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On non-integrability of general systems of differential equations

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1. Introduction

In 1983 H. Yoshida [Yos83] suggested a simple and easily verifiable criterion of the integrability of autonomous nonlinear systems of differential equations admitting a quasi-homogeneous group of symmetries. F. Gonzalez-Gascon [Gon88] criticized the Yoshida method and pointed out some imperfections of the proof. He also outlined some ideas with the help of which one could construct a counter-example to the Yoshida theorem. In his later works [Yos87, Yos88, Yos89] Yoshida, in fact, gave up his original method. By means of a rather complicated technique suggested by Ziglin [Zig83], he found a non-integrability criterion for quasi-homogeneous systems of a very special kind. He treated Hamiltonian systems where Hamiltonians were the sum of kinetic and potential energies, and the kinetic energy was proportional to the sum of squared impulses while the potential energy was a homogeneous polynomial. Though some model physical problems are of the above type (a one-dimensional three-body system, the Yang-Mills system, etc.), the domain of applicability of the Yoshida criterion became sufficiently narrow. In particular, one is not able to deal with systems the nature of which is essentially non-Hamiltonian because of the energy dissipation. The author thinks that some ideas of [Yos83] are quite fruitful and can become a ground for further scientific investigations in this direction. The main goal of this work is an attempt to use arithmetic properties of the Kowalevsky exponents to test whether a system of differential equations is integrable or not.

2. Basic lemma

Let us consider an analytic system of differential equations

$$\dot{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u}), \qquad \boldsymbol{u} \in \mathbb{C}^n, \qquad \boldsymbol{u} = (u^1, \dots, u^n) \tag{1}$$

in a neighbourhood of the trivial stationary solution $\boldsymbol{u} = \boldsymbol{0}$ ($\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$).

Let us denote the Jacobi matrix $\partial f/\partial \boldsymbol{u}(\mathbf{0})$ of the vector field $\boldsymbol{g}(\boldsymbol{u})$ at $\boldsymbol{u} = \mathbf{0}$ as A. For simplicity, we assume that A is diagonalizable and that the corresponding operator has already a diagonal form $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

The basic idea. If system (1.1) *has nontrivial integrals analytic in a neighbourhood of a trivial solution* $\boldsymbol{u} = \mathbf{0}$, *then eigen values of the matrix* A *have to satisfy certain resonant conditions.*

Lemma 1. Let $\det A \neq 0$, then if the eigen values $\lambda_1, \ldots, \lambda_n$ of the matrix A do not satisfy any resonant equality of the following type

$$\sum_{j=1}^{n} k_j \lambda_j = 0, \qquad k_j \in \mathbb{N} \cup \{0\}, \qquad \sum_{j=1}^{n} k_j \geq 1 \tag{2}$$

the system of equations (1) dos not have any nontrivial integral analytic in a neighbourhood of $\boldsymbol{u} = \mathbf{0}$.

Proof. We will use the proof by contradiction. Let the opposite statement hold. There is an analytic function $\phi(\boldsymbol{u})$ which is an integral of (1). Then this function has to satisfy the following partial differential equation

$$\left\langle \frac{\partial \phi}{\partial \boldsymbol{u}}(\boldsymbol{u}), \boldsymbol{f}(\boldsymbol{u}) \right\rangle \equiv 0, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ means the standard scalar product in $\mathbb{C}^n$.

Without any loss of generality, we assume that $\phi(\mathbf{0}) = 0$.

Let us expand the function $\phi(\boldsymbol{u})$ into the Maclaurin series

$$\phi(\boldsymbol{u}) = \phi_{(1)}(\boldsymbol{u}) + \phi_{(2)}(\boldsymbol{u}) + \cdots, \tag{4}$$

where $\phi_{(k)}$, $k = 1, 2, \ldots$ are homogeneous polynomials in $\boldsymbol{u}$.

Let us equate all the terms in (3) of the same order with respect to $\boldsymbol{u}$ to zero and consider the first form of the Maclaurin expansion of $\phi(\boldsymbol{u})$

$$\phi_{(1)}(\boldsymbol{u}) = \langle \boldsymbol{p}, \boldsymbol{u} \rangle. \tag{5}$$

Here $\boldsymbol{p} \in \mathbb{C}^n$ is a constant vector.

The corresponding equation for linear terms reads

$$\langle \boldsymbol{p}, A\boldsymbol{u} \rangle = 0 \tag{6}$$

for any $\boldsymbol{u}$.

Therefore, $\boldsymbol{p}$ is an eigen vector of the matrix $A^* = A$ with a zero eigen value, which contradicts the condition that $\det A^* \neq 0$.

Thus, the vector $\boldsymbol{p}$ has to be equal to zero.

We suppose now that we have proved that $\phi_{(1)} \equiv \cdots \equiv \phi_{(k-1)} \equiv 0$. Then it follows from (3) that $\phi_{(k)}$ has to satisfy the following linear partial differential equation

$$\left\langle \frac{\partial \phi_{(k)}}{\partial \boldsymbol{u}}(\boldsymbol{u}), A\boldsymbol{u} \right\rangle = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_{(k)}}{\partial u^j}(\boldsymbol{u}) u^j \equiv 0. \tag{7}$$

This means that the first nonzero term in the above expansion, for instance, the k-th differential of $\phi(\boldsymbol{u})$ at the point $\boldsymbol{u} = \mathbf{0}$ $\phi_{(k)}(\boldsymbol{u}) = D^{(k)}\phi(\mathbf{0}, \boldsymbol{u})$ is an integral of the linearized system

$$\dot{\boldsymbol{u}} = A\boldsymbol{u}. \tag{8}$$

Let us rewrite $\phi_{(k)}$ as a sum of elementary monomials

$$\phi_{(k)}(u^1, \dots, u^n) = \sum_{k_1 + \cdots + k_n = k} \phi_{k_1 \cdots k_n}(u^1)^{k_1} \cdots (u^n)^{k_n}. \tag{9}$$

Then, as follows from (7), a resonant condition of (2) type has to be fulfilled for any nonzero coefficient $\phi_{k_1 \cdots k_n}$ since

$$\sum_{k_1 + \cdots + k_n = k} (k_1\lambda_1 + \cdots + k_n\lambda_n)\phi_{k_1 \cdots k_n}(u^1)^{k_1} \cdots (u^n)^{k_n} \equiv 0, \tag{10}$$

which contradicts the conditions of Lemma 1. The lemma is proved.

Remark 1. The requirement of the analyticity of the vector field $\boldsymbol{f}(\boldsymbol{u})$ can be weakened. If we replace this condition with infinite differentiability, the statement on the non-existence of a nontrivial integral should be treated as the statement on the absence of a *formal* nontrivial integral.

