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Deterministic and stochastic Duffing–van der Pol oscillators are non-explosive

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Abstract. This paper is concerned with the (non-)explosion behavior of solutions of non-linear random and stochastic differential equations.

We primarily investigate the Duffing–van der Pol oscillator

$$\ddot{x} = (\alpha + \sigma_1 \xi_1)x + \beta\dot{x} - x^3 - x^2\dot{x} + \sigma_2\xi_2, \tag{1}$$

where α, β are bifurcation parameters, ξ_1, ξ_2 are either real or white noise processes, and σ_1, σ_2 are intensity parameters.

The notion of (strict) completeness (the rigorous mathematical formulation of "non-explosiveness") is introduced, and its scope is explained in detail. On the basis of the Duffing–van der Pol equation techniques for proving or disproving (strict) completeness are presented. It will turn out that the forward solution of (1) is strictly complete, but the backward solution is not complete in both the real and white noise case. This is in particular true for the deterministic Duffing–van der Pol oscillator.

In addition, some general results on the completeness of stochastic differential equations are given.

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1. Introduction

Any non-linear autonomous random or stochastic differential equation with locally Lipschitz continuous coefficients possesses a unique local solution. It is well-known that the only general condition ensuring globality (or non-explosiveness) of the solution is the linear growth condition of the coefficients. Hence if this condition does not hold, the solution may explode with positive probability for particular initial values.

A rigorous mathematical description of "non-explosiveness" is given by the notions of completeness, and strict completeness.

In the present paper we will introduce techniques to prove or disprove (strict) completeness of random and stochastic differential equations. We are primarily concerned with the investigation of the long-term behavior of the Duffing–van der Pol oscillator

$$\ddot{x} = (\alpha + \sigma_1 \xi_1)x + \beta\dot{x} - x^3 - x^2\dot{x} + \sigma_2\xi_2, \tag{1}$$

where α, β are bifurcation parameters, ξ_1, ξ_2 are either real or white noise processes, and σ_1, σ_2 are intensity parameters.

Our main question is: does the forward and backward solution of Eq. (1) explode (i.e., leave the state space in finite time) when time passes?

This is undoubtedly a classical textbook problem. However, we have not been able to detect any treatment of the problem in the literature, even for the deterministic case ($\sigma_1 = \sigma_2 = 0$).

We will prove that the forward solution of (1) is strictly complete, but the backward solution is not complete, in both the real and white noise case. This implies in particular the non-explosion property of the forward solution and the explosion property of the backward solution of the deterministic Duffing–van der Pol equation.

Although this is a study of a particular example we believe that the presented techniques are applicable to a much broader class of equations. In particular, we study the Duffing and the van der Pol equation. Both are "contained" in Eq. (1) in the following sense: the Duffing equation

$$\ddot{x} = (\alpha + \sigma_1 \xi_1)x + \beta\dot{x} - x^3 + \sigma_2\xi_2, \tag{2}$$

is obtained from (1) by omitting the term $-x^2\dot{x}$, while the van der Pol equation

$$\ddot{x} = (\alpha + \sigma_1 \xi_1)x + \beta\dot{x} - x^2\dot{x} + \sigma_2\xi_2, \tag{3}$$

is obtained by omitting the term $-x^3$.

Since in the stochastic case strict completeness is much harder to prove than completeness, more sufficient conditions are known for the latter. We present two general sufficient conditions for completeness in terms of the existence of moments and by using the generator. For stochastic Liénard equations we give a result ensuring strict completeness.

The Duffing–van der Pol oscillator is an interesting (and well-known) example, both from the mathematical and physical points of view. The bifurcation behavior of the deterministic equation, which exhibits pitchfork, Hopf, and global bifurcations, was investigated by Holmes and Rand [6], cf. also Guckenheimer and Holmes [4]. The stochastic system has been investigated recently by the author [13], where the main interest focused on the stochastic bifurcations of the generated random dynamical system.

A wide variety of applications ranging from flow induced vibrations and aeroelasticity to electronic circuits causes its importance in physics. There is a vast amount of publications by engineers investigating (1) from the diffusion process perspective, see e.g. the references in Lin and Cai [10] and Sri Namachchivaya [14, 15].

The paper is organized as follows. In Section 2 (strict) completeness is defined. Section 3 is devoted to the study of the forward solution of Eq. (1). Some general sufficient conditions for completeness are given in Section 4. Explosion of the backward solution is proved in Section 5. In Subsection 5.3 and 5.4 we consider the Duffing Eq. (2) and the van der Pol Eq. (3), respectively. At the end of each section we explain the meaning of the result for the corresponding flow.

2. Strict completeness and completeness

The general definitions are given for the white noise case. Below we explain their meaning for the real noise and the deterministic case.

Let an autonomous stochastic Itô differential equation

$$dx = f_0(x)\,dt + \sum_{i=1}^{m} f_i(x)\,dW_t^i, \tag{4}$$

on $\mathbb{R}^d$ be given, where $f_0, ..., f_m$ are vector fields on $\mathbb{R}^d$ and $W = (W_1, ..., W_m)$ is an m-dimensional Wiener processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $f_0, ..., f_m$ are locally Lipschitz continuous then a unique maximal solution exists up to an explosion time $\tau(\omega, x)$, see e.g. Kunita [7, Theorem 3.4.5].

The following definition is due to Kunita [7, p. 180].

Definition 2.1. The solution of (4) is called *complete* if $\mathbb{P}\{\tau(\omega,x) = \infty\} = 1$ for all $x \in \mathbb{R}^d$, and *strictly complete* if $\mathbb{P}\{\tau(\omega,x) = \infty \;\; \forall x \in \mathbb{R}^d\} = 1$.

The difference between these two notions of non-explosiveness is the dependence on null-sets. The solution of (4) is complete if every set $N_x := \{\omega \mid \tau(\omega,x) < \infty\}$ has measure zero. Note that this exceptional set is allowed to depend on the initial value x. To have strict completeness it is necessary that the exceptional null-set N is independent of the initial value x. Strict completeness is a stronger property than completeness because $\cup_{x\in\mathbb{R}^d} N_x$ is, in general, not a null-set or even measurable anymore.

In the literature completeness is also called regularity, conservativeness or non-explosiveness. Some authors call the generating differential equation complete and its maximal solution conservative.

Theorem 2.2. *Strict completeness implies completeness, and is equivalent to it for $d = 1$.*

For a proof see Kunita [7, p. 180f].

Counterexamples for the converse of this statement were given by Léandre [8, p. 159f]. These examples are in terms of differential equations on $\mathbb{R}^d$ ($d \geq 2$), whereas the "classical" counterexample, given by Elworthy, Kunita [7, Ex. 4.7.5], is only valid on the manifold $\mathbb{R}^d \setminus \{0\}$ with $d \geq 2$.

