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Interacting indentors on a poroelastic half-space

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Abstract. This paper examines the interaction between two rigid circular indentors on a poroelastic half-space. The resulting mixed boundary value problem, when formulated in the Laplace transform domain, yields an infinite set of Fredholm integral equations. These integral equations are then solved for some special cases. Numerical results for the case of a single indentor show a good agreement with those obtained by using Heinrich and Desoyer's assumption. For the case in which the radius of one indentor reduces to zero (interaction between a rigid indentor and an externally placed load), the resulting equations are solved by a semi-inverse method to give analytical solutions for the resultant force and moment required to maintain the indentor with no normal displacement. When the indentor is subjected to an axial load but allowed to undergo an additional settlement and tilt, numerical results are presented to demonstrate the manner in which Poisson's ratio and the drainage boundary conditions influence the consolidation of the half-space. Numerical results are also given to illustrate the interaction between two identical indentors when ratio of the radius to the spatial distance between them is small.

Keywords. Poroelasticity, contact problems, interacting indentors, fluid saturated media, circular punches, integral equations.

1. Introduction

The one-dimensional theory of the consolidation of a water saturated elastic porous geomaterial was first developed by Terzaghi [1] and later extended by Biot [2, 3] to develop the now classical theory of *poroelasticity* for a fluid-saturated medium. The generalized three-dimensional theory of poroelasticity developed by Biot [2, 3] has been successfully applied to the study of soil consolidation problems in geomechanics. By introducing two displacement functions, McNamee and Gibson [4, 5] studied the axisymmetric and plane strain problems for the cases where a deep clay stratum is subjected to uniform normal loading. By adding a new displacement function to include the asymmetry of deformation, Schiffman and Fungaroli [6] extended McNamee and Gibson's method and developed solutions for the consolidation of a half-space region with uniform tangential loads applied over a circular area. The analysis of mixed boundary value problem related to a fluid saturated poroelastic half-space region was given by a number of authors including Agbezuge and Deresiewicz [7], Chiarella and Booker [8] and Gaszynski and Szefer [9]. In these papers the problems treated were restricted to the axisymmetric states of deformation involving smooth contact between the plane indentor and the surface of a poroelastic halfspace region. More recently, Selvadurai and Yue [10], Yue and Selvadurai [11, 12] have examined poroelastic contact problems involving, respectively, the axisymmetric indentation of a poroelastic layer, the asymmetric indentation of halfspace region and a disc inclusion problem. In these studies the pore fluid is assumed to be compressible and the poroelastic constitutive formulation adopted is that given by Rice and Cleary [13]. A special feature in all poroelastic contact problems involves not only the specification of the displacement and traction boundary conditions, but also the specification of the appropriate boundary conditions related to the fluid pressure at the contacting plane. In a majority of previous studies the surface of the halfspace region is assumed to be either completely permeable or completely impermeable. Certain limited solutions have also been developed for the contact problems in which mixed pore pressure boundary conditions are prescribed within either the contact region or regions exterior to it. Account of recent developments and applications of theories of poroelasticity are given by Selvadurai [14].

To the authors' knowledge, solutions for contact problems associated with the consolidation of a porous medium have largely been restricted to single indentor problems and the majority of these solutions are given for axisymmetric cases. In this paper, we shall consider the problem of consolidation of a linear isotropic semi-infinite clay stratum indented by two circular indentors. The analysis is restricted to the case of a poroelastic medium which is saturated with an incompressible fluid. In section 2 a general formulation for the three-dimensional problem in poroelasticity is presented. This formulation can be considered as a generalization of Muki's [15] formulation in elasticity. It is shown that solution to a poroelasticity problem in the Laplace transform domain can be reduced to the determination of eight arbitrary functions. Therefore the formulation presented is capable of solving poroelastic problems with arbitrary boundary conditions provided the region of interest is either an infinite solid or a semi-infinite solid or a layer. Section 3 deals with two-indentor problems. Following the procedure given by Lan et al. [16] for solving the corresponding problems in classical elasticity, we show that, in the Laplace transform domain, solutions to the mixed boundary value problems associated with a poroelastic medium can be reduced to an infinite set of Fredholm integral equations of the second kind. Governing equations in the temporal domain can then be obtained by applying a Laplace inverse transform. This leads to a system of double Fredholm-Volterra integral equations. The resulting equations are solved for certain special cases in section 4. First in section 4 we consider a case of a single indentor. An accurate and efficient numerical scheme is developed to evaluate the time-dependent solutions of the resulting integral equations. Numerical results obtained are compared with those obtained by using Heinrich and Desoyer's assumption [17]. Results show that this assumption yields accurate solutions after the very early stage of the consolidation. We then show that, for a limiting case where the radius of one indentor approaches zero (this indentor reduces to a point force of limited magnitude), analytical solutions can be obtained by a semi-inverse method for the resultant force and moment required to maintain the indentor with zero normal displacement. A more practical case in which the indentor instead of being restrained with no normal displacement is subjected to a constant axial loading is also presented. Numerical results are presented to demonstrate the manner in which Poisson's ratio and the drainage boundary conditions influence the consolidation of the half-space. Also considered is the case where two identical indentors are subjected to the same loadings but rigidly connected to displace uniformly without rotation. Numerical results are provided to illustrate the interaction between these two indentors when the ratio of the radius to the spatial distance between them is small.

2. Formulation

Referring to a cylindrical coordinate system (r, θ, z), the displacement components (u_r, u_θ, u_z) in the isotropic elastic soil skeleton and the excess pore pressure p of incompressible fluid are governed by the following partial differential equations [2, 3]

$$\begin{aligned}
&\nabla^2 u_r + (2\eta - 1)\frac{\partial e}{\partial r} - \frac{1}{r^2}\left(2\frac{\partial u_\theta}{\partial \theta} + u_r\right) + \frac{1}{G}\frac{\partial p}{\partial r} = 0,\\
&\nabla^2 u_\theta + (2\eta - 1)\frac{1}{r}\frac{\partial e}{\partial \theta} + \frac{1}{r^2}\left(2\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{1}{Gr}\frac{\partial p}{\partial \theta} = 0,\\
&\nabla^2 u_z + (2\eta - 1)\frac{\partial e}{\partial z} + \frac{1}{G}\frac{\partial p}{\partial z} = 0, \qquad (1)
\end{aligned}$$

where $\eta = \frac{1-\nu}{1-2\nu}$, ν is Poisson's ratio and G is the shear modulus of the elastic soil skeleton, e refers to the dilatation

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

and ∇^2, the Laplacian operator, takes the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