Remark 2. The condition $\det A \neq 0$ can be omitted because, in fact, the requirement of the absence of the resonant relationships (2) contains the above condition.

We will use observations made in this Section to improve the Yoshida criterion. We will show, for instance, that a system of differential equations with a quasi-homogeneous symmetry group does not have "quite nice" integrals if eigen values of the Kowalevsky matrix are not resonant.

3. Quasi-homogeneous and semi-quasihomogeneous systems

Let us consider a system of differential equations smooth in a neighbourhood of the origin $\boldsymbol{x} = \mathbf{0}$

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^n. \tag{11}$$

Definition 1. System (11) is called a quasi-homogeneous one of degree m with exponents $s_1, \dots, s_n, s_1, \dots, s_n \in \mathbb{Z}, m \in \mathbb{N}, m < 1$, if for any $\varrho \in \mathbb{R}^+, \boldsymbol{x} = (x^1, \dots, x^n)$ all the components of the vector field $\boldsymbol{g} = (g^1, \dots, g^n)$ satisfy the following system of equalities

$$g^j(\varrho^{s_1}x^1, \dots, \varrho^{s_n}x^n) = \varrho^{s_j + m - 1}g^j(x^1, \dots, x^n). \tag{12}$$

We also give another definition.

Let $\boldsymbol{H} = \operatorname{diag}(h_1, \ldots, h_n)$ be a diagonal matrix with real elements. We will denote the following diagonal matrix $\operatorname{diag}(\mu^{h_1}, \ldots, \mu^{h_n})$ as $\mu^{\boldsymbol{H}}$ where $\mu \in \mathbb{R}^+$.

Definition 2. We will say that system (11) is quasi-homogeneous with respect to a symmetry group generated by the matrix $\boldsymbol{H}$ if

$$\boldsymbol{g}(\mu^{\boldsymbol{H}}\boldsymbol{x}) = \mu^{\boldsymbol{H}+\boldsymbol{E}}\boldsymbol{g}(\boldsymbol{x}), \tag{13}$$

where $\boldsymbol{E}$ is a unit matrix.

It is easy to see that equality (13) can be obtained from (12) by substituting $\varrho = \mu^{\alpha}$, $\alpha = 1/(m-1)$. It is also obvious that a quasi-homogeneous system is invariant under the following transformation group

$$\boldsymbol{x} \to \mu^{\boldsymbol{H}}\boldsymbol{x}, \qquad t \to \mu^{-1}t, \qquad \boldsymbol{H} = \alpha\boldsymbol{S}, \qquad \boldsymbol{S} = \operatorname{diag}(s_1, \ldots, s_n). \tag{14}$$

We will denote a quasi-homogeneous vector field as $\boldsymbol{g}_m$ bearing in mind its degree.

Let us consider the following construction. We rewrite each component g^j of the right-hand side of system (11) as a Maclaurin sum

$$g^j = \sum g^j_{k_1 \cdots k_n}(x^1)^{k_1} \cdots (x^n)^{k_n}. \tag{15}$$

A vector $\boldsymbol{k} \in \mathbb{R}^n$, $\boldsymbol{k} = (k_1, \ldots, k_n)$ will correspond to each nontrivial monomial of the above expansion.

Definition 3. The total collection of geometric points corresponding to vectors $\boldsymbol{k} \in \mathbb{R}^n$ representing elementary nontrivial monomials of the component g^j is called the j-th Newton diagram $\mathfrak{D}_j$ and its convex hull is called the j-th Newton polyhedron $\mathfrak{P}_j$.

It is easy to notice that if the system of equations (11) is quasi-homogeneous ($\boldsymbol{g} = \boldsymbol{g}_m$) then the corresponding Newton diagrams lie on hyperplanes defined by linear equations

$$s_1k_1 + \cdots + s_j(k_j - 1) + \cdots + s_nk_n = m - 1. \tag{16}$$

Definition 4. We will say that the system of equations (11) is semi-quasi-homogeneous if

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_m(\boldsymbol{x}) + \tilde{\boldsymbol{g}}(\boldsymbol{x}), \tag{17}$$

where $\boldsymbol{g}_m(\boldsymbol{x})$ is a quasi-homogeneous vector field, all the Newton diagrams of which lie on hyperplanes defined by equalities (16) and all the Newton diagrams of the vector field $\tilde{\boldsymbol{g}}(\boldsymbol{x})$ lie either in half-spaces defined by inequalities

$$s_1k_1 + \cdots + s_j(k_j - 1) + \cdots + s_nk_n < m - 1 \tag{18}$$

or in half-spaces defined by inequalities

$$s_1 k_1 + \cdots + s_j (k_j - 1) + \cdots + s_n k_n < m - 1. \tag{19}$$

If inequalities (18) hold, we say that system (11) is positively semi-quasihomogeneous. The quasi-homogeneous cut is usually said to be separated by positive faces of the Newton polyhedrons. In the opposite case (inequalities (19) hold) we will speak about negative semi-quasihomogeneity of the system under consideration. In this case the quasi-homogeneous cut is separated by negative faces of the Newton polyhedron.

If system (11) is semi-quasihomogeneous, then under the action of the quasi-homogeneous transformation group

$$\boldsymbol{x} \to \mu^{\boldsymbol{H}} \boldsymbol{x}, \qquad t \to \mu^{-1} t, \qquad \mu = \varrho^{m-1} \tag{20}$$

it becomes

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_m(\boldsymbol{x}) + \tilde{\boldsymbol{g}}(\boldsymbol{x}, \varrho), \tag{21}$$

where $\tilde{\boldsymbol{g}}(\boldsymbol{x}, \varrho)$ is a formal power series either with respect to ϱ (positive semi-quasihomogeneity) or with respect to ϱ^{-1} (negative semi-quasihomogeneity) without any constant term.

Let the system of equations (11) be semi-quasihomogeneous. First of all we consider its quasi-homogeneous cut

$$\dot{\boldsymbol{x}} = \boldsymbol{g}_m(\boldsymbol{x}). \tag{22}$$

We look for a particular solution of (22) in the quasi-homogeneous ray form

$$\boldsymbol{x}_0(t) = t^{-\boldsymbol{H}} \boldsymbol{c} = t^{-\alpha \boldsymbol{S}} \boldsymbol{c}, \tag{23}$$

where $\boldsymbol{c} \in \mathbb{C}^n$ is a constant vector.