In general strict completeness is harder to prove than completeness, due mainly to the fact that completeness is related to one-point motion and strict completeness to n-point motion, for arbitrary $n > 1$. For the class of Doléans–Dade equations strict completeness was proved by Thieullen [17] under a certain condition on the Lipschitz coefficient.

Remark 2.3. This phenomenon of different "strengths" of completeness is purely stochastic and does not occur for deterministic differential equations.

For stochastic differential equations these two different levels of completeness are of significant importance to the generation of stochastic flows. We denote the local stochastic flow generated through Eq. (4) by $\varphi_{s,t}(\omega) : D_{s,t}(\omega) \to R_{s,t}(\omega)$, where $D_{s,t}(\omega)$ and $R_{s,t}(\omega) (:= \varphi_{s,t}(\omega) D_{s,t}(\omega))$ are open non-empty subsets of $\mathbb{R}^d$. Following Kunita [7, Chap. 4.7] we have:

Theorem 2.4. (i) *Assume the solution of* (4) *is complete. Then* $\varphi_{s,t}(\omega)$ *is defined on an open dense subset of the state space, i.e.* $\overline{D_{s,t}(\omega)} = \mathbb{R}^d$ *for all* $t \geq s \geq 0$.

(ii) *Assume the solution of* (4) *is strictly complete. Then* $\varphi_{s,t}(\omega)$ *is defined on the whole state space, i.e.* $D_{s,t}(\omega) = \mathbb{R}^d$ *for all* $t \geq s \geq 0$.

It is worthwhile to note that a finer hierarchy of completeness can be defined. Elworthy and Li [9] introduced the notion of p-completeness with $0 \leq p \leq d$ where d is the dimension of the state space. A stochastic differential equation is called p-complete if $\mathbb{P}\{\tau(\omega, x) = \infty \;\; \forall x \in M\} = 1$ for each p-dimensional submanifold M of $\mathbb{R}^d$. Completeness in our sense thus corresponds to 0-completeness and strict completeness to d-completeness. An important result is that $d-1$-completeness implies d-completeness, for $d = 1$ this corresponds to Theorem 2.2. Nevertheless for the globality of stochastic flows only d-completeness is of importance.

Up to now we were only concerned with the forward motion of the solution, i.e. the time was $\mathbb{R}^+$. In fact, Kunita [7, Chap. 4.7], the maximal solution of (4) is defined on the random open interval $]\tau(\omega, x)^-, \tau(\omega, x)^+[\subset \mathbb{R}$ containing zero. We say the solution of (4) is *forward complete* if $\mathbb{P}\{\tau(\omega, x)^+ = \infty\} = 1$ for all $x \in \mathbb{R}^d$, and *backward complete* if $\mathbb{P}\{\tau(\omega, x)^- = -\infty\} = 1$ for all $x \in \mathbb{R}^d$. Strict forward/backward completeness is defined analogously.

In the real noise case the maximal solution of (1) is sample-wise defined, as explained in the next section. It is a deterministic equation defined for any trajectory of the noise process, so there is no need to use stochastic analysis to give a meaning to this equation. Nevertheless, all definitions given above make sense for this type of equation. We will see that due to this "deterministic" interpretation strict completeness is not harder to prove than completeness.

3. The forward solution

3.1. The real noise case

We call Eq. (1) with real noise excitations the "random Duffing–van der Pol equation". The word "random" refers to the sample-wise meaning opposed to the stochastic case for which equations do not, in general, have a path-wise meaning. More on random differential equations can be found in Arnold [2].

Throughout this work the phrase *real noise* means a locally integrable real-valued stochastic process $\xi_t : \Omega \to \mathbb{R}$, $t \in \mathbb{R}$, i.e. $(t, \omega) \mapsto \xi_t(\omega)$ is measurable and $\int_{-t}^{t} |\xi_s(\omega)|\, ds < \infty$ for any fixed $t > 0$ and any ω. In particular, any càdlàg process ξ_t (i.e. any trajectory is right-continuous and has limits from the left) is locally integrable. If ξ_t is a stationary process then it is locally integrable if $\xi_t \in \mathbb{L}^1$, i.e. it is integrable.

The further treatment of (1) is simpler if we rewrite it as a random Liénard system, which is different from the canonical system. Eq. (1) is equivalent to

$$\begin{aligned} dx_1 &= \left(x_2 + \beta x_1 - \frac{1}{3} {x_1}^3 \right)\, dt \\ dx_2 &= \left((\alpha + \sigma_1 \xi_1(t)) x_1 - {x_1}^3 + \sigma_2 \xi_2(t) \right)\, dt, \end{aligned} \tag{5}$$

where $x(t) = x_1(t)$. This differential equation has to be considered as formal representation of the corresponding integral equation, i.e. its solution is meant in the sense of Carathéodory. If $\xi_1(t)$, $\xi_2(t)$ are continuous then the above equation is a classical differential equation.

Theorem 3.1. *Assume that ξ_1, ξ_2 are real noises and that $|\xi_1(t)|^4$, $|\xi_2(t)|^2$ are locally integrable. Then the maximal solution of Eq.* (5)

(1) *exists and is unique;*
(2) *is continuous in (t, x), C^∞ with respect to $x = (x_1, x_2)$, $\alpha, \beta, \sigma_1, \sigma_2$; and*
(3) *is strictly complete for any fixed $(\alpha, \beta, \sigma_1, \sigma_2)$.*

Proof. Existence and uniqueness is a direct consequence of the deterministic theory and similarly for the continuity and C^∞ property, Amann [1, Chap. II]. It thus remains to verify assertion (3). We prove that on any set $[0, T] \subset \mathbb{R}^+$ for arbitrarily fixed sample path of the noise and initial value the solution is bounded. Since this estimate is independent of null-sets it follows that the solution is strictly complete.

