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Drag force and virtual mass of a cylindrical porous shell in potential flow

By Henry Power, Instituto de Mecánica de Fluídos, Facultad de Ingeniería, Universidad Central de Venezuela, and Peder A. Tyvand, Dept of Physics and Meteorology, Agricultural University of Norway, N-1432 Aas-NLH, Norway

1. Introduction

D'Alembert's Paradox states that if a steady uniform potential flow goes around a non-permeable arbitrary rigid body, then there would be no total force acting on the surface of the body. Power et al. (1984) solved the problem of potential flow past a porous body of arbitrary shape with constant permeability K_0, as well as the interior flow on the corresponding porous media. This interior flow was represented as a viscous potential flow with the corresponding pressure related to the seepage velocity by Darcy's Law. The solution of these flows was found by means of a pair of non-linear Fredholm integral equations of the second kind. A formal solution of the mentioned non-linear integral equations was given in terms of the solution of certain linear integral equation when the dimensionless parameter $K_* = \varrho K_0 V/\mu R_1$ is small; here μ is the fluid viscosity, ϱ the fluid density, V is the magnitude of the uniform velocity at infinity and R_1 is a characteristic radius of the arbitrary body, a similar dimensionless parameter was introduced by Chwang and Dong (1984) to wave dissipation due to a porous plate.

The exterior potential ϕ_1 was expressed as a linear combination of two auxiliary potential functions ϕ_0 and ϕ_1' as

$$\phi_1 = \phi_0 + K_* \phi_1' \tag{1.1}$$

in which ϕ_0 is the usual potential function for the flow around a non-permeable body of the same geometrical configuration as the given porous body, and ϕ_1' is the correction due to the fact that the body is porous.

The total force acting upon the porous body due to the uniform exterior potential flow was found to be:

$$F = \varrho K_* \int_S \nabla \phi_0 \frac{\partial \phi_1'}{\partial n} \, dS + O(K_*^2) \tag{1.2}$$

in which S is the surface of the body, and $\boldsymbol{n}$ is the normal unit vector directed outwardly from the body.

Power et al. (1984) solved the case of the two dimensional uniform flow around a porous circular cylinder of radius $R = R_1$ as an example for verification of the integral equation method, and found that in this case the flow exerts a drag force on the porous cylinder equal to:

$$\boldsymbol{F} = \frac{2\pi\varrho K_0}{\nu} V^3 \boldsymbol{e}_x + O(K_*^2) \tag{1.3}$$

in which $\boldsymbol{e}_x$ is the direction of the uniform flow at infinity. It can be noted that this force is independent of the cylinder size. Regarding the nice boundary geometry of the above mentioned case, its solution can be found in an elementary way using cylindrical Harmonic functions. Another case that can be solved without recourse to the integral equation formulation is the uniform potential flow past a porous sphere, Power and Garcia (1986) solved this case using spherical harmonic functions and found that the exterior flow exerts a drag force on the porous sphere which is linearly dependent on the radius of the sphere

$$\boldsymbol{F} = \frac{9}{16} \frac{\pi R_1 \varrho K_0}{\nu} V^3 \boldsymbol{e}_x + O(K_*^2). \tag{1.4}$$

Recently Power et al. (1990) solved the problem of uniform flow past a porous cylinder with a core of different permeability, and in particular give the solution for the case of a hollow core, whose limiting case when the thickness of the porous ring is very small yields the following expression for the total force exerted by the uniform flow upon the circular cylinder:

$$F_x = \frac{\varrho \pi V^3 K}{\nu \varepsilon}$$

where the thickness of the porous ring is $d = \varepsilon R$ with $\varepsilon \ll 1$ and K is the permeability of the porous ring. As in the previous work $K_* = \varrho V K / \mu d$ was assumed to be very small.

Here we will study the two dimensional potential flow due to a circular cylinder in motion relative to an unbounded fluid in terms of the dimensionless parameter $K_* = \varrho V K / \mu 2d$, with "$d$" as the characteristic shell thickness. The full nonlinear hydrodynamic problem, for arbitrary K_*, is solved by Fourier expansion of Green's theorem. For homogeneous porous shells, a maximal drag force occurs at the value 0.433 for the shell parameter, but the virtual mass is a monotonous function of the shell parameter. For an inhomogeneous shell, we have found a maximal value for the virtual mass which is 5% above the value for a rigid cylinder. Some of the results may be relevant to offshore engineering, especially in connection with porous coating of platform legs to reduce the total force.

2. Mathematical formulation and solution procedure

Let us consider a circular porous cylindrical shell of radius R, with physical permeability K, submerged in a uniform potential flow at infinity. We shall normalize all variables according to the following scales:

$$x' = \frac{x}{R}, \quad u'_k = \frac{u_k}{V}, \quad p'_k = \frac{2p_k}{\varrho V^2} \quad \text{and} \quad \phi'_k = \frac{\phi_k}{VR}$$

with $k = i, e$ for the interior and exterior regions respectively, here ϱ is the fluid density and V is the magnitude of the fluid velocity at infinity.