The fluid flow through the porous soil skeleton is assumed to be governed by Darcy's Law. If the fluid is considered to be incompressible, the volume change in an element of the poroelastic medium is identical to the excess volume of water leaving the element. Consequently, the conservation equation and Darcy's law yield the following differential equation governing the pore water pressure and the dilatation:

$$\nabla^2 p = -\frac{\gamma_\omega}{k}\frac{\partial e}{\partial t}, \tag{2}$$

where k is the permeability coefficient of the soil and γ_ω is the unit weight of the fluid. It can be verified by direct substitution that solutions to the partial differential equations (1) and (2) can be expressed in terms of three displacement functions $\phi(r,\theta,z,t)$, $\psi(r,\theta,z,t)$ and $\chi(r,\theta,z,t)$ as follows [6]

$$\begin{aligned}
u_r(r,\theta,z,t) &= \frac{\partial \phi}{\partial r} + z\frac{\partial \chi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta},\\
u_\theta(r,\theta,z,t) &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{z}{r}\frac{\partial \chi}{\partial \theta} - \frac{\partial \psi}{\partial r},\\
u_z(r,\theta,z,t) &= \frac{\partial \phi}{\partial z} + z\frac{\partial \chi}{\partial z} - \chi,\\
p(r,\theta,z,t) &= -2G\frac{\partial \chi}{\partial z} - 2G\eta\nabla^2\phi,
\end{aligned} \tag{3}$$

provided that these functions satisfy

$$\nabla^4\phi = \frac{1}{c}\frac{\partial}{\partial t}\nabla^2\phi, \quad \nabla^2\psi = 0, \quad \nabla^2\chi = 0, \tag{4}$$

where

$$c = \frac{2G\eta k}{\gamma_\omega}. \tag{5}$$

Stresses in the soil skeleton can also be expressed in terms of these three displacement functions. The stress components of interest to the problem formulation are given by

$$\begin{aligned}
\tau_{rz} &= G\left\{2\frac{\partial^2\phi}{\partial r\partial z} + 2z\frac{\partial^2\chi}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z}\right\},\\
\tau_{\theta z} &= G\left\{\frac{2}{r}\frac{\partial^2\phi}{\partial\theta\partial z} + \frac{2z}{r}\frac{\partial^2\chi}{\partial\theta\partial z} - \frac{\partial^2\psi}{\partial r\partial z}\right\},\\
\sigma_z &= 2G\left\{\frac{\partial^2\phi}{\partial z^2} - \nabla^2\phi + z\frac{\partial^2\chi}{\partial z^2} - \frac{\partial\chi}{\partial z}\right\}.
\end{aligned} \tag{6}$$

Now we seek solutions for the partial differential equations in which the dependence of $\phi(r,\theta,z,t)$, $\psi(r,\theta,z,t)$ and $\chi(r,\theta,z)$ on θ has the form

$$\begin{aligned}
\phi(r,\theta,z,t) &= \sum_{n=0}^{\infty}\phi_n(r,z,t)\cos n\theta,\\
\chi(r,\theta,z,t) &= \sum_{n=0}^{\infty}\chi_n(r,z,t)\cos n\theta,\\
\psi(r,\theta,z,t) &= \sum_{n=0}^{\infty}\psi_n(r,z,t)\sin n\theta.
\end{aligned} \tag{7}$$

Consequently, the displacements, stresses and the excess water pressure can also be expanded as either Fourier cosine or Fourier sine series in θ and let $u_r^n(r,z,t)$, $u_\theta^n(r,z,t)$, $u_z^n(r,z,t)$ and $\tau_{rz}^n(r,z,t)$, $\tau_{\theta z}^n(r,z,t)$, $\sigma_z^n(r,z,t)$ and $p_n(r,z,t)$ be the corresponding Fourier coefficients. Substituting (7) into (3) and (6), we obtain

$$\begin{aligned}
u_r^n + u_\theta^n &= (\frac{\partial}{\partial r} - \frac{n}{r})(\phi_n + z\chi_n - \psi_n),\\
u_r^n - u_\theta^n &= (\frac{\partial}{\partial r} + \frac{n}{r})(\phi_n + z\chi_n + \psi_n),\\
u_z^n &= \frac{\partial \phi_n}{\partial z} + z\frac{\partial \chi_n}{\partial z} - \chi_n,\\
\tau_{rz}^n + \tau_{\theta z}^n &= G(\frac{\partial}{\partial r} - \frac{n}{r})\left\{2\frac{\partial \phi_n}{\partial z} + 2z\frac{\partial \chi_n}{\partial z} - \frac{\partial \psi_n}{\partial z}\right\},\\
\tau_{rz}^n - \tau_{\theta z}^n &= G(\frac{\partial}{\partial r} + \frac{n}{r})\left\{2\frac{\partial \phi_n}{\partial z} + 2z\frac{\partial \chi_n}{\partial z} + \frac{\partial \psi_n}{\partial z}\right\},\\
\sigma_z^n &= 2G\left\{\frac{n^2}{r^2}\phi_n - \frac{\partial^2 \phi_n}{\partial r^2} - \frac{1}{r}\frac{\partial \phi_n}{\partial r} + z\frac{\partial^2 \chi_n}{\partial z^2} - \frac{\partial \chi_n}{\partial z}\right\},\\
p_n &= -2G\frac{\partial \chi_n}{\partial z} - 2G\eta\nabla_n^2\phi_n,
\end{aligned} \tag{8}$$

where

$$\nabla_n^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2} + \frac{\partial^2}{\partial z^2}.$$

Therefore the analysis of poroelasticity problems is reduced to the determination of the Fourier coefficients of three displacement functions. Let $\tilde{f}(r,z,s)$ be the Laplace transform of a function $f(r,z,t)$ with s being the transform parameter. Substituting (7) into equations (4) and taking Laplace transforms gives the following partial differential equations for the Fourier coefficients of the three displacement functions in the Laplace transform domain

$$\nabla_n^4\tilde{\phi}_n - \frac{s}{c}\nabla_n^2\tilde{\phi}_n = 0, \quad \nabla_n^2\tilde{\psi}_n = 0, \quad \nabla_n^2\tilde{\chi}_n = 0. \tag{9}$$