Fix $T > 0$ and apply the chain rule to ${x_t}^4 + 2{y_t}^2$, where $(x_t, y_t) := (x_1(t), x_2(t))$.

$${x_t}^4 + 2{y_t}^2 = \underbrace{{x_0}^4 + 2{y_0}^2}_{:=c_0} + 4 \int_0^t -\frac{1}{3} {x_s}^6 + \beta {x_s}^4 + (\alpha + \sigma_1 \xi_1(s)) x_s y_s + \sigma_2 y_s \xi_2(s)\, ds.$$

Using $\alpha xy \le (\alpha^2x^2+y^2)/2$, $\sigma_1 xy\xi_1 \le ({\sigma_1}^2x^2{\xi_1}^2+y^2)/2 \le (x^4+{\sigma_1}^4{\xi_1}^4)/4+y^2/2$, and $\sigma_2 y\xi_2 \le ({\sigma_2}^2{\xi_2}^2+y^2)/2$ this is

$$\le c_0 + 4\int_0^t \underbrace{-\frac{1}{3}{x_s}^6 + (\beta+\frac{1}{4}){x_s}^4 + \frac{\alpha^2}{2}{x_s}^2 + \frac{3}{2}{y_s}^2}_{\le c_1({x_s}^4+2{y_s}^2)+c_2} + \frac{{\sigma_1}^4}{4}\xi_1(s)^4 + \frac{{\sigma_2}^2}{2}\xi_2(s)^2\, ds,$$

with positive constants c_1, c_2 depending on α and β. For any fixed time interval $[0,T]$ we find, by the local integrability assumption, a positive constant $c_3(T)$ depending on σ_1, σ_2, and T such that

$${x_t}^4 + 2{y_t}^2 \le c_0 + c_2\,T + c_3(T) + 4\,c_1 \int_0^t {x_s}^4 + 2{y_s}^2\, ds$$

for all $0 \le t \le T$. The Gronwall lemma implies that

$${x_t}^4 + 2{y_t}^2 \le (c_0 + c_2 T + c_3(T))\, e^{4c_1 T}$$

for all $0 \le t \le T$. This holds true on any subset $[0,T]$ of $\mathbb{R}^+$, so the maximal solution satisfies $\tau(\omega,x) = \infty$ for any initial value x and any fixed ω. □

Remark 3.2. If ξ_1, ξ_2 are stationary processes then the conclusions of Theorem 3.1 are true provided $\xi_1 \in \mathbb{L}^4$, $\xi_2 \in \mathbb{L}^2$.

Corollary 3.3. *The random Duffing–van der Pol equation generates a flow (ω-wise defined) of local C^∞ diffeomorphisms $\varphi_{s,t}(\omega)$ which is global in the forward direction, i.e. $\varphi_{s,t}(\omega): \mathbb{R}^2 \to R_{s,t}(\omega)$ if $s \le t$.*

Proof. For fixed $\overline{\omega} := (\xi_1(t,\omega), \xi_2(t,\omega))_{t\in\mathbb{R}}$ define $\varphi_{s,t}(\overline{\omega})x$ to be the solution of (5) at time t starting at x at time s. By Theorem 3.1 this mapping has the claimed properties. □

Taking $\sigma_1 = \sigma_2 = 0$ this result carries over to the deterministic case.

Corollary 3.4. *The deterministic Duffing–van der Pol equation is forward complete, and generates a C^∞ flow which is global in the forward direction.*

3.2. The white noise case

In this subsection we prove that the stochastic Duffing–van der Pol Eq. (1) is strictly forward complete. In the previous subsection Eq. (1) had a sample-wise meaning, which does not hold for the equation excited by white noise and so one has to use stochastic analysis to make the equation meaningful.

Rewrite Eq. (1) as the equivalent Liénard system of Stratonovich equations

$$\begin{aligned} dx_1 &= \left(x_2 + \beta x_1 - \frac{1}{3}{x_1}^3\right)\, dt \\ dx_2 &= \left(\alpha x_1 - {x_1}^3\right)\, dt + \sigma_1 x_1 \circ dW_1 + \sigma_2 \circ dW_2, \end{aligned} \tag{6}$$

where $x(t) = x_1(t)$. In fact, Eq. (6) possesses solutions which are independent of its interpretation as an Itô or Stratonovich equation. One can check that the correction term

$$\frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\begin{pmatrix} \frac{\partial g_{1j}}{\partial x_k} \\ \frac{\partial g_{2j}}{\partial x_k} \end{pmatrix} g_{kj}, \quad \text{with} \quad g(x_1, x_2) = \begin{pmatrix} 0 & 0 \\ \sigma_1 x_1 & \sigma_2 \end{pmatrix}$$

is zero. For the following it is convenient to consider Eq. (1) as a Stratonovich equation.

Theorem 3.5. *The maximal solution of Eq.* (6)

(1) *exists and is unique;*

(2) *depends continuously (hence measurably) on* (t, W, x) *and is* C^∞ *with respect to* $x = (x_1, x_2), \alpha, \beta, \sigma_1,$ *and* σ_2*;*

(3) *has* $\mathbb{R}^+$ *as its maximal interval of existence for fixed* (W, x)*; and*

(4) *is strictly complete for any fixed* $(\alpha, \beta, \sigma_1, \sigma_2)$.

Proof. Consider (6) as a Stratonovich differential equation over the canonical dynamical system of Brownian motion defined e.g. in Arnold [2]. Take without loss of generality $\sigma_1 = \sigma_2 = 1$, cf. Remark 3.6.

We may transform (6) in such a way that the derivatives of $W(t) = (W_1(t), W_2(t))$ will be eliminated. Applying the transformation

$$y(t) := x_2(t) - x_1(t)W_1(t) - W_2(t),$$

gives the system

$$\begin{aligned} dx_1 &= \left(y + x_1 W_1(t) + W_2(t) + \beta x_1 - \frac{1}{3}{x_1}^3\right)\, dt \\ dy &= \left(\alpha x_1 - {x_1}^3 - \left(y + x_1 W_1(t) + W_2(t) + \beta x_1 - \frac{1}{3}{x_1}^3\right) W_1(t)\right)\, dt. \end{aligned} \tag{7}$$

This is a non-autonomous deterministic differential equation defined for any sample path $W \in C^0$. The key idea is to consider the right-hand side as a function of $(t, W) \in \mathbb{R} \times C^0$.

Define the right-hand side as $f(t, W, x_1, y)$ (where the dependence on the parameters α and β is surpressed) and $x := (x_1, y)$. Hence we have

$$\mathbb{R} \times C^0 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (t, W, x) \mapsto f(t, W, x).$$

We prove that the maximal solution of (7), say $\varphi(t, W, x)$, has the properties (1)–(4). Obviously, the re-transformation $x_2(t) = y(t) + x_1(t)W_1(t) + W_2(t)$ preserves all properties.

For any $W \in C^1(\mathbb{R}^+, \mathbb{R}^2)$ it is easily seen that the solution of (7) is equivalent to the solution of (6), which is of Stratonovich type. Generally, when $W \in C^0(\mathbb{R}^+, \mathbb{R}^2)$ this is a consequence of results of Sussmann [16, Thm. 8 and Sec. 7] (see also [3, Thm. 19]), because the coefficients of the noise commute. The general definition is: Eq. (4) is said to have commutative noise (or its coefficients of the noise commute), if $[f_i, f_j] = Df_i\, f_j - Df_j\, f_i \equiv 0$ for all $i, j \geq 1$.