The potential function ϕ_e describing the flow in the unbounded region Ω_e, exterior to the shell, satisfies the Laplace equation in its dimensionless form, and the following asymptotic conditions:

$$\nabla^2 \phi_e = 0 \quad \text{for all } r > 1 \tag{2.1-a}$$

$$\lim_{r \to \infty} \nabla \phi_e = e_x \tag{2.1-b}$$

where $r = (x_1 + x_2)^{1/2}$, and (x_1, x_2) are cartesian coordinates with fixed origin "0" chosen inside the circular shell, here for convenience the primes have been dropped in the dimensionless variables. Since the problem under consideration deals with a fluid of constant density ϱ, in an enclosed system, without free surface, the dynamic pressure p_e is given by Bernoulli's Law as:

$$p_e = 1 - (\nabla \phi_e)^2 = 1 - \left[\left(\frac{\partial \phi_e}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi_e}{\partial \theta} \right)^2 \right] \quad \text{for all } r > 1. \tag{2.1-c}$$

Part of the flow in Ω_e seeps through the shell into the bounded region Ω_i interior to the cylinder. The potential function ϕ_i describing the interior flow satisfies the following equations:

$$\nabla^2 \phi_i = 0 \quad \text{for all } r < 1 \tag{2.2-a}$$

and

$$p_i = C - (\nabla \phi_i)^2 = C - \left[\left(\frac{\partial \phi_i}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi_i}{\partial \theta} \right)^2 \right] \quad \text{for all } r < 1 \tag{2.2-b}$$

where C is an unknown dimensionless constant to be found.

At the shell surface, we have to satisfy the normal velocity matching condition:

$$\frac{\partial \phi_e}{\partial r} = \frac{\partial \phi_i}{\partial r} \quad \text{at } r = 1 \tag{2.3}$$

and Taylor's (1956) pressure jump condition, commonly known as the linear discharge law, which states that the normal velocity of the fluid at the porous thin-shell is linearly proportional to the pressure difference between the inner and the outer regions, such pressure jump condition can be written in a dimensionless form as:

$$\frac{\partial \phi_e}{\partial r} = K_*(p_i - p_e) \quad \text{at } r = 1 \tag{2.4}$$

here K_* is a dimensionless parameter equal to $\varrho VK/2\mu d$, where μ is the fluid viscosity and d is a characteristic shell thickness.

Baicorov (1951, 1952) solved the problem of uniform flow past porous circular ring of small thickness, h, and constant permeability K, for the cases of linear and quadratic discharge laws. Contrasting with the present problem, he assumes that the limiting value of the angular velocity at the interior porous wall is zero and then the interior pressure at the porous wall is just $p_i = C - (\partial \phi_i / \partial r)^2_{r=1}$, equation (6) in Baicorov (1951) paper. Substituting this interior pressure and the exterior one into the discharge law will give a relationship between the exterior radial and angular velocities at the porous wall, instead of a relation between the exterior radial velocity at the porous wall and the jump between the limiting value of the angular velocities at the wall coming from the exterior and the interior regions, as we will find below. It is important to point out that the Baicorov's formulation does not permit the interior flow as a solution of the Laplace' equation, since his method allows to find the exterior potential, then the interior radial velocity at the porous wall is determined by the normal velocity matching condition at the wall. Therefore, in his case we will have an interior flow with prescribed radial and angular velocities at the boundary, which over-determines the boundary condition for a potential problem.

Substituting equations (2.1-c) and (2.2-b) into equation (2.4) and using the normal velocity matching condition (2.3), we found that the pressure matching condition can be written as:

$$\frac{\partial \phi_e}{\partial r} = K_*\left[\left(\frac{\partial \phi_e}{\partial \theta}\right)^2 - \left(\frac{\partial \phi_i}{\partial \theta}\right)^2 + C - 1\right] \quad \text{at } r = 1. \tag{2.5}$$

From the non-flux condition

$$\int_\Gamma \frac{\partial \phi_e}{\partial r}\, d\sigma = \int_0^{2\pi} \left(r \frac{\partial \phi_e}{\partial r}\right)_{r=1} d\theta = 0 \tag{2.6}$$

where Γ is the circular curve of radius $r = 1$, we obtain the following relation between the constant C and the two potential functions ϕ_e and ϕ_i:

$$C = 1 - \frac{1}{2\pi} \int_0^{2\pi} \left[\left(\frac{\partial \phi_e}{\partial \theta}\right)^2 - \left(\frac{\partial \phi_i}{\partial \theta}\right)^2\right]_{r=1} d\theta. \tag{2.7}$$