If such a solution exists, the vector $\boldsymbol{c}$ has to satisfy the following algebraic system of equations

$$\boldsymbol{H}\boldsymbol{c} + \boldsymbol{g}_m(\boldsymbol{c}) = \boldsymbol{0}. \tag{24}$$

Let us make the following change of variables

$$\boldsymbol{x} = t^{-\boldsymbol{H}}(\boldsymbol{c} + \boldsymbol{u}) \tag{25}$$

perturbing the particular solution (23).

In these new variables, system (22) reads

$$t\dot{\boldsymbol{u}} = \boldsymbol{K}\boldsymbol{u} + \tilde{\boldsymbol{f}}(\boldsymbol{u}), \tag{26}$$

where

$$\boldsymbol{K} = \boldsymbol{H} + \frac{\partial \boldsymbol{g}_m}{\partial \boldsymbol{x}}(\boldsymbol{c}) \tag{27}$$

is the so-called Kowalevsky matrix and

$$\tilde{f}(u) = Hc + g_m(c + u) - \frac{\partial g_m}{\partial x}(c)u. \tag{28}$$

Obviously, the expansion of the vector field $\tilde{f}(u)$ begins with terms of at least second order.

If we make the logarithmic time change $\tau = \ln t$, the system of equations (26) becomes

$$u' = Ku + \tilde{f}(u), \tag{29}$$

where prime means the derivative with respect to the new "time" τ.

This system of equations is absolutely analogous to (1).

Let us now return to the perturbed system (11). We will consider two different problems depending on the "sign of semi-quasihomogeneity" of (11).

(A) Let system (11) be positively semi-quasihomogeneous. We will search for integrals of this system in the form of formal Maclaurin series

$$\phi(x^1, \ldots, x^n) = \sum_{k_1 \geq 0, \ldots, k_n \geq 0}^{\infty} \phi_{k_1 \cdots k_n}(x^1)^{k_1} \cdots (x^n)^{k_n}. \tag{30}$$

(B) Let system (11) be negatively semi-quasihomogeneous. We will search for polynomial integrals of this system, i.e. for those which have only a finite number of terms in (30).

Let the case (A) take place and the system (11) have a formal nontrivial integral of (30) type then this integral can be rescaled with the aid of the matrix S, i.e. it can be rewritten as follows

$$\phi(x) = \phi_l(x) + \phi_{l+1}(x) + \cdots, \tag{31}$$

where $\phi_{l+i}(x)$ are quasi-homogeneous functions of the degree $l + i$, i.e.

$$\phi_{l+i}(\varrho^S x) = \varrho^{l+i}\phi_{l+i}(x). \tag{32}$$

After changing variables $x \to \mu^H x$, $t \to \mu^{-1}t$, we can discover that in the case of positive semi-quasihomogeneity the system (21) has an integral

$$\phi(x, \varrho) = \phi_l(x) + \varrho\phi_{l+1}(x) + \cdots. \tag{33}$$

This integral exists for any value of ϱ, so the shortened system (22) has to have a quasi-homogeneous integral $\phi(x, 0) = \phi_l(x)$.

Let now the case (B) occur. Then the integral (30) can be rewritten in the following form

$$\phi(x) = \phi_L(x) + \phi_{L+1}(x) + \cdots + \phi_{l-1}(x) + \phi_l(x), \qquad L < l. \tag{34}$$

Then system (21) has an integral

$$\phi(x, \varrho) = \phi_l(x) + \varrho^{-1}\phi_{l-1}(x) + \cdots, \tag{35}$$

which is also defined for any ϱ. Thus, (22) has to have a quasi-homogeneous integral $\phi(\boldsymbol{x}, \infty) = \phi_l(\boldsymbol{x})$.

Let the shortened system (22) have a quasi-homogeneous integral of degree l. We make the change of variables (25). After this transformation, the integral becomes

$$\phi_l(\boldsymbol{x}) = t^{-\alpha l}\phi_l(\boldsymbol{c} + \boldsymbol{u}) = \Phi(u^0, \boldsymbol{u}), \tag{36}$$

where $u^0 = t^{-\alpha}$ is a new auxiliary variable.

The function $\Phi(u^0, \boldsymbol{u})$ is obviously an analytic integral of the augmented system of equations

$$u^{0'} = -\alpha u^0, \qquad \boldsymbol{u}' = \boldsymbol{K}\boldsymbol{u} + \tilde{\boldsymbol{f}}(\boldsymbol{u}). \tag{37}$$

Let the Kowalevsky matrix (27) be diagonalizable and $\lambda_1, \ldots, \lambda_n$ be its eigen values. As follows from Lemma 1, in order that system (37) may not have an analytic integral, it is sufficient that no resonant condition of the following type

$$-k_0\alpha + \sum_{j=1}^{n} k_j\lambda_j = 0, \qquad k_0, k_j \in \mathbb{N} \cup \{0\}, \qquad k_0 + \sum_{j=1}^{n} k_j \geq 1 \tag{38}$$

is fulfilled.

It is not difficult to prove that $\lambda = -1$ is an eigen value of the Kowalevsky matrix [Yos83]. The proof is rather simple. By differentiating (13) with respect to μ and putting $\mu = 1$, we obtain

$$\frac{\partial \boldsymbol{g}_m}{\partial \boldsymbol{x}}(\boldsymbol{x})\boldsymbol{H}\boldsymbol{x} = (\boldsymbol{H} + \boldsymbol{E})\boldsymbol{g}_m(\boldsymbol{x}). \tag{39}$$

Therefore, the vector $\boldsymbol{q} = \boldsymbol{H}\boldsymbol{c}$ is an eigen vector of the Kowalevsky matrix with the eigen value $\lambda = -1$.

Indeed, by using (24), we obtain

$$\boldsymbol{K}\boldsymbol{q} = \boldsymbol{H}\boldsymbol{q} + \frac{\partial \boldsymbol{g}_m}{\partial \boldsymbol{x}}(\boldsymbol{c})\boldsymbol{q} = \boldsymbol{H}^2\boldsymbol{c} + (\boldsymbol{H} + \boldsymbol{E})\boldsymbol{g}_m(\boldsymbol{c}) = -\boldsymbol{H}\boldsymbol{c} = -\boldsymbol{q}. \tag{40}$$

Since $\alpha = 1/(m-1)$, the resonance equalities (38) can be rewritten as follows

$$-1(k_0 + (m-1)k_1) + (m-1)\sum_{j=2}^{n} k_j\lambda_j = 0. \tag{41}$$

If such equalities do not hold, the system does not have the necessary integral.