To prove (1) and (2) we use

(i) $(t, W, x) \mapsto f(t, W, x)$ is continuous

(ii) f is locally Lipschitz continuous in x, i.e. for any (t_0, W, x_0) there exist a neighborhood $U \times V$ of (t_0, x_0) and a positive constant c such that $|f(t, W, x) - f(t, W, \bar{x})| \leq c|x - \bar{x}|$ for all $x, \bar{x} \in V$, $t \in U$

(iii) $W \mapsto f(t, W, x)$ is uniformly continuous in W, where (t, x) are taken from an arbitrary compact subset of $\mathbb{R}^+ \times \mathbb{R}^2$ and

(iv) f is C^∞ with respect to $x, \alpha, \beta, \sigma_1$, and σ_2 for fixed (t, W).

Properties (i) and (iii) are easily seen, because the time dependence of f enters only through $W(t)$ and C^0 is equipped with the topology of uniform convergence on compacta. (ii) can explicitly be calculated and (iv) is obvious (if one does not fix σ_1, σ_2).

(i) and (ii) imply (1) by Theorem 7.6 of Amann [1]. (i), (ii), and (iii) imply (2) part one by Theorem 8.3 together with Remark 8.5.b of Amann [1]. Measurability follows from the joint continuity because $\mathbb{R}^+$, C^0, and $\mathbb{R}^2$ are second countable metric spaces, hence $\mathcal{B}(\mathbb{R}^+ \times C^0 \times \mathbb{R}^2) = \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$. (i), (iii) and (iv) imply (2) part two by virtue of Theorem 9.4 and Remark 9.6.b of Amann [1] (simply augment the system by the equations $\dot{\alpha} = 0$, ..., $\dot{\sigma_2} = 0$). In particular, any derivative of the maximal solution with respect to x satisfies the corresponding variational equation and is continuous in (t, W, x).

To prove (3) we show: on any set $[0, T] \subset \mathbb{R}^+$ for fixed W and x there exists a finite constant $c(T)$ such that $|\varphi(t, W, x)| < c(T)$, i.e. the solution $\varphi(t, W, x)$ exists for all $t \in \mathbb{R}^+$.

Fix $T > 0$, W and x. First we apply the chain rule to ${x_t}^4 + 2{y_t}^2$, where $(x_t, y_t) := \varphi(t, W, x)$. Define $W_i(s) := W_s^i$.

$$\begin{aligned} {x_t}^4 + 2{y_t}^2 &= \underbrace{{x_0}^4 + 2{y_0}^2}_{:= c_0(x)} + 4 \int_0^t {x_s}^3 \dot{x}_s + y_s \dot{y}_s \, ds \\ &= c_0(x) + 4 \int_0^t -\frac{1}{3}{x_s}^6 + (\beta + W_1(s))\, {x_s}^4 + W_2(s){x_s}^3 + \frac{1}{3} W_1(s){x_s}^3 y_s \\ &\quad + \left(\alpha - \beta W_1(s) + W_1(s)^2\right) x_s y_s - W_1(s)W_2(s) y_s - W_1(s){y_s}^2 \, ds. \end{aligned}$$

Define $c_i := \sup_{s\in[0,T]} |W_i(s)|$, $i = 1,2$ and $c_3 := \sup_{s\in[0,T]} \left(|\alpha - \beta W_1(s) + W_1(s)^2|\right)$.

Hence

$$\begin{aligned}
&\leq c_0(x) + 4\int_0^t -\frac{1}{3}{x_s}^6 + (|\beta| + c_1){x_s}^4 + c_2\left({x_s}^4 + 1\right) \\
&\qquad + \frac{1}{2}\left(\left(\frac{1}{3}\right)^2 {x_s}^6 + W_1(s)^2{y_s}^2\right) + \frac{1}{2}c_3\left({x_s}^2 + {y_s}^2\right) \\
&\qquad + c_1c_2\left({y_s}^2 + 1\right) + c_1{y_s}^2\, ds \\
&= c_0(x) + 4\int_0^t -\frac{5}{18}{x_s}^6 + (|\beta| + c1 + c_2)\,{x_s}^4 + \frac{1}{2}c_3{x_s}^2 \\
&\qquad + \left(\frac{1}{2}W_1(s)^2 + c_1 + c_1c_2 + \frac{1}{2}c_3\right){y_s}^2 + c_1c_2 + c_2\, ds.
\end{aligned}$$

Let us consider the terms containing x_s^6 and x_s^2. Calculation of the maxima of $-\frac{5}{18}x^6 + \frac{1}{2}c_3x^2$ gives the upper bound $-\frac{5}{18}x^6 + \frac{1}{2}c_3x^2 \leq \left(\frac{1}{15}c_3{}^3\right)^{1/2}$ $(=: c_4)$. Thus we have

$$\begin{aligned}
{x_t}^4 + 2{y_t}^2 &\leq c_0(x) + 4\,(c_1c_2 + c_2 + c_4)\,t \\
&\quad + \int_0^t \underbrace{4\left(|\beta| + c_1 + c_2 + \frac{1}{4}c_1^2 + \frac{1}{2}c_1c_2 + \frac{1}{4}c_3\right)}_{=:c_5}\left({x_s}^4 + 2{y_s}^2\right)\, ds
\end{aligned}$$

for all $0 \leq t \leq T$. Now apply the Gronwall lemma to obtain

$${x_t}^4 + 2{y_t}^2 \leq \exp(c_5t)\left(c_0 + \int_0^t 4\,(c_1c_2 + c_2 + c_4)\exp(-c_5s)\, ds\right).$$

Therefore, ${x_t}^4 + 2{y_t}^2 \leq c(T) < \infty$ for all $0 \leq t \leq T$. Hence the solution of (7) exists on any interval $[0, T]$ and so its maximal interval of existence is $\mathbb{R}^+$.

Now the crucial property (4) follows readily, because the maximal non-explosive solution is defined for any $W \in C^0$ without any exceptional set. Therefore the solution is strictly complete. □

Remark 3.6. We are able to take without loss of generality $\sigma_1 = \sigma_2 = 1$ in the proof of Theorem 3.5 because the whole proof holds true for a non-standard Wiener process $W = (W_1, W_2)$ with an arbitrary covariance matrix.

Corollary 3.7. *The stochastic Duffing–van der Pol equation generates a stochastic flow of local C^∞ diffeomorphisms $\varphi_{s,t}(\omega)$ which is global in the forward direction, i.e. $\varphi_{s,t}(\omega) : \mathbb{R}^2 \to R_{s,t}(\omega)$ if $s \leq t$.*

Proof. For fixed $\omega := (W_t)_{t\in\mathbb{R}}$ define $\varphi_{s,t}(\omega)x$ to be the solution of (6) at time t starting at x at time s. By Theorem 3.5 this mapping has the claimed properties, see e.g. Kunita [7, p. 177] for the definition of stochastic flow. □

Again, taking $\sigma_1 = \sigma_2 = 0$ this result carries over to the deterministic case, cf. Corollary 3.4.