Applying Hankel transforms to the above equations reduces them to three ordinary differential equations, the general solutions of which take the form

$$\begin{aligned}
\tilde{\phi}_n(r,z,s) &= \int_0^\infty [A_n(\alpha,s)e^{-\alpha z} + \underline{A}_n(\alpha,s)e^{\alpha z} + B_n(\alpha,s)e^{-\beta z}\\
&\qquad + \underline{B}_n(\alpha,s)e^{\beta z}]\alpha J_n(\alpha r)d\alpha,\\
\tilde{\chi}_n(r,z,s) &= \int_0^\infty [C_n(\alpha,s)e^{-\alpha z} + \underline{C}_n(\alpha,s)e^{\alpha z}]\alpha J_n(\alpha r)d\alpha,\\
\tilde{\psi}_n(r,z,s) &= \int_0^\infty [D_n(\alpha,s)e^{-\alpha z} + \underline{D}_n(\alpha,s)e^{\alpha z}]\alpha J_n(\alpha r)d\alpha,
\end{aligned} \tag{10}$$

where

$$\beta^2 = \alpha^2 + \frac{s}{c}, \tag{11}$$

$A_n(\alpha, s)$, $\underline{A}_n(\alpha, s)$, etc. are arbitrary functions to be determined from the boundary conditions of a problem. Invoking certain properties of Hankel transforms [18, 19], the displacements, stresses and the excess pore pressure can be expressed in terms of these arbitrary functions as follows:

$$\begin{aligned}
\tilde{u}_r^n + \tilde{u}_\theta^n = -\int_0^\infty & \{[A_n + C_n z - D_n]e^{-\alpha z} + B_n e^{-\beta z} \\
& + [\underline{A}_n + \underline{C}_n z - \underline{D}_n]e^{\alpha z} + \underline{B}_n e^{\beta z}\}\, \alpha^2 J_{n+1}(\alpha r) d\alpha, \\
\tilde{u}_r^n - \tilde{u}_\theta^n = \int_0^\infty & \{[A_n + C_n z + D_n]e^{-\alpha z} + B_n e^{-\beta z} \\
& + [\underline{A}_n + \underline{C}_n z + \underline{D}_n]e^{\alpha z} + \underline{B}_n e^{\beta z}\}\, \alpha^2 J_{n-1}(\alpha r) d\alpha, \\
\tilde{u}_z^n = -\int_0^\infty & \{[\alpha A_n + (1 + \alpha z) C_n]e^{-\alpha z} + \beta B_n e^{-\beta z}\} \\
& - [\alpha \underline{A}_n + (\alpha z - 1)\underline{C}_n]e^{\alpha z} - \beta \underline{B}_n e^{\beta z}\}\, \alpha J_n(\alpha r) d\alpha, \\
\tilde{\tau}_{rz}^n + \tilde{\tau}_{\theta z}^n = G\int_0^\infty & [(2\alpha A_n + 2\alpha z C_n - \alpha D_n)e^{-\alpha z} + 2\beta B_n e^{-\beta z} \\
& - (2\alpha \underline{A}_n + 2\alpha z \underline{C}_n - \alpha \underline{D}_n)e^{\alpha z} - 2\beta \underline{B}_n e^{\beta z}]\alpha^2 J_{n+1}(\alpha r) d\alpha, \\
\tilde{\tau}_{rz}^n - \tilde{\tau}_{\theta z}^n = -G\int_0^\infty & [(2\alpha A_n + 2\alpha z C_n + \alpha D_n)e^{-\alpha z} + 2\beta B_n e^{-\beta z} \\
& - (2\alpha \underline{A}_n + 2\alpha z \underline{C}_n + \alpha \underline{D}_n)e^{\alpha z} - 2\beta \underline{B}_n e^{\beta z}]\alpha^2 J_{n-1}(\alpha r) d\alpha, \\
\tilde{\sigma}_z^n = 2G\int_0^\infty & \{[\alpha A_n + C_n(1 + \alpha z)]e^{-\alpha z} + \alpha B_n e^{-\beta z} \\
& + [\alpha \underline{A}_n + \underline{C}_n(-1 + \alpha z)]e^{\alpha z} + \alpha \underline{B}_n e^{\beta z}\}\alpha^2 J_n(\alpha r) d\alpha, \\
\tilde{p}_n = \int_0^\infty & [2G\alpha C_n e^{-\alpha z} - 2G\eta(\beta^2 - \alpha^2) B_n e^{-\beta z} \\
& - 2G\alpha \underline{C}_n e^{\alpha z} - 2G\eta(\beta^2 - \alpha^2)\underline{B}_n e^{\beta z}]\alpha J_n(\alpha r) d\alpha.
\end{aligned} \tag{12}$$

The set of equations (12) with eight arbitrary functions $A_n(\alpha, s)$, $\underline{A}_n(\alpha, s)$, etc. to be determined is quite general for the purpose of solving any three-dimensional problems in poroelasticity referred to an infinite space, a semi-infinite space or a layer.

We now examine a problem of a poroelastic half-space ($z \geq 0$) the surface of which is subjected to arbitrary normal stress and zero shear stresses. Solutions to this problem will be used to formulate the indentation problem in the next section. We will consider the following two types of drainage boundary conditions (i) a completely permeable surface, for which we have

$$p(r, \theta, 0, t) = 0 \tag{13}$$

and (ii) an impervious boundary, for which we have

$$\left[\frac{\partial}{\partial z}p(r,\theta,z,t)\right]_{z=0} = 0. \tag{14}$$

The regularity condition which requires that all the displacement components, stresses, and pore pressure vanish as $z \to \infty$, and the zero shear stresses boundary conditions at $z = 0$ impose the following restrictions on the eight arbitrary functions

$$\underline{A}_n = \underline{B}_n = \underline{C}_n = \underline{D}_n = D_n = 0, \quad \alpha A_n = -\beta B_n, \tag{15}$$

while the completely permeable drainage boundary condition (13) gives us

$$(\beta^2 - \alpha^2)B_n = c_e \alpha C_n, \tag{16}$$

and the impervious boundary (14) yields the following

$$\beta(\beta^2 - \alpha^2)B_n = c_e \alpha^2 C_n, \tag{17}$$

where $c_e = \frac{1}{\eta}$. In formulating the mixed boundary value problem for normal indentation of the surface, we require a relationship which relates the surface displacements to the stress, i.e.

$$\tilde{u}_z^n(r,0,s) = -\frac{1-\nu}{G}\int_0^\infty J_n(\alpha r)F_n(\alpha,s)[1+k(\alpha,s)]d\alpha, \tag{18}$$

with

$$k(\alpha,s) = \begin{cases} \frac{c_e}{2}\left[\frac{\alpha-\beta}{\beta+(1-c_e)\alpha}\right]; & \text{for free drainage boundary,} \\ \frac{c_e}{2}\left[\frac{2\alpha^2-\alpha\beta-\beta^2}{\alpha\beta+\beta^2-c_e\alpha^2}\right]; & \text{for impervious boundary,} \end{cases} \tag{19}$$

and $F_n(\alpha, s)$ is the Hankel transform of the Fourier coefficient for the normal stress at $z = 0$ in the Laplace domain; i.e.

$$F_n(\alpha,s) = \int_0^\infty \tilde{\sigma}_n(r,0,s) r J_n(\alpha r)dr. \tag{20}$$