In this way, equation (2.5) becomes:

$$\frac{\partial \phi_e}{\partial r} = K_* \left\{ \left[\left(\frac{\partial \phi_e}{\partial \theta} \right)^2 - \left(\frac{\partial \phi_i}{\partial \theta} \right)^2 - \frac{1}{2\pi} \int_0^{2\pi} \left[\left(\frac{\partial \phi_e}{\partial \theta} \right)^2 - \left(\frac{\partial \phi_i}{\partial \theta} \right)^2 \right] d\theta \right] \right\} \quad \text{at } r = 1. \tag{2.8}$$

Let us express the exterior potential ϕ_e in terms of the perturbed potential ϕ'_e, thus:

$$\phi_e = r \cos \theta + \phi'_e \tag{2.9-a}$$

where

$$\nabla^2 \phi'_e = 0 \quad \text{for all } r > 1 \tag{2.9-b}$$

and

$$\lim_{r \to \infty} \phi'_e = 0. \tag{2.9-c}$$

We now can use Green's integral representation formulae for the potential ϕ'_e and ϕ_i for the regions exterior and interior to the cylindrical shell (see Jawson and Symm (1977) page 57) for a fixed point $p \in \Omega_e$:

$$k + \int_\Gamma \phi'_e(Q) \frac{\partial}{\partial n} \log|p - Q| \, d\sigma_Q - \int_\Gamma \frac{\partial}{\partial n} (\phi'_e(Q)) \log|p - Q| \, d\sigma_Q = 2\pi \phi'_e(p) \tag{2.10-a}$$

where k is a constant accounting for the non-flux condition of ϕ'_e, and for a fixed point $p \in \Omega_i$

$$\int_\Gamma \phi_i(Q) \frac{\partial}{\partial n} \log|p - Q| \, d\sigma_Q - \int_\Gamma \frac{\partial}{\partial n} (\phi_i(Q)) \log|p - Q| \, d\sigma_Q = -2\pi \phi_i(p). \tag{2.10-b}$$

In the above two formulae $Q \in \Gamma$. The difference in sign between equations (2.10-a) and (2.10-b) comes from the orientation of the normal vector $\boldsymbol{n}$ with respect to the domains Ω_e and Ω_i, here the normal vector is outwardly directed from Γ.

A similar approach was used by Baicorov to represent the exterior potential, instead of the complete Green's formulae, he used a single layer potential, whose unknown density is found to be proportional to the square of the exterior angular velocity at the porous wall.

Using the well known continuity and discontinuity property across the curve Γ of the single-layer and double-layer potentials respectively, the above equations yield the following boundary formulae for a fixed point $p \in \Gamma$:

$$k + \int_\Gamma \phi'_e(Q) \frac{\partial}{\partial n} \log|p - Q| \, d\sigma_Q - \int_\Gamma \frac{\partial}{\partial n} (\phi'_e(Q)) \log|p - Q| \, d\sigma_Q$$
$$= \pi\phi'_e(p) \qquad (2.11\text{-a})$$

and

$$\int_\Gamma \phi_i(Q) \frac{\partial}{\partial n} \log|p - Q| \, d\sigma_Q - \int_\Gamma \frac{\partial}{\partial n} (\phi_i(Q)) \log|p - Q| \, d\sigma_Q = -\pi\phi_i(p). \qquad (2.11\text{-b})$$

Equations (2.11-a, b) can be simplified considerably when the curve Γ is a circle of radius $r = 1$. Thus, if $p = (\cos\theta, \sin\theta)$ and $Q = (\cos\alpha, \sin\alpha)$ are points on the circle $r = 1$, we have that equations (2.11-a, b) become (see Jaswon and Symm (1977) page 261):

$$k - \frac{1}{2}\int_0^{2\pi} \phi'_e(\alpha)\, d\alpha - \int_0^{2\pi} \left(\frac{\partial}{\partial r}\phi'_e(\alpha)\right)_{r=1} \log\left\{2 \sin \frac{|\theta - \alpha|}{2}\right\} d\alpha = \pi\phi'_e(\theta) \qquad (2.12\text{-a})$$

and

$$-\frac{1}{2}\int_0^{2\pi} \phi_i(\alpha)\, d\alpha - \int_0^{2\pi} \left(\frac{\partial}{\partial r}\phi_i(\alpha)\right)_{r=1} \log\left\{2 \sin \frac{|\theta - \alpha|}{2}\right\} d\alpha = -\pi\phi_i(\theta). \qquad (2.12\text{-b})$$

Differentiating (2.12-a, b) with respect to θ, bearing in mind that the first integrals in (2.12-a, b) are constant, we found

$$\frac{1}{2\pi}\int_0^{2\pi} \left(r\frac{\partial}{\partial r}\phi'_e(\alpha)\right)_{r=1} \cot\frac{(\theta - \alpha)}{2}\, d\alpha = \left(\frac{\partial}{\partial\theta}\phi'_e\right)_{r=1} \qquad (2.13\text{-a})$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \left(r\frac{\partial}{\partial r}\phi_i(\alpha)\right)_{r=1} \cot\frac{(\theta - \alpha)}{2}\, d\alpha = -\left(\frac{\partial}{\partial\theta}\phi_i\right)_{r=1}. \qquad (2.13\text{-b})$$

Equations (2.13-a, b) are Hilbert's integral formulae connecting the boundary value of a pair of conjugate harmonic functions at $r = 1$, internal and external to unit circle respectively (see Kanwal (1971) page 184).

