eliminate the radial displacements and circumferential stresses from (13)–(14) and the equations of radial equilibrium to obtain second-order differential equations for the radial stresses with appropriate boundary conditions. The circumferential stresses may then be expressed in terms of radial stresses and their derivatives. With the dimensionless quantities

$$\begin{gathered} S_i = \sigma_i h/T, \qquad \bar{S}_i = \bar{\sigma}_i h/\bar{T} \qquad (i = r, \theta), \\ R = r/a, \qquad H = h/h(0), \\ \gamma = \rho\omega^2 a^3/Eu_0, \qquad \bar{\gamma} = \rho\omega^2 a^3 h(0)/Ta = \gamma E u_0 h(0)/Ta, \end{gathered} \tag{15}$$

the procedure just described furnishes the following boundary-value problem:

$$R^2 S_r'' + (3 - RH'/H)\, RS_r' - (1 - \mu)(RH'/H)\, S_r + (3 + \mu)\, \bar{\gamma} R^2 H = 0, \tag{16}$$

$$S_\theta = (RS_r)' + \bar{\gamma} R^2 H, \tag{17}$$

$$S_r(0) = S_\theta(0) = \text{Finite}, \tag{18}$$

$$S_r(1) = 1, \tag{19}$$

where R is the independent variable and a prime indicates differentiation with respect to R. The governing equations and boundary conditions for the quantities $\bar{S}_r$ and $\bar{S}_\theta$ are obtained from (16)–(19) by setting $\bar{\gamma} = 0$ and replacing S_r, S_θ by $\bar{S}_r$, $\bar{S}_\theta$.

When displacement and strains in the optimality condition (12) are expressed in terms of stresses by the use of (13), this optimality condition becomes

$$\sigma_r\bar{\sigma}_r + \sigma_\theta\bar{\sigma}_\theta - \mu(\sigma_r\bar{\sigma}_\theta + \bar{\sigma}_r\sigma_\theta) - \rho\omega^2 r^2(\bar{\sigma}_\theta - \mu\bar{\sigma}_r) = Ek^2,$$

where $Ek^2 = 2(1 - \mu)\, \sigma_r(0)\, \bar{\sigma}_r(0)$ because $\sigma_r = \sigma_\theta$ and $\bar{\sigma}_r = \bar{\sigma}_\theta$ at $r = 0$. Accordingly, using the notations (15) and introducing the constant

$$K^2 = 2(1 - \mu)\, S_r(0)\, \bar{S}_r(0), \tag{20}$$

we have

$$H = \sqrt{\{[S_r\bar{S}_r + S_\theta\bar{S}_\theta - \mu(S_r\bar{S}_\theta + \bar{S}_rS_\theta)]/[K^2 + \bar{\gamma}(\bar{S}_\theta - \mu\bar{S}_r)\, R^2/H]\}}. \quad (21)$$

On the other hand, in view of (7)–(8), the constraint $C = 2\pi a u_0$ furnishes

$$h(0) = (Ta/Eu_0) \int_0^1 H^{-1}[S_r\bar{S}_r + S_\theta\bar{S}_\theta - \mu(S_r\bar{S}_\theta + \bar{S}_rS_\theta)]R\, dR. \quad (22)$$

For given $\bar{\gamma}$ (instead of γ), the relative thickness $H(R)$ must be determined to satisfy the optimality condition (21) while the stress resultants S_r, S_θ, $\bar{S}_r$, $\bar{S}_\theta$ are obtained from (16)–(19) and their analogs for the barred stress resultants, and K is found from (20). After the relative thickness and stress resultants are determined, $h(0)$ is calculated from (22) and the thickness h, the stresses σ_r, σ_θ, and the parameter γ follow from (15).

It can readily be verified that, for $\bar{\gamma} = 0$, we have the solution

$$\begin{aligned} H(R) &= 1, \\ S_r(R) = S_\theta(R) = \bar{S}_r(R) &= \bar{S}_\theta(R) = 1, \\ h(0) &= (1-\mu)\, Ta/(Eu_0). \end{aligned} \quad (23)$$

This indicates that, for this special static case, the optimal disk considered here is of uniform strength.