3. Two indentor problem

We now consider the problem of two rigid smooth circular indentors resting on a poroelastic halfspace saturated with an incompressible pore fluid. The indentors are subjected to individual central loads P_z and $\bar{P}_z$. The entire surface of the halfspace is assumed to be either completely permeable or completely impervious as previously indicated by equations (13) and (14). The radii of the two circular

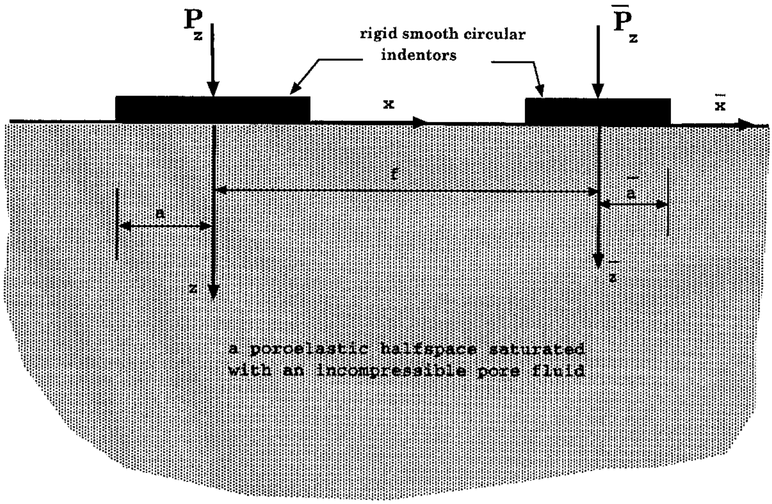

Figure 1. Two rigid circular smooth indentors on a poroelastic half space

contact regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ are a and $\bar{a}$ respectively and the distance between the two centers of these two indentors is denoted by f (Figure 1).

We choose two similarly oriented local cylindrical co-ordinate systems (r, θ, z) and $(\bar{r}, \bar{\theta}, \bar{z})$ such that the two contact areas occupy

$$\mathcal{R}: \; r < a, \; 0 \le \theta \le 2\pi, \; z = 0; \; \text{and} \; \bar{\mathcal{R}}: \; \bar{r} < \bar{a}, \; 0 \le \bar{\theta} \le 2\pi, \; \bar{z} = 0, \tag{21}$$

respectively. In terms of these coordinate systems, the stress and displacement boundary conditions of the problem can be written as

$$\begin{aligned} &\tau_{rz}^n(r,0,t) = \tau_{\theta z}^n(r,0,t) = 0, \;\; \text{for} \;\; r \ge 0; \\ &\sigma_z^n|_{z=0} = 0, \quad \text{for} \;\; r > a \;\; \text{or} \;\; \bar{r} > \bar{a}; \\ &u_z^n(r,0,t) = f_n(r,t), \quad \text{for} \;\; r \le a; \\ &u_{\bar{z}}^n(\bar{r},0,t) = \bar{f}_n(\bar{r},t), \;\; \text{for} \;\; \bar{r} \le \bar{a}, \end{aligned} \tag{22}$$

and

$$\int_0^a \int_0^{2\pi} \sigma_z(r,\theta,0,t) d\theta dr = P_z, \quad \int_0^{\bar{a}} \int_0^{2\pi} \sigma_{\bar{z}}(\bar{r},\bar{\theta},0,t) d\bar{\theta} d\bar{r} = \bar{P}_{\bar{z}}. \tag{23}$$

The above displacement and traction boundary conditions and the surface drainage conditions along with the initial condition $e(r,\theta,z,0) = 0$ (which implies that the volume change of the medium is zero at the instant of loading) constitute a complete mathematical statement of the problem.

In order to solve the two indentor problem, we first investigate the single indentor problem, for which the governing integral equations can be obtained from the surface stress and displacement relationship (18). By using equation (20) and the definition of the inverse Hankel transform, we have

$$\tilde{\sigma}_z^n(r,0,s) = \int_0^\infty F_n(\alpha,s)\alpha J_n(r\alpha)d\alpha. \tag{24}$$

By virtue of the identity [20]

$$\int_0^\infty J_\mu(x\alpha)J_\nu(y\alpha)\alpha^{1+\nu-\mu}d\alpha = \frac{2^{1+\nu-\mu}y^\nu H(x-y)}{x^\mu\Gamma(\mu-\nu)(x^2-y^2)^{1+\nu-\mu}}, \quad \text{for } \mu > \nu > -1, \tag{25}$$

we can show that the normal stress boundary condition on $z = 0$ which requires that tractions outside the contact region $\mathcal{R}$ be zero, is satisfied by choosing $F_n(\alpha, s)$ to be of the form

$$F_n(\alpha,s) = -2G\sqrt{\alpha}\int_0^a \sqrt{\rho}\tilde{X}_n(\rho,s)J_{n-1/2}(\alpha\rho)d\rho, \tag{26}$$

where $\tilde{X}_n(\rho, s)$ is a function to be determined on the interval $[0, a]$. Substituting the equation (26) into (18), changing the order of integration and applying the operator $\sqrt{\frac{2}{\pi}}\frac{d}{\rho^n d\rho}\int_0^\rho \frac{x^{n+1}dx}{\sqrt{\rho^2-x^2}}$ to both sides of the resulting equation lead to the following integral equation for $\tilde{X}_n(\rho, s)$

$$\tilde{X}_n(\rho,s) + \int_0^a \tilde{X}_n(y,s)K_n(\rho,y,s)dy = \tilde{U}_n(\rho,s), \tag{27}$$

where the kernel function is given by

$$K_n(\rho,y,s) = \sqrt{\rho y}\int_0^\infty \alpha J_{n-1/2}(\rho\alpha)J_{n-1/2}(y\alpha)k(\alpha,s)d\alpha, \tag{28}$$

and the right hand side takes the form

$$\tilde{U}_n(\rho,s) = \frac{1}{\sqrt{2\pi}(1-\nu)\rho^n}\frac{d}{d\rho}\int_0^\rho \frac{x^{n+1}\tilde{f}_n(x,s)dx}{\sqrt{\rho^2-x^2}}. \tag{29}$$

Here we have used the displacement boundary conditions in (22). It is evident that equivalent equations for the indentation problem involving single indentor $\bar{\mathcal{R}}$ can be obtained in a similar way with $\tilde{X}_n$, $\tilde{f}_n$ and $\tilde{U}_n$ replaced by their counterparts $\tilde{\bar{X}}_n$, $\tilde{\bar{f}}_n$ and $\tilde{\bar{U}}_n$ in equations (27) and (29).