Substituting the decomposition given by equation (2.9-a) into equation (2.13-a), we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} \left(r\frac{\partial}{\partial r}\phi_e(\alpha)\right)_{r=1} \cot\frac{(\theta - \alpha)}{2}\, d\alpha = 2\sin\theta + \left(\frac{\partial}{\partial\theta}\phi_e\right)_{r=1} \qquad (2.13\text{-c})$$

where we have used the following relation coming from Hilbert's integral formulae for any exterior harmonic function of the form $\phi = r^{-n}\begin{pmatrix}\cos(n\theta)\\ \sin(n\theta)\end{pmatrix}$, at the unit circle:

$$\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\cos(n\alpha)\\ \sin(n\alpha)\end{pmatrix}\cot\frac{(\theta-\alpha)}{2}\,d\alpha = \pm\begin{pmatrix}\sin(n\theta)\\ \cos(n\theta)\end{pmatrix}. \tag{2.14}$$

From equations (2.13-b) and (2.13-c) and the normal velocity matching condition (2.3), we found the following jump condition between the tangential derivative of the exterior and interior potentials, ϕ_e and ϕ_i, at the cylindrical curve:

$$\frac{\partial\phi_e}{\partial\theta} = -\left(2\sin\theta + \frac{\partial\phi_i}{\partial\theta}\right) \quad \text{at } r = 1. \tag{2.15}$$

Substituting (2.15) and (2.3) into (2.8), we found:

$$\frac{\partial\phi_i}{\partial r} = 4K_*\left\{\sin^2\theta + \sin\theta\,\frac{\partial\phi_i}{\partial\theta} - \frac{1}{2} - \frac{1}{2\pi}\int_0^{2\pi}\sin\theta\,\frac{\partial\phi_i}{\partial\theta}\,d\theta\right\} \quad \text{at } r = 1. \tag{2.16}$$

It is interesting to observe that the jump in the angular velocity given by (2.15) transforms the nonlinear relation (2.8) into a linear one. This simplification cannot be found in the case when the interior angular velocity is neglected, and therefore equation (2.8) yields a nonlinear relation between the exterior radial and angular velocities.

Finally, substituting (2.13-b) into the above equation, we found the following Fredholm integral equation of the second kind for the unknown density $(\partial\phi/\partial r)$ at $r = 1$:

$$\begin{aligned}4K_*\left(\sin^2\theta - \frac{1}{2}\right) = {} & \frac{\partial}{\partial r}\phi_i(\theta) + 4K_*\left\{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial r}\phi_i(\alpha)\right)\sin\theta\cot\frac{(\theta-\alpha)}{2}\,d\alpha\right.\\ & -\frac{1}{2\pi}\int_0^{2\pi}\sin\theta\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial r}\phi_i(\alpha)\right)\right.\\ & \left.\left.\times\cot\frac{(\theta-\alpha)}{2}\,d\alpha\right)d\theta\right\} \quad \text{at } r = 1\end{aligned} \tag{2.17}$$

which can be written as follows, after using the relation given by equation (2.14) in the second integral:

$$\begin{aligned}4K_*\left(\sin^2\theta - \frac{1}{2}\right) = {} & \frac{\partial}{\partial r}\phi_i(\theta) + \frac{4K_*}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial r}\phi_i(\alpha)\right)\\ & \times\left[\sin\theta\cot\frac{(\theta-\alpha)}{2} - \cos\alpha\right]d\alpha \quad \text{at } r = 1.\end{aligned} \tag{2.18}$$

An integral equation similar to the above, but nonlinear, was found by Baicorov. In his (1952) paper, he found an iterative solution of the nonlinear integral equation for the case of small parameter $\lambda = \varrho K/2R^2$, and gives an explicit expression up to order λ^6.

Without lost of generality, we can write the internal potential as:

$$\phi_i(r, \theta) = -\sum_{n=1}^{\infty} B_n r^n \cos(n\theta) \quad \text{for } r \leq 1 \tag{2.19}$$

where the coefficients in the above series can be found from integral equation (2.18), in this way, we obtain the following trigonometric series for the coefficients B_n, $n = 1, 2, 3, \ldots, \infty$, valid for arbitrary values of K_*.

$$2K_* \cos(2\theta) = B_1(\cos\theta + 4K_* \sin^2\theta - 2K_*) + \sum_{n=2}^{\infty} nB_n(\cos(n\theta) + 4K_* \sin\theta \sin(n\theta)) \tag{2.20}$$

where we have used the integral relation (2.14). It can be observed, that the above formulation allows us to study the case of variable permeability i.e. $K_* = K_*(\theta)$, that has not been discussed previously in the literature. In the next section we will present some numerical results for the cases where the shell parameter K_* has a cosine variation.