4. Completeness of stochastic differential equations

4.1. A sample-wise technique

The "Sussmann" method used in the proof of Theorem 3.5, where it was possible to "eliminate" the derivatives of the Wiener process by applying a transformation, can be generalized in the following way. In particular, this technique furnishes a general method to prove strict completeness.

Consider the Stratonovich differential equation

$$\ddot{x} = g(x) + f(x)\dot{x} + \sum_{j=1}^{m} h_j(x)\dot{W}_j, \tag{8}$$

where g and f are locally Lipschitz continuous and $h_1, \ldots, h_m$ are continuously differentiable.

Eq. (8) is equivalent to the system

$$dx = (y + F(x))\,dt, \qquad dy = g(x)\,dt + \sum_{j=1}^{m} h_j(x) \circ dW_j, \tag{9}$$

where $F(x) = \int_0^x f(s)\,ds$. Application of the transformation

$$z(t) = y(t) - \sum_{j=1}^{m} h_j(x(t))W_j(t), \quad \text{where } \dot{z} = \dot{y} - \sum_{j=1}^{m} Dh_j(x)\dot{x}W_j - \sum_{j=1}^{m} h_j(x)\dot{W}_j,$$

yields the system

$$\begin{aligned} dx &= (z + F(x) + \sum_{j=1}^{m} h_j(x)W_j(t))\,dt \\ dz &= (g(x) - \sum_{j=1}^{m} Dh_j(x)(z + F(x) + \sum_{i=1}^{m} h_i(x)W_i(t))\; W_j(t))\,dt, \end{aligned} \tag{10}$$

which is a non-autonomous deterministic differential equation.

We have equivalence of the stochastic solution of the Stratonovich Eq. (9) and of the sample-wise defined solution of (10) because the coefficients of the noise of Eq. (9) commute, Sussmann [16, Thm. 8 and Sec. 7]. This condition (see proof of Theorem 3.5 is satisfied because

$$\left[\begin{pmatrix}0\\h_i\end{pmatrix},\begin{pmatrix}0\\h_j\end{pmatrix}\right] = D\begin{pmatrix}0\\h_i\end{pmatrix}\begin{pmatrix}0\\h_j\end{pmatrix} - D\begin{pmatrix}0\\h_j\end{pmatrix}\begin{pmatrix}0\\h_i\end{pmatrix} \equiv \begin{pmatrix}0\\0\end{pmatrix}$$

for any (i,j).

Consequently we have

Corollary 4.1. *Assume the solution of Eq.* (10) *is defined on* $[0,\infty[$ *for any sample* W *and any initial value* (x,z). *Then Eq.* (9) *is strictly complete.*

The above method to prove strict completeness is only applicable in particular cases. But, as denoted before, completeness is a property of the one-point motion and thus this problem is more tractable. In the next two subsection we present two different sufficient conditions which ensure completeness.

4.2. Finite moment implies completeness

The condition we are going to present is given in terms of the expected value of a function of the maximal solution. Let us give a preparation.

Let $(x_t)_{t\in\mathbb{R}^+}$ be an a.s. continuous stochastic process with values in $\mathbb{R}^d \cup \{\infty\}$, the one-point-compactification of $\mathbb{R}^d$. Further, let ∞ be an absorbing point, i.e. if $x_s(\omega) = \infty$ then $x_t(\omega) = \infty \ \forall\, t \geq s$. Define $\tau(\omega) := \inf\{t \geq 0 \mid x_t(\omega) = \infty\}$.

Lemma 4.2. *Let* $f : \mathbb{R}^d \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$ *be a continuous function with* $f(\infty) = \infty$. *Assume* $\mathbb{E}f(x_t) < \infty$ *for any* $t \in \mathbb{R}^+$. *Then* $x_t \in \mathbb{R}^d$ *for all* t *a.s. (where the exceptional set does not depend on* t*). Or equivalently* $\tau(\omega) = \infty$ *a.s.*

Proof. By the continuity of x_t it is sufficient to prove: If for fixed t, $\mathbb{E}f(x_t) < \infty$ then $\tau(\omega) > t$ a.s. Assume $\mathbb{P}\{\tau(\omega) > t\} < 1$, i.e. $A := \{\tau(\omega) \leq t\}$ satisfies $\mathbb{P}(A) > 0$. By the absorbtion property of ∞ we have $x_u(\omega) = \infty$ for all $u \geq \tau(\omega)$ and $\omega \in A$. Hence $\mathbb{E}f(x_t) \geq \int_{\{\omega:\tau(\omega)\leq t\}} f(x_t)\, d\mathbb{P} = \infty$. □

Denote by $x_t(\omega, x)$ the maximal continuous solution of a given stochastic differential equation on $\mathbb{R}^d$ with local Lipschitz continuous coefficients, cf. Eq. (4). Here $x \in \mathbb{R}^d$, $t \in [0, \tau(\omega,x)[$ and $\tau(\omega,x) \in]0,\infty]$ is the unique (measurable) explosion time. For arbitrarily fixed initial value x this fits into the above case if we extend x_t to a continuous process on $\mathbb{R}^d \cup \{\infty\}$ by letting $x_t(\omega,x) = \infty$ for all $t \geq \tau(\omega,x)$.

Remark 4.3. Any of the functions $f(x) = \log^+(|x|), \log(1+|x|), |x|^n$ with $n \geq 1$ satisfies the assumption of Lemma 4.2. Consequently, if the maximal solution

$x_t(\omega, x)$ satisfies $\mathbb{E}|x_t(\omega, x)|^n < \infty$ for any $t \in \mathbb{R}^+$ then $\tau(\omega, x) = \infty$ a.s. (where in general the exceptional set depends on the initial value x). Hence existence of an arbitrary moment of the solution ensures completeness.

4.3. Infinitesimal generator condition

In this subsection we obtain a result for the completeness of dissipative second order stochastic differential equations. Similar conditions have been used by Khasminskii [5, III.4].

Consider a d-dimensional second order stochastic Itô differential equation

$$\ddot{x} = f(x, \dot{x}) + g(x, \dot{x})\dot{W}, \tag{11}$$

where $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous functions and W is an m-dimensional Brownian motion.