Now consider the normal indentation problem involving two indentors. It is clear that a superposition of the solutions for the two separate (either $\mathcal{R}$ or $\bar{\mathcal{R}}$) normal indentation problems satisfies all the boundary conditions except the displacement boundary conditions on the indentation surface $z = 0$. It will be shown that these displacement conditions result in a system of coupled Fredholm integral equations for $\tilde{X}_n(\rho,s)$ and $\tilde{\bar{X}}_n(\rho,s)$. In deriving these integral equations, expressions of relation (18) and its analogue for indentor $\bar{\mathcal{R}}$ in both local co-ordinate systems are required. As shown in the previous section, all the displacements, stresses and excess pore presure are generated from three displacements functions. For the indentation problem involving the half-space ($z \geq 0$), these displacement functions take the following form in the Laplace transform domain

$$\begin{aligned}
\tilde{\phi}(r,\theta,z,s) &= \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} [A_n(\alpha,s)e^{-\alpha z} + B_n(\alpha,s)e^{-\beta z}]\alpha J_n(\alpha r)d\alpha, \\
\tilde{\chi}(r,\theta,z,s) &= \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} C_n(\alpha,s)e^{-\alpha z}\alpha J_n(\alpha r)d\alpha, \\
\tilde{\psi}(r,\theta,z,s) &= \sum_{n=0}^{\infty} \sin(n\theta) \int_0^{\infty} D_n(\alpha,s)e^{-\alpha z}\alpha J_n(\alpha r)d\alpha. \qquad (30)
\end{aligned}$$

Note that all displacement and stress components derived from the above functions approach zero as $z \to \infty$. With the aid of the addition formula for Bessel functions (e.g. Watson [20]

$$J_m(r\alpha) \begin{matrix} \sin(m\theta) \\ \cos(m\theta) \end{matrix} = \sum_{n=-\infty}^{\infty} J_{m+n}(f\alpha)J_n(\bar{r}\alpha) \begin{matrix} \sin[n(\pi-\bar{\theta})] \\ \cos[n(\pi-\bar{\theta})], \end{matrix} \qquad (31)$$

we can express these three functions in terms of the second local co-ordinates $(\bar{r},\bar{\theta},\bar{z})$ as follows:

$$\begin{aligned}
\tilde{\phi}(r,\theta,z,s) &= \tilde{\phi}(\bar{r},\bar{\theta},\bar{z},s) \\
&= \sum_{n=0}^{\infty} \cos(n\bar{\theta}) \int_0^{\infty} \alpha J_n(\bar{r}\alpha)[A_n^*(\alpha,s)e^{-\alpha\bar{z}} + B_n^*(\alpha,s)e^{-\beta\bar{z}}]d\alpha, \\
\tilde{\chi}(r,\theta,z,s) &= \tilde{\chi}(\bar{r},\bar{\theta},\bar{z},s) = \sum_{n=0}^{\infty} \cos(n\bar{\theta}) \int_0^{\infty} \alpha J_n(\bar{r}\alpha)C_n^*(\alpha,s)e^{-\alpha\bar{z}}d\alpha, \\
\tilde{\psi}(r,\theta,z,s) &= \tilde{\psi}(\bar{r},\bar{\theta},\bar{z},s) = \sum_{n=1}^{\infty} \sin(n\bar{\theta}) \int_0^{\infty} \alpha J_n(\bar{r}\alpha)D_n^*(\alpha,s)e^{-\alpha\bar{z}}d\alpha, \qquad (32)
\end{aligned}$$

where

$$A_n^*(\alpha,s) = (-1)^n \sum_{m=0}^{\infty}{}' A_m(\alpha,s)[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)],$$

$$B_n^*(\alpha,s) = (-1)^n \sum_{m=0}^{\infty}{}' B_m(\alpha,s)[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)],$$

$$C_n^*(\alpha,s) = (-1)^n \sum_{m=0}^{\infty}{}' C_m(\alpha,s)[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)],$$

$$D_n^*(\alpha,s) = (-1)^{n+1} \sum_{m=1}^{\infty}{}' D_m(\alpha,s)[J_{m+n}(f\alpha) - (-1)^n J_{m-n}(f\alpha)]. \tag{33}$$

Here the prime on the summation sign implies that the $(-1)^n J_{m-n}(f\alpha)$ terms do not appear when $n = 0$. This shows that all the three functions take the same form in the two systems of cylindrical co-ordinates (r, θ, z) and $(\bar{r}, \bar{\theta}, \bar{z})$ and therefore the displacement and stress components also take the same form in these two co-ordinate systems. With this observation, the stress-displacement relationship equivalent to (18) expressed in terms of the second system of coordinates $(\bar{r}, \bar{\theta}, \bar{z})$, takes the form

$$u_{\bar{z}}^n(\bar{r},0) = -\frac{(1-\nu)}{G}\int_0^\infty F_n^*(\alpha,s)[1+k(\alpha,s)]J_n(\bar{r}\alpha)d\alpha, \tag{34}$$

with

$$F_n^*(\alpha,s) = (-1)^n \sum_{m=0}^{\infty}{}' F_m(\alpha,s)[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)]. \tag{35}$$

Similarly we can find the stress-displacement relationship for the second indentor $\bar{R}$ in terms of the first local coordinate system

$$\bar{u}_z^n(r,0) = -\frac{(1-\nu)}{G}\int_0^\infty \bar{F}_n^*(\alpha,s)[1+k(\alpha,s)]J_n(r\alpha)d\alpha, \tag{36}$$

where

$$\bar{F}_n^*(\alpha,s) = \sum_{m=0}^{\infty}{}' (-1)^m \bar{F}_m(\alpha,s)[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)]. \tag{37}$$

Equations (34) and (36) are results of special importance. Superposing the above two normal displacement fields in the two local coordinate systems respectively and substituting them into the normal displacement boundary conditions (22), results in the following system of coupled Fredholm integral equations for $\tilde{X}_n(\rho, s)$

and $\tilde{\bar{X}}_n(\rho, s)$,

$$\tilde{X}_n(\rho, s) + \int_0^a \tilde{X}_n(y, s)K_n(\rho, y, s)dy + \sum_{m=0}^{\infty}{}'(-1)^m \int_0^{\bar{a}} \tilde{\bar{X}}_m(y, s)K_{nm}^p(\rho, y, s)dy = \tilde{U}_n(\rho, s), \quad \text{on } \mathcal{R},$$

$$\tilde{\bar{X}}_n(\rho, s) + \int_0^{\bar{a}} \tilde{\bar{X}}_n(y, s)K_n(\rho, y, s)dy + (-1)^n \sum_{m=0}^{\infty}{}' \int_0^a \tilde{X}_m(y, s)K_{nm}^p(\rho, y, s)dy = \tilde{\bar{U}}_n(\rho, s), \quad \text{on } \bar{\mathcal{R}}, \tag{38}$$

where kernels $K_n(\rho, y, s)$ are given by (28), $K_{nm}^p(\rho, y, s)$ are defined as the following

$$K_{nm}^p(\rho, y, s) = \sqrt{\rho y} \int_0^\infty J_{m-1/2}(y\alpha)J_{n-1/2}(\rho\alpha)\alpha[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)][1 + k(\alpha, s)]d\alpha \tag{39}$$

and the right hand sides of the second integral equation are the same as that of (29) for the first equation with $\tilde{\bar{f}}_n(x)$ replacing $\tilde{f}_n(x)$.