In a similar way we can write the exterior potential as:

$$\phi_e(r, \theta) = r\cos\theta + \sum_{n=1}^{\infty} \frac{A_n}{r^n} \cos(n\theta) \quad \text{for } r \geq 1 \tag{2.21}$$

where the relation between the coefficients A_n and B_n is found from the normal velocity matching condition at $r = 1$, and thus $1 - A_1 = -B_1$ and $A_n = B_n$, $n = 2, 3, \ldots, \infty$. Hence, the potential jump between the exterior and interior potentials across the cylindrical curve is

$$(\phi_e - \phi_i)_{r=1} = 2\left[(1 + B_1)\cos\theta + \sum_{n=2}^{\infty} B_n \cos(n\theta)\right]. \tag{2.22}$$

The total dimensionless force $\boldsymbol{F}$ exerted by a steady uniform flow surrounding the porous cylindrical shell is:

$$\boldsymbol{F} = \int_\Gamma (p_e - p_i)\boldsymbol{n}\, d\sigma \tag{2.23}$$

which can be written as

$$\boldsymbol{F} = \frac{1}{K_*}\int_\Gamma \left(\frac{\partial}{\partial r}\phi_e\right)\boldsymbol{n}\, d\sigma = \frac{1}{K_*}\int_\Gamma \left(\frac{\partial}{\partial r}\phi_i\right)\boldsymbol{n}\, d\sigma \tag{2.24}$$

after substitution of the thin shell pressure jump condition given by equation (2.4) and the normal velocity matching condition given by (2.3).

Therefore, the total force is given in terms of the B_n coefficients as:

$$F_y = 0, \quad F_x = \frac{1}{K_*} \sum_{n=1}^{\infty} \int_0^{2\pi} nB_n \cos(n\theta) \cos\theta \, d\theta = -\frac{B_1}{K_*}\pi. \tag{2.25}$$

The virtual mass coefficient due to an unsteady uniform flow at infinity is given in terms of the potential jump between the exterior and interior potentials across the cylindrical curve as (see Newman (1977), page 139):

$$m_{1j} = \int_\Gamma (\phi_e - \phi_i) n_j \, d\sigma \tag{2.26}$$

which, after substitution of equation (2.22), yields:

$$m_{11} = \int_0^{2\pi} 2(1 + B_1)\cos^2\theta \, d\theta + \sum_{n=2}^{\infty} \int_0^{2\pi} nB_n \cos(n\theta)\cos\theta \, d\theta = 2(1 + B_1)\pi. \tag{2.27}$$

Therefore, the total force and virtual mass coefficient can be written only in terms of the exterior dipole moment, A_1, as:

$$F_x = -\frac{1}{K_*}(A_1 - 1)\pi \quad \text{and} \quad m_{11} = 2A_1\pi. \tag{2.28}$$

It can be observed, that Fourier theory can not be applied to find the coefficients $B_n, n = 1, 2, 3, \ldots, \infty$, in equation (2.20) due to the term $\sin\theta \sin(n\theta)$. However, since the equation must be satisfied at all points over the unit circle, we can solve numerically the following truncated approximation

$$2K_* \cos(2\theta) = B_1(\cos\theta + 4K_* \sin^2\theta - 2K_*) + \sum_{n=2}^{N} nB_n(\cos(n\theta) + 4K_* \sin\theta \sin(n\theta)) \tag{2.29}$$

by applying the above equation at N different points $\theta_j, j = 1, 2, \ldots, N$, over the unit circle. Once the coefficients B_n are found, the total force and virtual mass are given in terms of B_1 by equations (2.25) and (2.27) respectively. In the next section we will present numerical results for different values of the parameter K_*.

3. Numerical results

In this chapter we will present numerical results for the dimensionless drag force and virtual mass. We solve the system of linear equations coming from equation (2.29) by a standard computer library routine based on the

Gauss-Jordan elimination method. We recall that this set of equations results from sampling this equation in a finite number N of sample points around the circular contour. By solving these equations, we find the Fourier coefficients of truncated versions of the solutions (2.19) and (2.21), for the internal and external flow potential, respectively.

In the previous chapter, we introduced the dimensionless virtual mass tensor m_{ij} for the porous shell. However, in the tables we have chosen to replace it by the virtual mass coefficient m_{ij}/π, which is defined as the virtual mass of the cylinder divided by the fluid volume inside the cylinder. The unit of dimensionless force per length unit of the cylinder is given by the dynamic pressure $\varrho V^2/2$ multiplied by the radius R.