We reformulate this result in the following form

Theorem 1. Let system (11) be semi-quasihomogeneous. If the Kowalevsky matrix of its quasi-homogeneous cut is diagonalizable and its eigen values $\lambda_1, \ldots, \lambda_n$ do not satisfy any resonant condition

$$\sum_{j=1}^{n} k_j' \lambda_j = 0, \qquad k_j' \in \mathbb{N} \cup \{0\}, \qquad \sum_{j=1}^{n} k_j' \geq 1, \tag{42}$$

then the system of equations under consideration does not have any nontrivial polynomial integral. Moreover, if the system is positively semi-quasihomogeneous, there exist no smooth integrals which can be expanded into nontrivial formal Maclaurin series in a neighbourhood of the origin $\boldsymbol{x} = \mathbf{0}$ too.

4. How can one use the above method if there exists a single nontrivial integral?

We are going to consider now a more "tricky" situation when the system under consideration (11) has a single nontrivial integral. That means that the quasi-homogeneous cut (22) must also have a nontrivial integral. As was shown in [Yos83], the following statement holds.

Lemma 2. Let the truncated system (22) have a quasi-homogeneous integral $\phi_l(\boldsymbol{x})$ of degree l such that $\boldsymbol{p} = \partial\phi_l/\partial\boldsymbol{x}(\boldsymbol{c}) \neq \mathbf{0}$. Then among the Kowalevsky exponents there is the following number $\lambda = \alpha l$.

So, in a general situation at least one resonant relationship of (42) type must be satisfied. Further, without any loss of generality, we suppose that $\lambda = \alpha l$ is the last Kowalevsky exponent λ_n.

By using the modified technique suggested by Yoshida in [Yos87], we find a more refined criterion of non-integrability.

Theorem 2. If system (11) has a nontrivial integral $\phi(\mathbf{x})$ such that its quasi-homogeneous cut $\phi_l(\boldsymbol{x})$ (a quasi-homogeneous integral of the shortened system (22)) is nondegenerate on the "ray direction" $\boldsymbol{c}$ ($\boldsymbol{p} \neq \mathbf{0}$), and the first $n-1$ Kowalevsky exponents $\lambda_1, \ldots, \lambda_{n-1}$ are not in an integer resonance, i.e. no equality

$$\sum_{j=1}^{n-1} k_j'' \lambda_j = 0, \qquad k_j'' \in \mathbb{Z}, \qquad \sum_{j=1}^{n-1} |k_j''| \neq 0 \tag{43}$$

is fulfilled, then any other nontrivial integral $\psi(\boldsymbol{x})$ of (11) is a function of $\phi(\boldsymbol{x})$, i.e.

$$\psi = \mathscr{F}(\phi), \tag{44}$$

where $\mathscr{F}$ is a smooth function.

The proof is based on several lemmas.

Lemma 3. Let system (11) have a nontrivial integral $\phi(\boldsymbol{x})$ and any nontrivial quasi-homogeneous integral $\psi_q(\boldsymbol{x})$ of the cut system (22) be a smooth function of the integral $\phi_l(\boldsymbol{x})$, i.e. $\psi_q = \mathscr{G}(\phi_l)$, then any nontrivial smooth integral $\psi(\boldsymbol{x})$ of the entire system (11) is a smooth function of $\phi(\boldsymbol{x})$ (see (44)).

Proof. With the help of the quasi-homogeneous change of variables $\mathbf{x} \to \mu^{\boldsymbol{H}}\boldsymbol{x}$, $t \to \mu^{-1}t$ let us rewrite system (11) in the following form

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}, \varepsilon) = \boldsymbol{g}_m(\boldsymbol{x}) + \sum_{j=1} \varepsilon^j \boldsymbol{g}_{m+\sigma j}(\boldsymbol{x}), \tag{45}$$

where $\varepsilon = \varrho$, $\sigma = 1$ if the case (A) (positive quasi-homogeneity) takes place and $\varepsilon = \varrho^{-1}$, $\sigma = -1$ in the case (B) (negative quasi-homogeneity), $\boldsymbol{g}_{m+\sigma j}$ are quasi-homogeneous vector fields.

We can also rewrite the integral $\phi(x)$ of system (11) as follows

$$\phi(\boldsymbol{x}, \varepsilon) = \phi_l(\boldsymbol{x}) + \sum_{j=1} \varepsilon^j \phi_{l+\sigma j}(\boldsymbol{x}), \tag{46}$$

where $\phi_{l+\sigma j}(\boldsymbol{x})$ are corresponding quasi-homogeneous functions.

By analogy, after the above change of variables any other smooth integral $\psi(\boldsymbol{x})$ of (11) reads

$$\psi(\boldsymbol{x}, \varepsilon) = \psi_q(\boldsymbol{x}) + \sum_{j=1} \varepsilon^j \psi_{q+\sigma j}(\boldsymbol{x}). \tag{47}$$

Since $\psi_q = \mathscr{G}^{(0)}(\phi_l)$, $\mathscr{G}^{(0)} = \mathscr{G}$, the function

$$\psi^{(1)}(\boldsymbol{x}, \varepsilon) = \psi(\boldsymbol{x}, \varepsilon) - \mathscr{G}^{(0)}(\phi(\boldsymbol{x}, \varepsilon)) \tag{48}$$

is also a smooth integral of system (45). This function is obviously of first order with respect to ε.

Let us consider the following function

$$\psi^{(1)}_{q_1}(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \psi^{(1)}(\boldsymbol{x}, \varepsilon), \tag{49}$$

which is obviously a quasi-homogeneous function of a certain integer degree q_1. This function is an integral of system (45) as $\varepsilon = 0$, i.e. an integral of the truncated system (22). According to the assumptions of the lemma, $\psi^{(1)}_{q_1} = \mathscr{G}^{(1)}(\phi_l)$. So the function

$$\psi^{(2)}(\boldsymbol{x}, \varepsilon) = \psi^{(1)}(\boldsymbol{x}, \varepsilon) - \mathscr{G}^{(1)}(\phi(\boldsymbol{x}, \varepsilon)) \tag{50}$$

is also an integral of (45) which has second order with respect to ε.

By repeating infinitely this process, we obtain that

$$\psi(\boldsymbol{x}, \varepsilon) = \sum_{j=0}^{\infty} \mathscr{G}^{(j)}(\phi(\boldsymbol{x}, \varepsilon)), \tag{51}$$

which is equivalent to the fact that

$$\psi = \mathscr{F}(\phi) \tag{52}$$

for a certain smooth function $\mathscr{F}$. The lemma is proved.