Governing equations in the time domain can then be obtained by taking Laplace inverse transforms of the above integral equation set. They are

$$X_n(\rho, t) + \int_0^a \int_0^t X_n(y, t - \tau)K_n^1(\rho, y, \tau)d\tau dy + \sum_{m=0}^{\infty}{}'(-1)^m \int_0^{\bar{a}} \left\{\bar{X}_m(y, t)K_{nm}^2(\rho, y) + \int_0^t \bar{X}_m(y, t - \tau)K_{nm}^3(\rho, y, \tau)d\tau\right\} dy = U_n(\rho, t), \quad \text{on } \mathcal{R},$$

$$\bar{X}_n(\rho, t) + \int_0^{\bar{a}} \int_0^t \bar{X}_n(y, t - \tau)K_n^1(\rho, y, \tau)d\tau dy + (-1)^n \sum_{m=0}^{\infty}{}' \int_0^a \left\{X_m(y, t)K_{nm}^2(\rho, y) + \int_0^t X_m(y, t - \tau)K_{nm}^3(\rho, y, \tau)d\tau\right\} dy = \bar{U}_n(\rho, t), \quad \text{on } \bar{\mathcal{R}}, \tag{40}$$

where the kernels are

$$K_n^1(\rho, y, t) = \sqrt{\rho y} \int_0^\infty \alpha J_{n-1/2}(\rho\alpha)J_{n-1/2}(y\alpha)K(\alpha, t)d\alpha,$$

$$K_{nm}^2(\rho, y) = \sqrt{\rho y} \int_0^\infty J_{m-1/2}(y\alpha)J_{n-1/2}(\rho\alpha)\alpha[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)]d\alpha,$$

$$K_{nm}^3(\rho, y, t) = \sqrt{\rho y} \int_0^\infty J_{m-1/2}(y\alpha)J_{n-1/2}(\rho\alpha)\alpha[J_{m+n}(f\alpha) + (-1)^n J_{m-n}(f\alpha)]K(\alpha, t)d\alpha$$

and function $K(\alpha,t)$ is the inverse Laplace transform of $k(\alpha,s)$. Now we have reduced the two-indentor problem for a poroelastic half-space to an infinite set of integral equations. Once $X_n(\rho,t)$, $\bar{X}_n(\rho,t)$ or $\tilde{X}_n(\rho,s)$, $\tilde{\bar{X}}_n(\rho,s)$ are known, pressures in the contact regions can be obtained from equations (24) and (26) and their analogue for $\bar{\mathcal{R}}$. For example, pressure $p(r,\theta,t)$ under indentor $\mathcal{R}$ takes the following form

$$p(r,\theta,t) = 2G\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty} r^{n-1}\frac{d}{dr}\int_r^a \frac{X_n(\rho,t)d\rho}{\rho^{n-1}\sqrt{\rho^2-r^2}}\cos(n\theta). \tag{42}$$

The total force P_z in the z direction exerted by indentor $\mathcal{R}$ can be obtained by integrating the pressure (42) over the contact area $\mathcal{R}$; i.e.

$$P_z(t) = -\int_0^{2\pi}\int_0^a p(r,\theta,t)rdrd\theta = 4G\sqrt{2\pi}\int_0^a X_0(\rho,t)d\rho, \tag{43}$$

and the resultant moment M_y about y direction can also be found from (42)

$$M_y(t) = \int_0^{2\pi}\int_0^a p(r,\theta,t)r^2\cos(\theta)drd\theta = -4G\sqrt{2\pi}\int_0^a \rho X_1(\rho,t)d\rho. \tag{44}$$

Similar expressions for the force and moment resultants in the region $\bar{\mathcal{R}}$ can be obtained by considering the equilibrium of the indentor $\bar{\mathcal{R}}$.

4. Specific solutions

In this section we solve the general system of integral equations derived in the previous section for three special cases, namely the case of normal indentation by a single indentor, the case of interaction between an indentor and a point force and the case of symmetric indentation by two identical indentors.

<u>**A single indentor problem:**</u> We consider first the special case in which the poroelastic half-space is indented by a single flat indentor subjected to the constant force P_z. Due to the axial symmetry of the problem, the only unknown is $\tilde{X}_0(\rho,s)$ which satisfies

$$\tilde{X}_0(\rho,s) + \int_0^a \tilde{X}_0(y,s)K_0(\rho,y,s)dy = \tilde{U}_0(s), \tag{45}$$

and

$$4G\sqrt{2\pi}\int_0^a \tilde{X}_0(\rho,s)d\rho = \frac{P_z}{s}. \tag{46}$$

Considering the structure of the kernel functions $K_0(\rho,y,s)$ given by (28), it is unlikely that the system of complex integral equations has analytical solutions.

There are some numerical methods available in the literature for poroelastic contact problems. These methods can be divided into two categories. The first method, e.g. [8], solves the governing equations directly in the time domain while the second, e.g. [10, 11], deals with the integral equations in the Laplace domain first and then applies Laplace transform inversion procedure. The advantage of the first method is that it avoids the procedure involved in the numerical inversion of Laplace transforms, which in some circumstances could be unstable. The main shortcoming of this method is that we have to deal with a system of double integral equations, the kernels of which themselves are infinite integrals. Also it is computing intensive to find solution for a large time, since it requires (due to the convolution nature) all the solutions at previous times. By using the second method we can find solutions at a specific time without knowing solutions at other time.