In Table 1 we show results for the virtual mass coefficient and the dimensionless drag force, which are produced by the first term in the Fourier series. The virtual mass is a monotonous function of the shell parameter, while the drag force has a maximal value at $K_* = 0.433$. The results for very small values of K_*, $B_1 = -4K_*^2$ coincide with the value found analytically in the appendix. In our numerical solution, the convergence depends on the location of the sample points around the circular contour. This is investigated in Table 2. We always restrict ourselves to a constant distance between two neighboring sample points. In Table 2(a) we have $N = 36$ and we investigate further the case of maximal drag force. We find that optimal convergence requires that the sample points are not placed

Table 1
Numerical results for a thin-shell porous cylinder with K_* constant along its perimeter. The first sampling angle is here chosen as $\theta_1 = 2°$. The sampling angles are evenly distributed with $N = 30$.

Shell parameter	First Fourier	Virtual mass coefficient	Dimensionless drag force/π
K_*	B_1	$m_{11}/\pi = 2(1 + B_1)$	B_1/K_*
0.01	−0.0004	1.9992	0.0040
0.1	−0.0371	1.9250	0.3709
0.2	−0.123	1.754	0.615
0.3	−0.219	1.561	0.731
0.4	−0.307	1.386	0.769
0.433	−0.333	1.333	0.770
0.5	−0.382	1.236	0.764
0.6	−0.444	1.111	0.741
0.707	−0.500	1.000	0.707
0.8	−0.540	0.919	0.676
1.0	−0.609	0.781	0.609
2.0	−0.774	0.452	0.387
5.0	−0.891	0.217	0.178
10.0	−0.9503	0.0994	0.0950
100.0	−0.9993	0.0014	0.00999
1000.0	−0.999993	0.000014	0.00100

Table 2
Effects of varying the first sampling angle. Results for the dimensionless drag force divided by π are shown. The sampling angles are evenly distributed.

First sampling angle (θ)	Dimen. ionless drag force/π = $-B_1/K_*$		
	a) $K_* = 0.433$ $N = 36$	b) $K_* = 1.$ $N = 30$	c) $K_* = 1.$ $N = 31$
0°	−0.6599	1.6963	0.8592
1°	0.7698	0.6092	0.6093
2°	0.7698	0.6094	0.6094
3°	0.7698	0.6097	0.6096
4°	0.7698	0.6098	0.6098
5°	0.7145	0.6100	0.6099
6°	0.7698	0.6515	0.6100

symmetrically with respect to coordinate axes. The system of equations may degenerate due to such symmetry, as indicated by the cases of sampling angles equal to 0° or 5°. However, it does not seem that symmetry with respect to the origin will influence the convergence. If so, we would have been forced to choose N as an odd number, which is done in Table 2(c). Here $N = 31$, while in Table 2(b) we have the same case ($K_* = 1$) with $N = 30$. The divergence at a sampling angle of 6° when $N = 30$ disappears for $N = 31$ due to the loss of symmetry. But the divergence at a sampling angle equal to zero will persist although it is less severe. All the results in Table 2 indicate that divergence is easy to identify, as the variations between the converged solutions are very small in comparison. In most of the calculations we choose the first sampling angle equal to 2°. An exception is Table 3, where we got some divergence with a choice of 2°, and the choice of 5° gave good convergence.

In Tables 3 and 4 we have investigated the case of a non-homogeneous circular shell. We study only the cases of a cosine variation of the shell parameter $K_*(\theta)$, with one wavelength in Table 4. We now calculate the principal values of the tensor of virtual mass coefficients m_{ij}/π. As in the case of constant shell parameter, these virtual mass coefficients are given by $2(1 + B_1)\pi$. However, the drag force for a non-homogeneous shell involves the full (truncated) Fourier series, and is omitted here. One striking fact from Tables 3 and 4 is that we cannot in general say whether maximum virtual mass occurs for a flow incident on the point with maximal permeability, or perpendicular to that direction. Both possibilities are practicable, and we have not found any physical argument for selecting one or the other.

Comparing Tables 3 and 4, we see the anisotropy in the virtual mass is strongest where there is just one wavelength of variation around the contour. This makes sense, because the shell is closer to homogeneous the larger the number of wavelengths around the contour. The greatest an-

Table 3
Principal values of the tensor of virtual mass coefficient m_{ij}/π for the function $K_*(\theta) = a + b\cos\theta$. $N = 31$ and the first sampling angle is 5°.

a	0.5	0.75	0.9	0.25	0.4	0.1
b	0.5	0.25	0.1	0.25	0.1	0.1
$K_{*\max}$	1.0	1.0	1.0	0.5	0.5	0.2
$K_{*\min}$	0.0	0.5	0.8	0.0	0.3	0.0
m_{11}/π	1.767	1.041	0.850	2.000	1.450	2.103
m_{22}/π	1.601	1.171	0.929	1.703	1.459	1.799

Table 4
Principal values of the tensor of virtual mass coefficient m_{ij}/π for the function $K_*(\theta) = a + b\cos 2\theta$. $N = 36$ and the first sampling angle is 2°.