Rewrite Eq. (11) as a $2d$-dimensional system of Itô differential equations.

$$\begin{aligned} dx &= y\,dt \\ dy &= f(x, y)\,dt + g(x, y)\,dW_t, \end{aligned} \tag{12}$$

where $(x, y) = (x, \dot{x})$. Its infinitesimal generator is

$$L = \sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{d} f_i(x, y) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{d} (g(x, y)g(x, y)^T)_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}.$$

The following theorem tells us that a stochastic differential equation whose generator fulfills a dissipativity condition has non-explosive solutions. This theorem goes back to Narita [12] and Khasminskii [5, Thm. III.4.1].

Theorem 4.4. *If the stochastic Eq.* (12) *satisfies the condition*

(D) *Let* $V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $V(x, y) := E(x, y) + |y|^2$ *be a* C^2 *function, where* $E(x, y) \geq 0 \ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. *Assume there exist constants* $c_1, c_2, c_3 \geq 0$, *and an* $\epsilon \in]0, 2[$ *such that*

$$LV(x, y) \leq c_1 + c_2 V(x, y) + c_3 |x|^{2-\epsilon} \quad \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

then the maximal solution is complete.

Proof. The assertion follows along the lines of the proof of Markus and Weerasinghe [11, Thm. 2.1] (their $U(x, y)$ replaced by $V(x, y) = E(x, y) + |y|^2$). However, let us point out that one has to use Dynkin's formula and not Itô's formula here. In addition the joint measurability of $(x(s \wedge \tau_m), y(s \wedge \tau_m))$ is needed to be able to interchange the integration with respect to the probability measure (taking expected value) and the integration with respect to time. □

Remark 4.5. Theorem 4.4 implies that the stochastic Duffing–van der Pol oscillator is complete for arbitrarily fixed $(\alpha, \beta, \sigma_1, \sigma_2)$. Eq. (1) is derived from (12) by taking $d = 1$, $m = 2$, $f(x, y) = \alpha x + \beta y - x^3 - x^2 y$ and $g(x, y) = (\sigma_1 x, \sigma_2)$. Elementary calculations show that for fixed $(\alpha, \beta, \sigma_1, \sigma_2) \in \mathbb{R}^4$

$$E(x,y) = x^4/2,\ c_1 = |\alpha| + \sigma_1^2 + \sigma_2^2,\ c_2 = 2(|\alpha| + |\beta| + \sigma_1^2),\ \text{and}\ c_3 = 0$$

fulfill the assumptions of Theorem 4.4.

5. The backward solution

In this section we answer the question: Is it possible that for particular initial values the solution of the Duffing–van der Pol equation explodes with a positive probability when time tends to $-\infty$? The answer is: Eq. (1) is not backward complete in both the real and white noise cases. Remember that backward completeness is defined as $\mathbb{P}\{\tau(\omega, x)^- = -\infty\} = 1$ for all $x \in \mathbb{R}^d$. In fact we will prove in the white noise case that $\mathbb{P}\{\tau(\omega, x)^- > -\infty\} > 1 - \epsilon$ for arbitrary small $\epsilon > 0$ for a set of initial values of infinite Lebesgue measure.

To avoid the occurrence of too many minus signs in the following we apply the time transformation (time reversion) $t \mapsto -t$, i.e. $y(t) := x(-t)$, where $dx(t) = f(x(t), \xi(t))dt$, satisfies

$$\frac{dy(t)}{dt} = \frac{dx(-t)}{dt} = \frac{dx(-t)}{d(-t)} \frac{d(-t)}{dt} = -f(x(-t), \xi(-t)) = -f(y(t), \xi(-t)).$$

Since $\xi(t)$ is either a stochastic process or white noise the dependence on $-t$ versus t is not essential, nor is the minus sign in front of the stochastic terms. We omit them in the following.

Application of this transformation to the canonical system

$$\dot{x} = y, \quad \dot{y} = (\alpha + \sigma_1 \xi_1)x + \beta y - x^3 - x^2 y + \sigma_2 \xi_2$$

corresponding to (1) yields the differential equation

$$\dot{x} = -y, \quad \dot{y} = -\alpha x + \sigma_1 \xi_1 x - \beta y + x^3 + x^2 y + \sigma_2 \xi_2.$$

To simplify the further treatment, we again transform this equation by letting $x_1 = x$, $x_2 = -y$, giving the backward Duffing–van der Pol equation

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= ((\alpha + \sigma_1 \xi_1(t))x_1 - \beta x_2 - {x_1}^3 + {x_1}^2 x_2 + \sigma_2 \xi_2(t))\, dt. \end{aligned} \tag{13}$$

Note that the Duffing equation remains unchanged under time reversion, whereas in the van der Pol equation the term $-{x_1}^2 x_2$ changes its sign. The equations are treated in Subsection 5.3 and 5.4, respectively.

To prove explosion of the backward Duffing–van der Pol equation we need a proposition concerning non-autonomous deterministic differential equations.

Proposition 5.1. *Let a non-autonomous deterministic differential equation* $dx = f(x,t)dt$ *on* $\mathbb{R}^d$ *be given, possessing a unique local continuous solution (in the sense of Carathéodory), denoted by* x_t.

Assume there exists an unbounded domain $G \subset \mathbb{R}^d$ *and a function* $V \in C^1(G, \mathbb{R}^+)$ *such that*

(i) *G is invariant under the local flow,*

(ii) $LV(x) \leq -1$ *for all* $x \in G$.

Then $\tau(x) < \infty$ *for all* $x \in G$. *More precisely, one has* $\tau(x) \leq V(x)$.

It suffices if condition (i) *is fulfilled for all* $t \leq \sup\{V(x) \mid x \in G\}$. *Thus, if V is uniformly bounded on G then $\tau(x)$ is also uniformly bounded and condition* (i) *has only to be valid on a finite time interval.*

$LV(x)$ is the derivative of V along the vector field f, i.e. $LV(x) = \langle DV(x), f(x)\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$.

(ii) can be replaced by $LV(x) \leq -c$ for all $x \in G$ with an arbitrary $c > 0$.

Proof. Define $\tau_n := \inf\{t \geq 0 \mid \|x_t\| \geq n\}$. $\tau_n \uparrow \tau$ when $n \uparrow \infty$. (ii) implies

$$V(x_{t\wedge\tau_n}) - V(x_0) = \int_0^{t\wedge\tau_n} LV(x_s)\, ds \leq -(t \wedge \tau_n).$$

Letting $n \to \infty$, the continuity and positivity of V gives

$$-V(x_0) \leq V(x_{t\wedge\tau}) - V(x_0) \leq -(t \wedge \tau).$$

Hence $\tau \leq V(x_0)$. □

Although the proof is written down in five lines, in applications it can be very difficult to find an invariant region G describing the "escape route" and a decreasing function V to provide an estimate from above on the explosion time.