In this paper we adopt a second type numerical scheme for the evaluation of the time-dependent solutions of the integral equations. After some minor changes, the technique described below also applies to the more complicated indentation problem such as the one considered later in this section. This numerical algorithm consists of two major steps. The first step involves solving the integral equations in the Laplace domain for each given Laplace transform parameter s. Rewriting equations (45) and (46) in a non-dimensional form, we have

$$\underline{X}_0(r,\underline{s}) + \int_0^1 \underline{X}_0(y,\underline{s})\underline{K}_0(r,y,\underline{s})dy = \underline{U}_0(\underline{s}),$$
$$\int_0^1 \underline{X}_0(\underline{s})dr = 1/\underline{s}, \tag{47}$$

where the new non-dimensional notations are

$$\underline{X}_0(r,\underline{s}) = \frac{4G\sqrt{2\pi}a\tilde{X}_0(\rho,s)}{P_z}; \quad \underline{U}_0(\underline{s}) = \frac{4G\sqrt{2\pi}a}{P_z}\tilde{U}_0(s); \quad r = \frac{\rho}{a}; \quad \underline{s} = \frac{a^2 s}{c};$$
$$\underline{K}_0(r,y,\underline{s}) = a^2 K_0(\rho, ay, s) = \frac{2}{\pi}\int_0^\infty \cos(r\alpha)\cos(y\alpha)k(\alpha,\underline{s})d\alpha. \tag{48}$$

Dividing the integral $[0,1]$ into N equal segments and letting the collocation points x_i be the midpoints of each segments, the integral equations can be converted into two systems of linear algebraic equations of the form

$$AX - BY = R_1; \quad AY + BX = R_2, \tag{49}$$

where A and B are two $(N+1)\times(N+1)$ matrices, X is an $(N+1)$ dimensional vector with the first N components being the real part of $\underline{X}_0(x_i,\underline{s})$, $i = 1,2,...,N$, and the last component being the real part of $\underline{U}_0(\underline{s})$, Y is also an $(N+1)$ dimensional vector with the first N components being the imaginary part of $\underline{X}_0(x_i,\underline{s})$,

$i = 1, 2, \dots, N$, and the last component being the imaginary part of $\underline{U}_0(\underline{s})$, and right hand sides R_1 and R_2 are two $(N+1)$ dimensional vectors with the first N components being zeros and the last components being the real and imaginary parts of $1/\underline{s}$, respectively.

In solving the coupled linear systems (49), the most expensive computation occurs in forming the two $(N+1) \times (N+1)$ matrices A and B. Each entry of these two matrices involves numerical evaluation of an infinite integral (48). Due to the presence of the oscillatory factor $\cos(\rho s)\cos(ys)$ in the integrand of this infinite integral, accurate integration requires adding more Gauss integration points. This can be very computing intensive considering that this numerical integration process has to be done for every element in matrices A and B. An effort is made to reduce the computation time by replacing function $k(\alpha, s)$ with a linear combination of a properly chosen function set, which best fits $k(\alpha, s)$ at some selected points. These fitting functions are chosen in such a way that analytical expressions for the kernels are available, and the coefficients of these fitting functions are determined by a least squares method. The accuracy with which the kernel (48) is evaluated depends on the selection of the fitting functions and the fitting points. Since the fitting data are virtually all the data required to evaluate the infinite integral, the best possible choice of the fitting points is the Gauss points. Numerical evaluations show that the following combination gives a good fit to function $k(\alpha, s)$

$$k(\alpha, s) = c_0(s) + c_1(s)e^{-\alpha} + c_2(s)e^{-2\alpha} + \sum_{n=4}^{M} c_n(s)\frac{1}{\alpha^2 + (n+2)^2}, \tag{50}$$

and that $M = 6$ is sufficient to give satisfactory results. The advantage of this method is that only one of these fitting processes is required for each given Laplace transform parameter s.

The second step of this numerical scheme involves the inversion of Laplace transforms. In the present paper we adopt the modified version [10, 11] of the method proposed by Crump [21].

Some of the numerical results are shown in figures 2 and 3. Also shown in these two figures are solutions obtained by using Heinrich and Desoyer's assumption [17]. Guided by the observation that the initial and final contact stress distribution under the indentor are the same, they assume that the contact pressure remains unchanged throughout consolidation process and identify the consolidation of the clay stratum as the average of the normal displacement under the indentor. This assumption dramatically simplifies the numerical procedure for the problem. Figure 2 shows the influence of Poisson's ratio on the non-dimensional time-dependent settlement behaviour of the single indentor and Figure 3 shows the effect of the surface drainage boundary conditions. In both these figures, $\Delta_0^e = \frac{(1-\nu)P_z}{4Ga}$. From these two figures, we can see that Poisson's ratio has significant influences on the consolidation of the poroelastic half-space. Results also show that Heinrich and

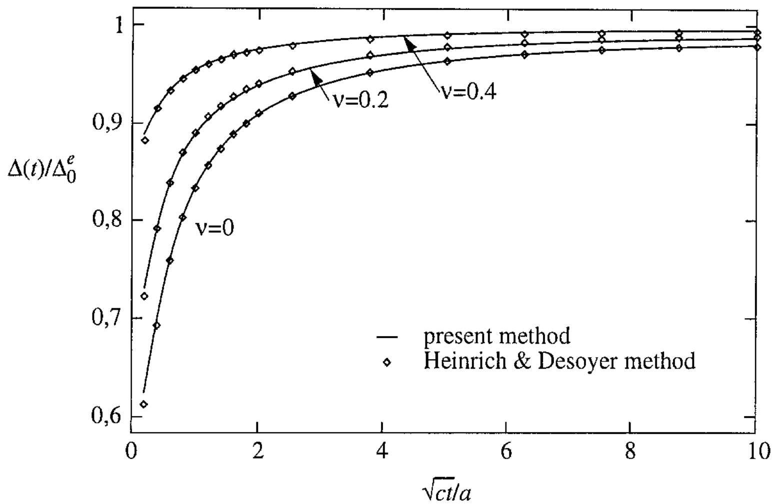

Figure 2. Single Indentor: Effect of the Poissson's ratio on the non-dimensional settlement $\Delta(t)/\Delta_0^e$ when the surface of the poroelastic half-space is completely permeable.

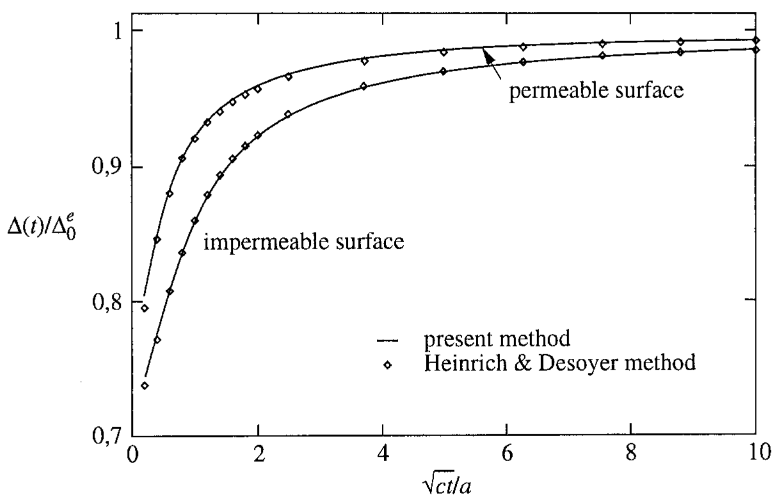

Figure 3. Single Indentor: Effect of the surface drainage conditions on the settlement $\Delta(t)/\Delta_0^e$ for $\nu = 0.3$.