Approximate solutions for sufficiently small $\bar{\gamma}$ can be obtained from a perturbation scheme that starts with the relations (23). One finds

$$\begin{aligned} H(R) &= 1 - \bar{\gamma}[(3+\mu)/4(1+\mu)]\, R^2, \\ S_r(R) &= 1 + \bar{\gamma}[(3+\mu)^2/16(1+\mu)](1-R^2), \\ S_\theta(R) &= 1 + \bar{\gamma}\{R^2 + [(3+\mu)^2/16(1+\mu)](1-3R^2)\}, \\ h(0) &= (Ta/Eu_0)(1-\mu)[1 + \bar{\gamma}(5+3\mu)/8(1+\mu)], \end{aligned} \quad (24)$$

within higher-order terms in $\bar{\gamma}$.

For greater values of $\bar{\gamma}$, the solutions for relative thickness and stress resultants may be obtained by an iterative scheme. Starting from Eq. (23-1) or Eq. (24-1), the relative thickness $H_{n+1}(R)$ at the $(n+1)$th step is calculated from the right-hand side of (21), where the stress resultants are obtained from (16)–(19) and their analogs for the barred stress resultants using the relative thickness $H_n(R)$ at nth step, and K is evaluated from (20). This procedure is repeated until two successive results of H agree within the desired accuracy. This procedure was carried out for $\mu = 0.3$ and the values $\bar{\gamma} = 0.1$, 0.5, and 1, which correspond to $\gamma = 0.132$, 0.529, and 0.887, respectively. A finite-difference method with the step length $\Delta R = 0.02$ was used to solve (16)–(19). Five iterations were required to obtain H to four significant digits starting from Eq. (24-1). It was found that, for $\bar{\gamma} \leqslant 0.1$, the approximate solutions (24) agree with the iterative results within at least two significant digits.

Figure 1 shows typical profiles of optimal disks. The distributions of radial and circumferential stresses for these disks are shown in Fig. 2. It is seen that, for $\bar{\gamma} > 0$, the optimal disks are far from exhibiting uniform strength.

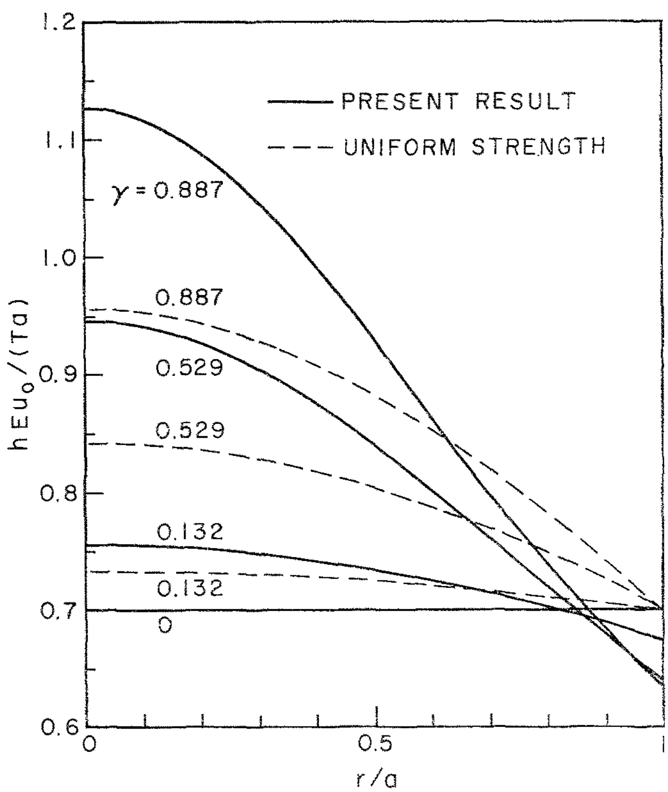

Fig. 1. Thickness profiles of optimal disks and disks of uniform strength for $\mu = 0.3$.