Let us now apply the change of variables (25) to the cut system (22), which results in system (26). By changing the "time" $\tau = \ln t$ and linearizing the system obtained, we face the following linear system of equations

$$\boldsymbol{u}' = \boldsymbol{K}\boldsymbol{u}. \tag{53}$$

This system has obviously two nontrivial nonautonomous integrals

$$\exp(-\alpha l\tau)D^{(1)}\phi_l(\boldsymbol{c}, \boldsymbol{u}) = \exp(-\alpha l\tau)\langle \boldsymbol{p}, \boldsymbol{u}\rangle, \tag{54}$$

since $\boldsymbol{p} = \partial\phi_l/\partial\boldsymbol{x}(\boldsymbol{c}) \neq \boldsymbol{0}$ and

$$\exp(-\alpha q\tau)D^{(k)}\psi_q(\boldsymbol{c}, \boldsymbol{u}) \tag{55}$$

for a certain integer $k \geq 1$.

That is why, if we introduce an auxiliary variable $u^0 = e^{-\alpha\tau}$, the following augmented linear system of equations

$$u^{0'} = -\alpha u^0, \qquad \boldsymbol{u}' = \boldsymbol{K}\boldsymbol{u} \tag{56}$$

has two integrals

$$\Phi(u^0, \boldsymbol{u}) = (u^0)^l\langle \boldsymbol{p}, \boldsymbol{u}\rangle \quad \text{and} \quad \Psi(u^0, \boldsymbol{u}) = (u^0)^q\psi_{(k)}(\boldsymbol{u}), \tag{57}$$

where $\psi_{(k)}$ is a certain homogeneous form of degree k such that

$$\psi_{(k)}(\boldsymbol{u}) = D^{(k)}\psi_q(\boldsymbol{c}, \boldsymbol{u}). \tag{58}$$

Since $\boldsymbol{p}$ is an eigen vector of the Kowalevsky matrix $\boldsymbol{K}$ with an eigen value $\lambda = \alpha l$, and the Kowalevsky matrix $\boldsymbol{K}$ is diagonal, as was agreed earlier, we can assume that $\boldsymbol{p}$ is parallel to the last basis vector $\boldsymbol{e}_n = (0, \ldots, 0, 1)$ which is an eigen vector of $\boldsymbol{K}$ with the eigen value $\lambda_n = \alpha l$. So the first integral Φ of (56) can be rewritten as follows

$$\Phi(u^0, \boldsymbol{u}) = (u^0)^l u^n \tag{59}$$

with the accuracy up to a multiplicative constant.

We should also bear in mind the fact that, as was also agreed earlier, the first eigen value of the Kowalevsky matrix $\boldsymbol{K}$ is equal to $\lambda_1 = -1$ and the corresponding eigen vector $\boldsymbol{q} = \boldsymbol{H}\boldsymbol{c}$ is parallel to the first basis vector $\boldsymbol{e}_1 = (1, \ldots, 0, 0)$.

Lemma 4. Any integral of the augmented linear system of equations (56) which is generated by a quasi-homogeneous integral of system (22) does not depend on the first variable u^1.

Proof. Let us consider the quasi-homogeneous integral $\psi_q(\boldsymbol{x})$ of (22).

Let $\dot{\psi}_q(\boldsymbol{x})$ be the derivative of $\psi_q(\boldsymbol{x})$ calculated via (22) with respect to time

$$\dot{\psi}_q(\boldsymbol{x}) = \left\langle \frac{\partial \psi_q}{\partial x}(\boldsymbol{x}), \boldsymbol{g}_m(\boldsymbol{x}) \right\rangle \equiv 0. \tag{60}$$

Therefore, for any natural number p, by using the Leibnitz rule, we obtain that

$$D^{(p)}\dot{\psi}_q(\boldsymbol{c}, \boldsymbol{u}) = \sum_{i=0}^{p} C_p^i \left\langle D^{(p-i)} \frac{\partial \psi_q}{\partial \boldsymbol{x}}(\boldsymbol{c}, \boldsymbol{u}), D^{(i)}\boldsymbol{g}_m(\boldsymbol{c}, \boldsymbol{u}) \right\rangle \equiv 0. \tag{61}$$

Since the operators D and $\partial/\partial \boldsymbol{x}$ are permutative, by choosing p equal to the first natural number k for which the differential of $\psi_q(\boldsymbol{x})$ calculated at $\boldsymbol{x} = \boldsymbol{c}$ is nontrivial, we can get the following formula

$$D^{(k)}\dot{\psi}_q(\boldsymbol{c}, \boldsymbol{u}) = \left\langle \frac{\partial}{\partial \boldsymbol{u}}(D^{(k)}\psi_q(\boldsymbol{c}, \boldsymbol{u})), \boldsymbol{g}_m(\boldsymbol{c}) \right\rangle = \left\langle \frac{\partial}{\partial \boldsymbol{u}}\psi_{(k)}(\boldsymbol{u}), \boldsymbol{Hc} \right\rangle \equiv 0. \tag{62}$$

Since $\boldsymbol{Hc} = \boldsymbol{q} = q\boldsymbol{e}_1 = (q, 0, \ldots, 0)$,

$$\frac{\partial \psi_{(k)}}{\partial u^1}(\boldsymbol{u}) \equiv 0, \tag{63}$$

which results in the fact that $\Psi(u^0, \boldsymbol{u})$ does not depend on the variable u^1. The lemma is proved.

Lemma 5. Let all the assumptions about the integral $\phi(\boldsymbol{x})$ of system (11) be fulfilled, then if system (11) has an integral functionally independent of $\phi_l(\boldsymbol{x})$, the augmented system (56) has an integral which does not depend on u^n. This integral will have the following form

$$\Psi^*(u^0, \boldsymbol{u}) = (u^0)^r \psi_{(h)}(\boldsymbol{u}) = (u^0)^r \psi_{(h)}(u^2, \ldots, u^{n-1}) \tag{64}$$

for a certain $h \in \mathbb{N}$, where $r \in \mathbb{Z}$ is possibly negative.

Proof. Let us rewrite the integral $\Psi(u^0, \boldsymbol{u})$ in the following form

$$\Psi(u^0, \boldsymbol{u}) = (u^0)^q \sum_{j=0}^{k} \tilde{\psi}_{(k-j)}(u^2, \ldots, u^{n-1})(u^n)^j. \tag{65}$$

Since $(u^0)^l u^n = c$ is an integral of (56) where c is an arbitrary constant, Ψ can be rewritten as follows

$$\Psi(u^0, \boldsymbol{u}) = \sum_{j=0}^{k} (u^0)^{q-lj} \tilde{\psi}_{(k-j)}(u^2, \ldots, u^{n-1}) c^j. \tag{66}$$

Therefore, all the terms of the kind

$$\Psi^*(u^0, \boldsymbol{u}) = (u^0)^{q-lj} \tilde{\psi}_{(k-j)}(u^2, \ldots, u^{n-1}) \tag{67}$$

are nontrivial integrals of (56) if only all the corresponding homogeneous forms $\widetilde{\psi}_{(k-j)}$ are not identically equal to zero. Unfortunately, we cannot stop the proof of the lemma here since the case when

$$\Psi(u^0, \boldsymbol{u}) = a_k(u^0)^q(u^n)^k, \tag{68}$$

where a_k is a certain constant, is also theoretically possible.