5.1. The real noise case

In this subsection we prove that the random Duffing–van der Pol equation is not backward complete. As before assume that ξ_1, ξ_2 are locally integrable.

Theorem 5.2. *Assume there exist constants* $c_1, c_2 > 0$ *such that the event* $A := \{|\xi_1(t)| + |\xi_2(t)| \leq c_1, \ \forall\, t \in [0, c_2]\}$ *has positive probability, i.e.* $\mathbb{P}(A) > 0$. *Then* *the random Duffing–van der Pol equation is not backward complete for arbitrary* $\alpha, \beta, \sigma_1, \sigma_2$.

More precisely, for each initial value

$$x \in G(c) := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq c,\ x_2 \geq {x_1}^{3-\epsilon}\},$$

(where $\epsilon \in]0,1[$ *is arbitrarily fixed, and* $c = c(c_1, c_2, \epsilon, \alpha, \beta)$*) one has*

$$\mathbb{P}\{\tau(\omega, x)^- \geq -c_2\} \geq \mathbb{P}(A) > 0.$$

Proof. Fix $\epsilon \in]0,1[$ arbitrary. We show for any $\omega \in A$ that $V(x_1, x_2) = \frac{1}{x_1}$ and $G(c)$ satisfy the assumptions of Prop. 5.1 when $c > 0$ is chosen sufficiently large.

Step 1. There exists a $c > 0$ such that $G(c)$ is invariant for any $\omega \in A$. First, consider the boundary $x_1 = c$, $x_2 \geq c^{3-\epsilon}$. Then $\dot{x}_1 = x_2 \geq c^{3-\epsilon} > 0$, i.e. the vector field along this boundary is directed inward G for $x_1 = c$, $x_2 > c^{3-\epsilon}$.

Second, consider the boundary $x_1 \geq c$, $x_2 = {x_1}^{3-\epsilon}$. Since $\dot{x}_1 > 0$ on this boundary we have to show that

$$\frac{\dot{x}_2}{\dot{x}_1} > (3-\epsilon){x_1}^{2-\epsilon} \quad \forall (x_1, x_2) \in \{(x_1, {x_1}^{3-\epsilon}) \mid x_1 \geq c\}.$$

One has for all times $t \leq c_2$ and all $\omega \in A$

$$\begin{aligned} \frac{\dot{x}_2}{\dot{x}_1} &\geq \frac{{x_1}^{5-\epsilon} - {x_1}^3 - |\beta|{x_1}^{3-\epsilon} - (|\alpha| + |\sigma_1|c_1)x_1 - |\sigma_2|c_1}{{x_1}^{3-\epsilon}} \\ &= {x_1}^2 - {x_1}^{\epsilon} - |\beta| - \frac{|\alpha| + |\sigma_1|c_1}{{x_1}^{2-\epsilon}} - \frac{|\sigma_2|c_1}{{x_1}^{3-\epsilon}}. \end{aligned}$$

This is larger than $(3-\epsilon){x_1}^{2-\epsilon}$ for sufficiently large x_1. Hence if c is chosen large enough then $G(c)$ is invariant as long as $t \leq c_2$.

Step 2 is to show that $LV(x_1, x_2) \leq -1$ on $G(c)$. Observe that

$$LV(x_1, x_2) = -\frac{\dot{x}_1}{{x_1}^2} = -\frac{x_2}{{x_1}^2} \leq -1 \iff x_2 \geq {x_1}^2.$$

Hence $LV(x) \leq -1$ on $G(c)$ if $c \geq 1$.

Choose a $c > 0$ such that steps 1 and 2 hold, and in addition $\frac{1}{c} \leq c_2$ (which implies $V(x_1, x_2) \leq c_2$ for all $(x_1, x_2) \in G(c)$). Then by Prop. 5.1 the random Duffing–van der Pol equation explodes in finite time for all initial values $x \in G(c)$ and any $\omega \in A$. The explosion time is uniformly bounded by $\tau(\omega, x) \leq c_2$. □

Putting this result and Corollary 3.3 together gives

Corollary 5.3. *The random Duffing–van der Pol equation generates a flow (ω-wise defined) of local* C^∞ *diffeomorphisms* $\varphi_{s,t}(\omega)$ *which is global in the forward direction and local in the backward direction, i.e.* $D_{s,t}(\omega) = \mathbb{R}^2$, *but* $R_{s,t}(\omega) \subsetneqq \mathbb{R}^2$ *for all* $s < t$.

Taking $\sigma_1 = \sigma_2 = 0$ this result carries over to the deterministic case.

Corollary 5.4. *The deterministic Duffing–van der Pol equation is forward complete, but not backward complete.*

Remark 5.5. The assumption of Theorem 5.2 holds true e.g. if ξ_1, ξ_2 are càdlàg processes. An example for which the assumption of Theorem 5.2 does not hold is the locally integrable function $\xi(t) = \begin{cases} 1/t & t \in \mathbb{Q} \setminus \{0\} \\ 0 & \text{otherwise.} \end{cases}$

5.2. The white noise case

In this subsection we prove that the stochastic Eq. (1) is not backward complete.

We apply the same transformations as before and obtain the stochastic backward Duffing–van der Pol equation

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= \left(\alpha x_1 - \beta x_2 - {x_1}^3 + {x_1}^2 x_2\right) dt + \sigma_1 x_1 \circ dW_1 + \sigma_2 \circ dW_2. \end{aligned} \tag{14}$$

Theorem 5.6. *The stochastic Duffing–van der Pol equation is not backward complete for arbitrary* $\alpha, \beta, \sigma_1, \sigma_2$.

More precisely, the event

$$A(c_1, c_2) := \{|\sigma_1 W_1(t)| + |\sigma_2 W_2(t)| \le c_1, \ \ \forall\, t \in [0, c_2]\}$$

satisfies $\mathbb{P}(A(c_1, c_2)) > 0$ *for arbitrary* $c_1, c_2 > 0$, *and for any initial value*

$$x \in G(c) := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge c,\ x_2 \ge {x_1}^{3-\epsilon}\},$$

(where $\epsilon \in]0,1[$ *is arbitrarily fixed,* $c = c(c_1, c_2, \epsilon, \alpha, \beta)$*) one has*

$$\mathbb{P}\{\tau(\omega, x)^- \ge -c_2\} \ge \mathbb{P}(A(c_1, c_2)) > 0.$$

In particular, for arbitrary small $\delta > 0$ *and any fixed* c_1 *(resp.* c_2*) there exists a* c_2 *(resp.* c_1*) such that* $\mathbb{P}\{\tau(\omega, x)^- \ge -c_2\} \ge \mathbb{P}(A(c_1, c_2)) > 1 - \delta$.