a	0.5	0.75	0.9	0.25	0.4	0.1
b	0.5	0.25	0.1	0.25	0.1	0.1
$K_{*\max}$	1.0	1.0	1.0	0.5	0.5	0.2
$K_{*\min}$	0.0	0.5	0.0	0.0	0.3	0.0
m_{11}/π	1.359	0.920	0.822	1.729	1.402	1.944
m_{22}/π	1.355	1.041	0.072	1.576	1.387	1.845

isotropy obviously occurs when the minimum permeability is zero. In this case we have shown that:

$$\frac{m_{11}}{\pi} > 2 \quad \text{for } K_{*\max} < 0.5. \tag{3.1}$$

This means that if the shell parameter is small enough, and has zero minimum, the maximal virtual mass will exceed that of a rigid cylinder. The corresponding values of m_{22}/π are always below 2. The case with the maximal virtual mass is given by:

$$\frac{m_{11}}{\pi}, \frac{m_{22}}{\pi} = (2.1029, 1.7963) \tag{3.2}$$

and occurs at:

$$(K_{*\max}, K_{*\min}) = (0.206, 0). \tag{3.3}$$

Both the maximum and the minimum values of the shell parameter are then located at $y = 0$.

4. Summary and conclusions

We have studied the two dimensional potential flow due to a circular porous shell in an infinite fluid. The motion of this porous cylinder is an arbitrary function of time. We have introduced a dimensionless shell parameter K_* and solved the hydrodynamic problem numerically. The hydrodynamic problem is non-linear, as we retain the full convective term in Bernoulli's equation for the dynamic boundary condition across the shell. Still our numerical problem is simply to solve a linear system of algebraic equations. This truncated set of equations arises from the sampling of a Fourier series expansion of Green's theorem.

We have calculated the drag force and the virtual mass for various values of K_*, constant along the perimeter of the shell. Both of these quantities are determined by the first term B_1 of the Fourier series. The virtual mass tensor is calculated for some cases where K_* varies along the shell perimeter.

There is a maximal drag force at $K_* = 0.433$. This is remarkably close to the corresponding value $K_* = 0.5$ which gives maximal damping of gravity waves on a porous plate in a canal (Chwang and Dong (1984)). However, in this comparison we must reinterpret V as the phase velocity of the shallow-water gravity waves.

Within potential flow theory, we have an exact Morison-type equation, where the virtual mass force and drag force are added together for a flow varying arbitrarily in time. The coefficients of this equation are constant, as a contrast to the usual Morison equation for viscous flow (Sarpkaya and Isaachson, page 9, (1981)). Our Morison-type equation may be written (with dimension) as follows (valid for a homogeneous shell K_* constant):

Total force per unit length of the cylinder

$$= -B_1 \frac{\pi\mu R d}{2K} V + 2(1 + B_1)\varrho\pi R^2 \frac{dV}{dt}. \tag{4.1}$$

Let us recall that the Fourier coefficient B_1 is a function of K_*, see Table 1. In equation (4.1) we note that the relative importance of the inertia force increases with the radius of the shell. If we take into account viscosity in the surrounding fluid, a more realistic version of equation (4.1) would result from replacing the first term by a squared-velocity term, but keep the second (inertial) term. The drag force due to viscous flow around a porous cylinder must be determined experimentally.

Recently Molin (1989, 1990) studied the virtual mass and drag force due to a homogeneous porous shell in potential flow with a quadratic discharge law, he solved the problems of a circular cylinder undergoing harmonic and biharmonic motion, and found a maximal drag force when the virtual mass was about 1/2 times its value for a solid cylinder i.e. a virtual mass ratio equal to 1/2. The corresponding virtual mass ratio for the case of uniform flow past a homogeneous circular cylindrical porous shell with a linear discharge law is 2/3, see Table 1.

In offshore applications it is important that the virtual mass is as small as possible. On the other hand, we have found that the virtual mass coefficient may exceed its classical value (2.0) for a rigid cylinder. By introducing an angle-dependent permeability of a shell, we are able to increase the inertia force by 5%. This increment is probably too small to have any practical importance.

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Appendix

Let us use Power et al. (1984) decomposition, when $K_* = (\varrho VK/2\nu d)$ is very small, for the potential functions ϕ_e and ϕ_i:

$$\phi_e = \phi_{e0} + K_* \phi_{e1} + K_*^2 \phi_{e2} + \cdots \qquad \text{(A-1,a)}$$

and

$$\phi_i = K_* \phi_{i1} + K_*^2 \phi_{i2} + \cdots \qquad \text{(A-1,b)}$$

where

$$\phi_{e0} = r \cos\theta \left(1 + \frac{1}{r^2}\right) \quad \text{for all } r \geq 1 \qquad \text{(A-2)}$$

is the usual potential due to a uniform flow around a solid circular cylinder. Therefore:

$$\left.\frac{\partial \phi_{e0}}{\partial r}\right|_{r=1} = 0. \qquad \text{(A-3)}$$