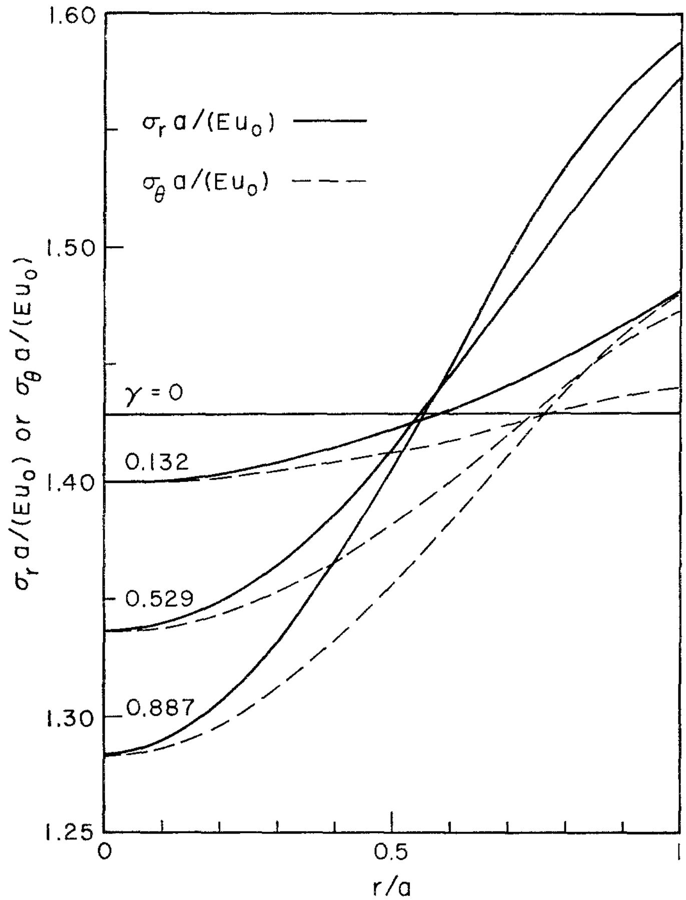

Fig. 2. Stress distributions of optimal disk for $\mu = 0.3$.

4. Comparison of Optimal Disks with Disks of Uniform Strength

In this section, we shall compare the optimal elastic disks obtained in the preceding section with elastic-plastic disks in which the von Mises yield limit is simultaneously reached throughout the disk. The yield condition of von Mises

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta = \sigma_0^2 \tag{25}$$

may be satisfied by setting

$$\sigma_r = \sigma_0[\cos\phi + (1/\sqrt{3})\sin\phi], \qquad \sigma_\theta = \sigma_0[\cos\phi - (1/\sqrt{3})\sin\phi], \tag{26}$$

where $\phi = \phi(r)$ must satisfy the boundary condition

$$\phi(0) = 0, \tag{27}$$

because $\sigma_r = \sigma_\theta$ at $r = 0$.

Substitution of strains in terms of stresses from (13) into the compatibility condition (14) and use of (26) furnish the differential equation

$$\{(1 - \mu) \sin \phi + [(1 + \mu)/\sqrt{3}] \cos \phi\}(d\phi/dr) + [2(1 + \mu)/r\sqrt{3}] \sin \phi = 0, \tag{28}$$

which can be integrated to yield

$$e^{\phi}(r^2 \sin \phi)^{(1+\mu)/[(1-\mu)\sqrt{3}]} = \text{Const.} \tag{29}$$

In view of (27), the constant of integration in (29) vanishes so that

$$\phi(r) = 0. \tag{30}$$

It then follows from (26) that the elastic-plastic disk in which the von Mises yield limit is reached simultaneously throughout the disk satisfies the constraint (1) of uniform strength.

The profile of the disk of uniform strength is (see Refs. 1–2)

$$h(r) = h(0) \exp(-\tfrac{1}{2}\rho\omega^2 r^2/\sigma_0), \tag{31}$$

and the conditions at the edge $r = a$ furnish

$$\sigma_0 = Eu_0/(1 - \mu)a = T/h(a). \tag{32}$$

With γ and R defined as in (15), one finds

$$h(R) = [(1 - \mu)\, Ta/Eu_0] \exp[\tfrac{1}{2}\gamma(1 - \mu)(1 - R^2)], \tag{33}$$

and the volume $\bar{V}$ of the disk of uniform strength is obtained as

$$\bar{V} = (2\pi Ta^3/Eu_0)\{\exp[\tfrac{1}{2}\gamma(1 - \mu)] - 1\}/\gamma. \tag{34}$$

The dashed lines in Fig. 1 show the profiles (33) for the values $\gamma = 0.132$, 0.529, and 0.887, which correspond to the values $\bar{\gamma} = 0.1$, 0.5, and 1.0 considered in Section 3. While the *profiles* of the two types of disk differ appreciably, their *volumes* are found to be practically the same. For $\mu = 0.3$ and the γ-values mentioned above, the disks designed according to Section 3 respectively are 0.3%, 0.5%, and 1.0% lighter than the corresponding disks of uniform strength.

References

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