Proof. The idea is to transform (14) into a deterministic sample-wise defined equation as done in the proof of Theorem 3.5. The transformed equation contains the

Wiener processes as continuous time dependent functions. Therefore the same procedure as in the proof of Theorem 5.2 is applicable.

The transformation $x = x_1$, $y = x_2 - x_1\sigma_1 W_1(t) - \sigma_2 W_2(t)$ yields the equivalent system (cf. proof of Theorem 3.5)

$$\begin{aligned} dx &= (y + \sigma_1 x W_1(t) + \sigma_2 W_2(t))\ dt \\ dy &= \left(\alpha x - x^3 + (-\beta + x^2 - \sigma_1 W_1(t))(y + \sigma_1 x W_1(t) + \sigma_2 W_2(t))\right)\ dt. \end{aligned}$$

We want to apply Proposition 5.1 and therefore have to show the same steps as in the proof of Theorem 5.2.

Fix $\epsilon \in]0,1[$, $c_1, c_2 > 0$ arbitrary. We prove for any event from $A(c_1, c_2)$ that the set $G(c)$ and the function $V(x,y) := \frac{1}{x}$ satisfy the assumptions of Proposition 5.1 when $c > 0$ is chosen sufficiently large.

Step 1. There exists a $c > 0$ such that $G(c)$ is invariant for any $\omega \in A$. First, consider the boundary $x = c$, $y \geq c^{3-\epsilon}$. Then for any $(W_1, W_2) = \omega \in A$

$$\dot{x} = y + \sigma_1 x W_1(t) + \sigma_2 W_2(t) \geq c^{3-\epsilon} - c_1\, c - c_1 > 0$$

for c sufficiently large, i.e. the vector field along $x = c$, $y > c^{3-\epsilon}$ is directed inward $G(c)$ as long as the time $t \leq c_2$.

Second, consider the boundary $x \geq c$, $y = x^{3-\epsilon}$. Since $\dot{x} > 0$ on this boundary we have to show that

$$\frac{\dot{y}}{\dot{x}} > (3-\epsilon)x^{2-\epsilon} \quad \forall (x,y) \in \{(x, x^{3-\epsilon}) \mid x \geq c\}.$$

One has for all times $t \leq c_2$

$$\frac{\dot{y}}{\dot{x}} \geq \frac{x^{5-\epsilon} - (c_1+1)x^3 - (|\beta| + c_1)x^{3-\epsilon} - c_1 x^2 - (|\alpha| + |\beta| c_1 + {c_1}^2)x - (|\beta| + c_1)c_1}{x^{3-\epsilon} + c_1 x + c_1}$$

This is larger than $(3-\epsilon)x^{2-\epsilon}$ for sufficiently large x, because the above expression is of order x^2. Hence, if c is chosen large enough then $G(c)$ is invariant as long as $t \leq c_2$.

Step 2 is to show that $LV(x,y) \leq -1$ on $G(c)$ for any $\omega \in A$. Observe that

$$LV(x,y) = -\frac{y + \sigma_1 x W_1(t) + \sigma_2 W_2(t)}{x^2} \leq -1 \iff y + \sigma_1 x W_1(t) + \sigma_2 W_2(t) \geq x^2.$$

Since $y \geq x^{3-\epsilon}$ on $G(c)$ it is sufficient to have $x^{3-\epsilon} - c_1 x - c_1 \geq x^2$. This holds true if c is large enough. Hence there exists a c such that $LV(x) \leq -1$ on $G(c)$.

Choose a $c > 0$ such that steps 1 and 2 hold, and in addition $\frac{1}{c} \leq c_2$. Then by Prop. 5.1 the stochastic Duffing–van der Pol equation explodes in finite time for all initial values $(x,y) \in G(c)$ and any $\omega \in A$, where the explosion time is uniformly bounded by $\tau(\omega, x, y) \leq c_2$. □

Putting this result and Corollary 3.7 together gives

Corollary 5.7. *The stochastic Duffing–van der Pol equation generates a stochastic flow of local C^∞ diffeomorphisms $\varphi_{s,t}(\omega)$ which is global in the forward direction and local in the backward direction, i.e. $D_{s,t}(\omega) = \mathbb{R}^2$, but $R_{s,t}(\omega) \subsetneqq \mathbb{R}^2$ for all $s < t$.*

5.3. The Duffing equation

In this section we prove that the Duffing Eq. (2) (i.e. the Duffing–van der Pol equation without the term $-x^2\dot{x}$) is strictly forward and backward complete. This result is not covered by the general theory because this equation contains a term with cubic growth.

Theorem 5.8. *Assume in the real noise case that the conditions from Theorem 3.1 hold. (There is no assumption in the white noise case.)*

Then the Duffing Eq. (2) is strictly forward and backward complete for arbitrary $\alpha, \beta, \sigma_1, \sigma_2$.

Proof. We rewrite Eq. (2) in the canonical form as

$$\dot{x} = y, \qquad \dot{y} = (\alpha + \sigma_1 \xi_1)x + \beta y - x^3 + \sigma_2 \xi_2.$$

Following Section 5 the corresponding backward equation is

$$\dot{x} = y, \qquad \dot{y} = (\alpha + \sigma_1 \xi_1)x - \beta y - x^3 + \sigma_2 \xi_2.$$

Hence it is sufficient to prove strict forward completeness.

In the real noise case this is completely analogous to the proof of Thm. 3.1. And in the white noise case it is almost analogous to the proof of Thm. 3.5. Since in the estimate of ${x_t}^4 + 2{y_t}^2$ the term $-x^6$ is missing, one has to use the estimate $cx^n \leq |c|(x^{n+1} + 1)$ for the linear and cubic terms of x and y. □

5.4. The van der Pol equation

The following theorem shows that the van der Pol Eq. (3) (i.e. the Duffing–van der Pol equation without the term $-x^3$) has essentially the same (non-) explosion behavior as the Duffing–van der Pol equation.

Theorem 5.9. *Assume in the real noise case that the conditions from Theorem 3.1 hold. (There is no assumption in the white noise case.)*

Then the van der Pol Eq. (3) is strictly forward complete, but not backward complete for arbitrary $\alpha, \beta, \sigma_1, \sigma_2$.

Proof. The strict forward completeness of the van der Pol equation follows as in the proofs of Theorems 3.1 and 3.5. One simply has to carry out the estimates for the term ${x_t}^2 + {y_t}^2$ instead of ${x_t}^4 + 2{y_t}^2$ in the real noise case, and for ${x_t}^4 + {y_t}^2$ instead of ${x_t}^4 + 2{y_t}^2$ in the white noise case.

The proof of backward non-completeness is exactly the same as for the Theorems 5.2 and 5.6. □

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