In this way the exterior and interior pressure are given by

$$pe = 1 - (\nabla \phi_{e0})^2 - 2K_* \nabla \phi_{e0} \nabla \phi_{e1} + O(K_*^2) \qquad \text{(A-4,a)}$$

and

$$pi = C - O(K_*^2). \qquad \text{(A-4,b)}$$

Substituting the above expressions into the pressure jump condition and using the normal velocity matching condition, we get

$$K_* \frac{\partial \phi_{e1}}{\partial r} + K_*^2 \frac{\partial \phi_{e2}}{\partial r} + \cdots \Big|_{r=1}$$

$$= K_* \frac{\partial \phi_{i1}}{\partial r} + K_*^2 \frac{\partial \phi_{i2}}{\partial r} + \cdots \Big|_{r=1}$$

$$= (C - 1 + (\nabla \phi_{e0})^2 + 2K_* \nabla \phi_{e0} \nabla \phi_{e1} + \cdots)\Big|_{r=1}. \quad \text{(A-5)}$$

Separating terms of equal order of K_*, we obtain:

$$\frac{\partial \phi_{e1}}{\partial r}\Big|_{r=1} = \frac{\partial \phi_{i1}}{\partial r}\Big|_{r=1} = C - 1 + (\nabla \phi_{e0})^2 \quad \text{(A-6,a)}$$

$$\frac{\partial \phi_{e2}}{\partial r}\Big|_{r=1} = \frac{\partial \phi_{i2}}{\partial r}\Big|_{r=1} = 2 + (\nabla \phi_{e0} \nabla \phi_{e1}). \quad \text{(A-6,b)}$$

Substitution of equation (A-2) in equation (A-6,a), yields:

$$\frac{\partial \phi_{e1}}{\varphi r}\Big|_{r=1} = \frac{\partial \phi_{i1}}{\partial r}\Big|_{r=1} = C - 1 + 4\sin^2\theta = 1 + C - 2\cos(2\theta). \quad \text{(A-7)}$$

As before, we found from the non-flux condition

$$\int_S \frac{\partial \phi_{e1}}{\partial r}\, dS = 0, \quad C = -1.$$

Then, the exterior and interior potentials ϕ_{e1} and ϕ_{i1} satisfying the above Newman conditions are:

$$\phi_{e1} = \frac{1}{r^2}\cos(2\theta) \quad \text{(A-8,a)}$$

and

$$\phi_{i1} = -r^2 \cos(2\theta). \quad \text{(A-8,b)}$$

Substituting equations (A-2) and (A-8,a) into equation (A-6,b), we obtain:

$$\frac{\partial \phi_{e2}}{\partial r}\Big|_{r=1} = \frac{\partial \phi_{i2}}{\partial r}\Big|_{r=1} = 8\sin\theta\sin(2\theta) = 4(\cos\theta - \cos(3\theta)). \quad \text{(A-9)}$$

Therefore, we have

$$\phi_{e2} = -\frac{4}{r}\cos\theta + \frac{4}{3}\frac{\cos(3\theta)}{r^3} \quad \text{(A-10,a)}$$

and

$$\phi_{i1} = 4r \cos\theta + \frac{4}{3} r^3 \cos(3\theta). \qquad \text{(A-10,b)}$$

In this way, we have from equations (A-1,a,b) that

$$\phi_e = r\cos\theta + (1 - 4K_*^2)\frac{\cos\theta}{r} + K_*\frac{\cos(2\theta)}{r^2} + \frac{4}{3}K_*^2\frac{\cos(3\theta)}{r^3} + O(K_*^3) \qquad \text{(A-11,a)}$$

and

$$\phi_i = 4K_*^2 r\cos\theta - K_* r^2\cos(2\theta) - \frac{4}{3}K_*^2 r^3 \cos(3\theta) + O(K_*^3). \qquad \text{(A-11,b)}$$

Finally from equation (2.19), we conclude that in this case

$$B_1 = -4K_*^2, \quad \text{then} \quad F'_x = 4K_*^2\pi = \frac{\pi\varrho VK}{\mu d} \qquad \text{(A-12)}$$

or in terms of the original variables

$$F_x = \varrho\frac{V^2}{2}RF'_x = \frac{\pi\varrho V^3 K}{\nu\varepsilon} \qquad \text{(A-13)}$$

which is in agreement with Power's et al. (1990) solution, in the case of a porous circular cylinder with a hollow core of very small thickness, $d = \varepsilon R$.

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Abstract

We study the two-dimensional potential flow due to a circular cylinder in motion relative to an unbounded fluid. The cylinder consists of a thin, circular porous shell with fluid inside. The full nonlinear hydrodynamic problem is solved by Fourier expansion of Green's theorem. The truncated series is determined numerically by sampling points around the circle. A dimensionless shell parameter is introduced. For homogeneous porous shells, a maximal drag force occurs at the value 0.433 for the shell parameter, but the virtual mass is a monotonous function of the shell parameter. For an inhomogeneous shell, we have found a maximal value for the virtual mass which is 5% above the value for a rigid cylinder. Some of the results may be relevant to offshore engineering, especially in connection with porous coating of platform legs to reduce the total force.

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