To overcome this situation, we consider another integral of (22)

$$\psi_{q_1}^{(1)} = \psi_{q_0}^{(0)} - \frac{a_{k_0}}{k_0!}(\phi_l)^{k_0}, \tag{69}$$

where $k_0 = k$, $q_0 = q$, $\psi_{q_0}^{(0)} = \psi_q$ and q_1 is the degree of the quasi-homogeneous function $\psi_{q_1}^{(1)}$.

The Maclaurin expansion of this integral at the point $\boldsymbol{x} = \boldsymbol{c}$ has to vanish at least up to $(k_0 + 1)$-th order. Hence, the augmented system of linear equations (56) has an integral of the following kind

$$\Psi^{(1)}(u^0, \boldsymbol{u}) = (u^0)^{q_1}\psi_{(k_1)}^{(1)}(\boldsymbol{u}), \tag{70}$$

where $\psi_{(k_1)}^{(1)}(\boldsymbol{u}) = D^{(k_1)}\psi_{q_1}^{(1)}(\boldsymbol{c}, \boldsymbol{u})$ and $k_1 > k_0$. Again, if the corresponding integral of (56) has the form

$$\Psi^{(1)}(u^0, \boldsymbol{u}) = a_{k_1}(u^0)^{q_1}(u^n)^{k_1}, \tag{71}$$

we will consider the following integral of (22)

$$\psi_{q_2}^{(2)} = \psi_{q_1}^{(1)} - \frac{a_{k_1}}{k_1!}(\phi_l)^{k_1} \tag{72}$$

and so on.

As was pointed out in [Yos87], *this process must terminate in a finite number of steps.* In the opposite case there would exist an infinite increasing sequence of positive integers $\{k_j\}_{j=0}^{\infty}$ such that the original integral $\psi_q(\boldsymbol{x})$ could be expressed as follows

$$\psi_q = \sum_{j=0}^{\infty} \frac{a_{k_j}}{k_j!}(\phi_l)^{k_j}. \tag{73}$$

But it would mean that the integral ψ_q is a function of the integral ϕ_l which contradicts to the lemma's conditions. The lemma is proved.

Proof of Theorem 2. If system (22) has a nontrivial quasi-homogeneous integral $\psi_q(\boldsymbol{x})$ functionally independent of $\phi_l(\boldsymbol{x})$, in accordance with Lemmas 4 and 5, there exists a nontrivial homogeneous integral of the augmented system (56) in the following form

$$\Psi^*(u^0, \boldsymbol{u}) = (u^0)^r\psi_{(h)}(u^2, \ldots, u^{n-1}), \tag{74}$$

where the function $\psi_{(h)}(\boldsymbol{u})$ has to satisfy the following partial differential equation

$$-\alpha r\psi_{(h)}(\boldsymbol{u}) + \left\langle \frac{\partial \psi_{(h)}}{\partial \boldsymbol{u}}, \boldsymbol{K}\boldsymbol{u} \right\rangle = -\alpha r\psi_{(h)}(\boldsymbol{u}) + \sum_{j=2}^{n-1} \lambda_j u \frac{\partial \psi_{(k)}}{\partial u^j}(\boldsymbol{u}) = 0. \tag{75}$$

If, as earlier, we rewrite the function $\psi_{(h)}$ as a sum of elementary monomials

$$\psi_{(h)}(u^2, \ldots, u^{n-1}) = \sum_{k_2 + \cdots + k_{n-1} = h} \psi_{k_2 \cdots k_{n-1}} (u^2)^{k_2} \cdots (u^{n-1})^{k_{n-1}}, \quad (76)$$

we realize that for any nonzero coefficient $\psi_{k_2 \cdots k_{n-1}}$ the following resonant equality has to be fulfilled

$$-r + (m-1) \sum_{j=2}^{n-1} k_j \lambda_j = 0, \quad (77)$$

which can be interpreted as a condition of (43) type since $\lambda_1 = -1$. Of course, that contradicts the theorem's condition that the first $n-1$ Kowalevsky exponents are not in an integer resonance. Finally, to completely finish the proof, we should apply Lemma 3. The theorem is proved.

Unfortunately, this theorem is of little use for Hamiltonian systems because in the Hamiltonian case there are always some resonances of a special type between eigen values of the Kowalevsky matrix [Yos83]. V. Kozlov [Koz92] has shown that the above resonances result from the fact that any Hamiltonian system possesses an invariant measure.

5. Examples

1. Let us consider a generalized two-dimensional system of the Volterra-Lotka type

$$\dot{x} = x(\alpha + ax + by), \qquad \dot{y} = y(\beta + cx + dy). \quad (78)$$

This system describes interaction between two communities (populations of animals, branches of industry etc.) [Bar67, GMM71].

Let us find arithmetic conditions which coefficients of (78) should satisfy so that the system under consideration does not have smooth integrals in a neighbourhood of the equilibrium $x = y = 0$.

Of course, this two-dimensional system with analytic right-hand sides cannot show chaotic behaviour and it is integrable in a sense. But nevertheless, it is possible to assert that under some conditions integrals of this system are more or less "bad".

First, as follows from Lemma 1, system (78) has no integrals which can be represented as formal power series if for any $k_1, k_2 \in \mathbb{N} \cup \{0\}$, $k_1 + k_2 \geq 1$

$$k_1 \alpha + k_2 \beta \neq 0. \quad (79)$$

The inequality (79) can be rewritten as follows

$$-\frac{\alpha}{\beta} \notin \mathbb{Q}^+. \quad (80)$$

On the other hand, system (78) is negatively semi-homogeneous. Its cut

$$\dot{x} = x(ax + by), \qquad \dot{y} = y(cx + dy) \tag{81}$$

is a (quasi-)homogeneous system of degree 2 and with exponents $s_x = s_y = 1$.

System (81) has a particular solution of the ray type

$$x = \xi/t, \qquad y = \eta/t, \qquad \xi = \frac{b-d}{ad-cb}, \qquad \eta = \frac{c-a}{ad-cb} \tag{82}$$

if $ad - bc \neq 0$.

The Kowalevsky matrix has obviously the following form

$$\boldsymbol{K} = \begin{pmatrix} 1 + 2a\xi + b\eta & b\xi \\ c\eta & 1 + 2d\eta + c\xi \end{pmatrix}. \tag{83}$$

The corresponding characteristic equation $\det(\boldsymbol{K} - \lambda \boldsymbol{E}) = 0$ has two roots.

$$\lambda_1 = -1, \qquad \lambda_2 = \lambda = \frac{(b-d)(c-a)}{ad-cb}. \tag{84}$$

Therefore, according to Theorem 1, the system of equations under consideration has no polynomial integral, if

$$k_1 \lambda \neq k_2. \tag{85}$$

The inequality (85) can be rewritten as follows

$$\frac{(b-d)(c-a)}{ad-cb} \notin \mathbb{Q}^+. \tag{86}$$

2. Let us consider a perturbed oregonator model [Tys78].

$$\begin{aligned} \dot{x} &= \alpha(y - xy + x - \varepsilon xz - gx^2) \\ \dot{y} &= \alpha^{-1}(-y - xy + fz) \\ \dot{z} &= \beta(x - \varepsilon xz - z). \end{aligned} \tag{87}$$

This system of equations describes a hypothetical chemical reaction of the Belousov-Zhabotinsky type where variables x, y, z mean concentrations of reagents. First, the unperturbed problem ($\varepsilon = 0$) was investigated [FN74]. It turned out that properties of the perturbed system were completely different. Let us find arithmetic relationships on the system's coefficients guaranteeing that the system under consideration (87) has no integrals represented as nontrivial formal Maclaurin series.

It is just impossible to obtain explicit formulas for eigen values of the linearized problem. Nevertheless, if all the eigen values have either strictly positive or strictly negative real parts, there is no integral which can be presented in the form of Maclaurin series. To obtain guaranteeing conditions, we should apply the Routh-Hurwitz criterion (see, e.g. [Che61]). We are not going to give precise formulas here.

Let us concentrate on the modified Yoshida method. System (87) is negatively semi-quasihomogeneous. Its cut can be written as follows

$$\begin{aligned} \dot{x} &= -\alpha x(y+gx) \\ \dot{y} &= -\alpha^{-1}xy \\ \dot{z} &= \beta x(1-\varepsilon z). \end{aligned} \tag{88}$$

This system is quasi-homogeneous of degree 2 with exponents $s_x = s_y = 1, s_z = 0$. It is easy to find a particular solution of the ray type of system (88).

$$\begin{aligned} &x = \xi/t, \qquad y = \eta/t, \qquad z = \zeta, \\ &\xi = \alpha, \qquad \eta = \alpha^{-1} - \alpha g, \qquad \zeta = \varepsilon^{-1}. \end{aligned} \tag{89}$$

After some simple calculations we obtain the following expressions for the elements of the Kowalevsky matrix

$$\boldsymbol{K} = \begin{pmatrix} -g\alpha^2 & -\alpha^2 & 0 \\ -\alpha^{-2}+g & 0 & 0 \\ 0 & 0 & -\alpha\beta\varepsilon \end{pmatrix}. \tag{90}$$

This matrix has obviously the following eigen values

$$\lambda_1 = -1, \qquad \lambda_2 = 1 - g\alpha^2, \qquad \lambda_3 = -\alpha\beta\varepsilon. \tag{91}$$

According to Theorem 1, the system of equations (87) does not have any polynomial integrals if there is no resonant equality of the kind

$$-k_1 + (1 - g\alpha^2)k_2 - \alpha\beta\varepsilon k_3 = 0, \tag{92}$$

where $k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}, k_1 + k_2 + k_3 \geq 1$.

It follows from (92) that for fixed g, α, β one can find a set of small numbers ε such that there will be no resonance of the above type.

3. As a nontrivial example illustrating Theorem 2, let us consider a problem of integrability of the Euler-Poincaré equations on Lie algebras [Arn66] studied in different branches of mathematical physics. Those equations are also often called the Euler-Arnold equations. They are a quite natural generalization of the famous Euler equations describing dynamics of a rigid body with a fixed point when no external forces act. The Euler-Poincaré equations can be written as follows

$$\dot{m}_k = C^j_{ik}\omega^i m_j. \tag{93}$$

Here standard tensor notations are used when the summation with respect to repeating indexes has to be done. The constant numbers C^j_{ik} are the so-called structural constants of a finite-dimensional Lie algebra $\mathfrak{g}$, $\omega = (\omega^1, \ldots, \omega^n)$ are n-dimensional vectors of "angular velocities" connected with co-vectors of "moments" $\boldsymbol{m} = (m_1, \ldots, m_n)$ as follows

$$m_j = I_{ij}\omega^i, \tag{94}$$

where I_{ij} is a positive definite symmetric tensor analogous to the inertia tensor for a usual rigid body. In this usual situation $\mathfrak{g} = \mathfrak{so}(3)$.

Equations (93) always have an "energy integral":

$$T = \frac{1}{2} I_{ij}\omega^i\omega^j. \tag{95}$$

As usual there appears a question whether any additional integrals of (93) exist. A lot of works is devoted to the above problem (see, e.g., the monograph [FT88] and the literature cited therein). V. Kozlov [Koz88] has studied a related problem whether system (93) possesses an integral invariant (invariant measure with an infinitely smooth positive density). For low dimensions he gave an exhausting answer to the question stated. It turns out that for $n = 3$ only solvable Lie algebras can provide us with examples of the Euler-Poincaré systems without integral invariants. It goes without saying that such algebras are also suspicious as being the cause of non-integrability.

Thus, let us confine ourselves to the case of $n = 3$ and solvable algebras.

According to [Koz88], there is a canonical basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such that the only nonzero structural constants C^j_{ik} are:

$$\begin{aligned} C^1_{13} &= -C^1_{31} = \alpha, & C^2_{13} &= C^2_{31} = \beta \\ C^1_{23} &= -C^1_{32} = \gamma, & C^2_{23} &= C^2_{32} = \delta. \end{aligned} \tag{96}$$

We will also suppose that

$$\alpha\delta - \beta\gamma \neq 0. \tag{97}$$

To preserve the analogy with the rigid body dynamics, let us denote the angular velocity as $\omega = (p, q, r)$. For the inertia tensor $\boldsymbol{I}$ and its inverse $\boldsymbol{I}^{-1}$, we introduce the following notations:

$$\boldsymbol{I} = \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix} \qquad \boldsymbol{I}^{-1} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}. \tag{98}$$

We denote the angular momentum as $\boldsymbol{m} = (x, y, z)$.

Then the corresponding Euler-Poincaré equations read

$$\begin{aligned} \dot{x} &= -r(\alpha x + \beta y) \\ \dot{y} &= -r(\gamma x + \delta y) \\ \dot{z} &= p(\alpha x + \beta y) + q(\gamma x + \delta y), \end{aligned} \tag{99}$$

where $p = ax + dy + ez$, $q = dx + by + fz$, $r = ex + fy + cz$.

The energy integral can be written as follows

$$T = \frac{1}{2}(xp + yq + zr)$$
$$= \frac{1}{2}(Ap^2 + Bq^2 + Cr^2 + 2Dpq + 2Epr + 2Fqr) \qquad (100)$$
$$= \frac{1}{2}(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz).$$

System (99) is homogeneous with quadratic right/hand sides so it can be treated as quasi-homogeneous and semi-quasihomogeneous. Integral (100) is, of course, nondegenerate on any ray direction $\boldsymbol{c}$.

It is more or less easy to prove that equations (99) have the following particular rectilinear solution

$$x = \xi/t, \qquad y = \eta/t, \qquad z = \zeta/t, \qquad (101)$$

where ξ, η, ζ are, strictly speaking, complex numbers $|\xi|^2 + |\eta|^2 + |\zeta|^2 \neq 0$.

But it is difficult to find values of ξ, η, ζ explicitly. So, to calculate the Kowalevsky exponents, we have to do the following technical trick. Generally, the Kowalevsky matrix reads as follows

$$\boldsymbol{K} = \begin{pmatrix} 1 - eX - \alpha w & -fX - \beta w & -cX \\ -eY - \gamma w & 1 - fY - \delta w & -cY \\ aX + dY + \alpha u + \gamma v & dX + bY + \beta u + \delta v & 1 + eX + fY \end{pmatrix}, \qquad (102)$$

where the following notations are introduced

$$u = a\xi + d\eta + e\zeta, \qquad v = d\xi + b\eta + f\zeta, \qquad w = e\xi + f\eta + c\zeta,$$
$$X = \alpha\xi + \beta\eta, \qquad Y = \gamma\xi + \delta\zeta. \qquad (103)$$

We know *a priori* two roots of its characteristic polynomial $\Delta_K(\lambda)$

$$\lambda_1 = -1, \qquad \lambda_3 = 2. \qquad (104)$$

The second one appears due to the existence of the nondegenerate homogeneous integral (100) of degree 2. Therefore, the quadratic polynomial $P(\lambda) = \lambda^2 - \lambda - 2$ must divide $\Delta_K(\lambda)$. We can easily calculate the quotient of $\Delta_K(\lambda)$ divided by $P(\lambda)$

$$Q(\lambda) = \lambda - 2 + (\alpha + \delta)w. \qquad (105)$$

Hence,

$$\lambda_2 = \lambda(w) = 2 - (\alpha + \delta)w, \qquad (106)$$

and we only need to calculate the magnitude w.

To solve this problem, we should notice that equations for ξ, η have an "almost" closed form

$$\begin{aligned} &(1-\alpha w)\xi - \beta w\eta = 0 \\ &-\gamma w\xi + (1-\delta w)\eta = 0. \end{aligned} \tag{107}$$

The above system of linear equations has nontrivial solutions if and only if its determinant is equal to zero. Therefore,

$$w = \frac{1}{2}\frac{\alpha+\delta \pm \sqrt{(\alpha-\delta)^2 + 4\beta\gamma}}{\alpha\delta - \beta\gamma}. \tag{108}$$

If

$$\lambda(w) \notin \mathbb{Q}, \tag{109}$$

where the magnitude w is given by (108), there is no resonance of (43) type, and any smooth integral of (99) which can be expanded into formal power series with respect to x, y, z or p, q, r is functionally dependent on T.

Let us consider two interesting particular cases. Let

$$\alpha = \delta = 0. \tag{110}$$

Then

$$\lambda_2 = 2 \tag{111}$$

and system (99) becomes suspicious as being integrable.

As follows from (111), there may exist an additional quadratic integral. In this case it is evident

$$F = \frac{1}{2}(\gamma x^2 - \beta y^2). \tag{112}$$

It is interesting to notice that as was showed in [Koz88], if $\beta\gamma < 0$, the system under consideration has an integral invariant (the algebra $\mathfrak{g}$ is unimodular). If, vice versa, $\beta\gamma > 0$, there are no integral invariants but there exists an invariant measure, the density of which has arbitrarily large (but finite!) order of smoothness.

Let us further suppose that

$$\beta = \gamma = 0. \tag{113}$$

In this case

$$\lambda_2 = 1 - \frac{\alpha}{\delta} \quad \text{or} \quad 1 - \frac{\delta}{\alpha}, \tag{114}$$

and the condition to non-integrability is

$$\frac{\alpha}{\delta} \notin \mathbb{Q}. \tag{115}$$

As follows from [Koz88], if $\alpha\delta < 0$ the Euler-Poincaré system (99) does not have an invariant measure even with a summable density.

5. Concluding remarks

Finally, let us briefly discuss the obtained conditions of non-integrability and compare them with the Yoshida criterion. According to Yoshida, if at least one Kowalevsky exponent was not rational, the quasi-homogeneous system under consideration was not algebraically integrable. The above statement allows complex analytical interpretation connected with Ziglin's ideas [Zig83] that the branching of solutions in the complex plane is an obstacle to integrability. Let us consider the Fuchsian system of equations obtained by means of the linearization of system (26)

$$t\dot{\boldsymbol{u}} = \boldsymbol{K}\boldsymbol{u}. \tag{116}$$

The monodromy operator acting on the functional space of solutions of (116) has the following matrix:

$$\boldsymbol{M} = \exp(2\pi i\boldsymbol{K}), \tag{117}$$

which can not be a rational root of the unit matrix $\boldsymbol{E}$ if the Yoshida conditions hold. This means that there exist solutions of (116) with an infinitely sheeted Riemann surface.

As follows from lemma 2, any $n-1$ quasi-homogeneous integrals of the quasi-homogeneous system (22) must be functionally dependent at the point $\boldsymbol{x} = \boldsymbol{c}$ if the Yoshida conditions hold. But it is not clear enough whether this "domain" of dependence can be significantly extended. The conditions of Theorems 1 and 2 require much more from the Kowalevsky exponents, which allows us to obtain those precise assertions on non-integrability.

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Abstract

The article is aimed at finding an algebraic criterion of non-integrability of non-Hamiltonian systems of differential equations. The main idea is to use the so-called Kowalevsky exponents to reveal whether the system under consideration is integrable or not. The method used in this article is based on previous works by H. Yoshida. The article suggests improving the above technique in such a way that it can be applied to a wider class of differential equations.

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