Desoyer's assumption gives satisfactory results after the very early stage of consolidation. However, their method does not provide an indication of the variation of contact stress distribution with time and is not applicable to more complicated problems discussed later.

Interaction between an indentor and an external concentrated force:

We now consider the interaction between a rigid circular indentor and a concentrated normal force $\bar{P}_z$ applied at point $(f,0,0)$ in the coordinates (r,θ,z). In this case the radius of the second indentor approaches zero in such a way that functions $\tilde{\bar{X}}_n(\rho,s)$ satisfy

$$4G\sqrt{2\pi}\lim_{\bar{a}\to 0}\int_0^{\bar{a}}\tilde{\bar{X}}_0(\rho,s)d\rho = \frac{\bar{P}_z}{s};\quad \text{and}\ \tilde{\bar{X}}_n(\rho,s)=0,\ n=1,2,3,.. \tag{51}$$

in the Laplace domain. As the first example, we examine the resultant force and moment required to maintain the indentor with zero normal displacement. In the Laplace transform domain the governing equations for this problem are as follows

$$\tilde{X}_n(\rho,s)+\int_0^a \tilde{X}_n(y,s)K_n(\rho,y,s)dy = \frac{-\bar{P}_z}{4Gs\sqrt{2\pi}}K^p_{n0}(\rho,0,s). \tag{52}$$

Before solving this problem, it is instructive to first examine two limiting cases, the initial $(t\to 0^+)$ and the final $(t\to\infty)$ stress distribution under the indentor. Results show that the stress distributions in these two limiting cases are the same, and they are determined by

$$\begin{aligned} X_0(\rho,0^+) = X_0(\rho,\infty) &= \frac{-\bar{P}_z}{2\pi G\sqrt{2\pi}}\frac{1}{\sqrt{f^2-\rho^2}};\\ X_n(\rho,0^+) = X_n(\rho,\infty) &= \frac{-\bar{P}_z}{\pi G\sqrt{2\pi}}\frac{\rho^n}{f^n\sqrt{f^2-\rho^2}},\quad \text{for } n\geq 1. \end{aligned} \tag{53}$$

This result suggests that

$$X_0(\rho,t) = \frac{-\bar{P}_z}{2\pi G\sqrt{2\pi}}\frac{1}{\sqrt{f^2-\rho^2}};\quad X_n(\rho,t) = \frac{-\bar{P}_z}{\pi G\sqrt{2\pi}}\frac{\rho^n}{f^n\sqrt{f^2-\rho^2}} \tag{54}$$

could be a solution for equation (52). With the aid of two special cases of the identity (25), we can show that (54) does satisfy equation (52) and therefore is indeed the solution. Using the above solutions, the force and moment required to maintain zero normal displacement of the indentor follow from (43) and (44)

$$\begin{aligned} P_z &= -\frac{2\bar{P}_z}{\pi}\arcsin(\frac{a}{f}),\\ M_y &= \frac{2}{\pi}\bar{P}_z f\left[\arcsin(\frac{a}{f}) - \frac{a}{f}\sqrt{1-\frac{a^2}{f^2}}\right]. \end{aligned} \tag{55}$$

These results indicate that the force and moment required to maintain the indentor with no normal displacement are time-independent and they are exactly the same as those for the associated elastic half-space problem given by Selvadurai [12].

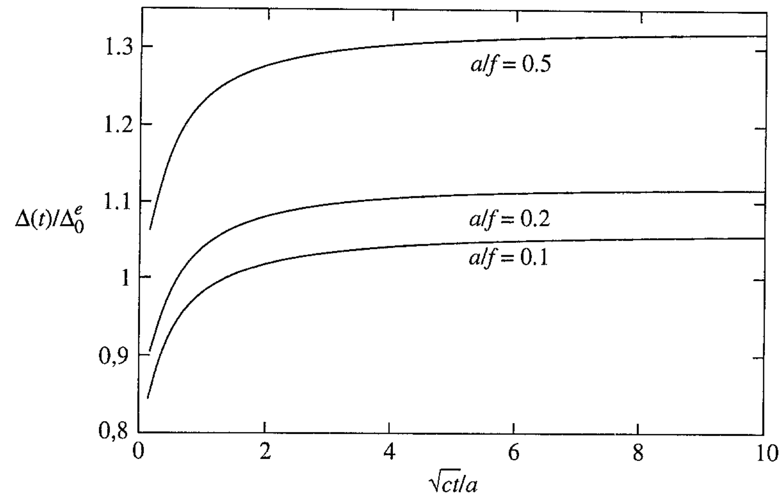

Figure 4. Interaction between a rigid indentor and an externally placed normal force: Variation of the non-dimensional consolidation $\Delta(t)/\Delta_0^e$ with respect to time for $\bar{P}_z/P_z = 1$, $\nu = 0.3$ and various values of $\frac{a}{f}$ when the surface of the poroelastic half-space is completely permeable.

As the second example, we consider a more practically relevant problem in which the indentor, instead of being restrained with zero normal displacement, is subjected to a constant axial loading P_z. Hence we have

$$u_z(r,\theta,0,t) = \Delta(t) + \Omega(t) r\cos(\theta), \tag{56}$$

and

$$4G\sqrt{2\pi}\int_0^a \tilde{X}_0(\rho,s)d\rho = \frac{P_z}{s}; \quad \text{and } M_y = 4G\sqrt{2\pi}\int_0^a \rho\tilde{X}_1(\rho,s)d\rho = 0, \tag{57}$$

with the settlement $\Delta(t)$ and the tilt angle $\Omega(t)$ to be determined by

$$\tilde{X}_n(\rho,s) + \int_0^a \tilde{X}_n(y,s)K_n(\rho,y,s)dy = \tilde{U}_n(\rho,s) - \frac{\bar{P}_z}{4Gs\sqrt{2\pi}}K_{n0}^p(\rho,0,s), \quad n = 0,1. \tag{58}$$

As for the single indentor problem, the unknowns decouple and therefore we can solve the resulting equations for $\tilde{X}_0$ and $\tilde{X}_1$ separately. For both $n = 0$ and $n = 1$ the algebraic equations arising from the integral equations are in the form of (49). After some minor changes in the matrices A and B and the right hand sides R_1 and R_2, these linear algebraic equations are solved similarly as those for the single indentor problem. Some of the numerical results are shown in figures 4–6. In figures 5 and 6, $\Omega_0^e = \frac{3(1-\nu)\bar{P}_z}{4\pi G a^2}$. Unlike the settlement behaviour as shown in Figures 2, 3 and 4, the tilt angle reduces to its elastic limit much faster than the

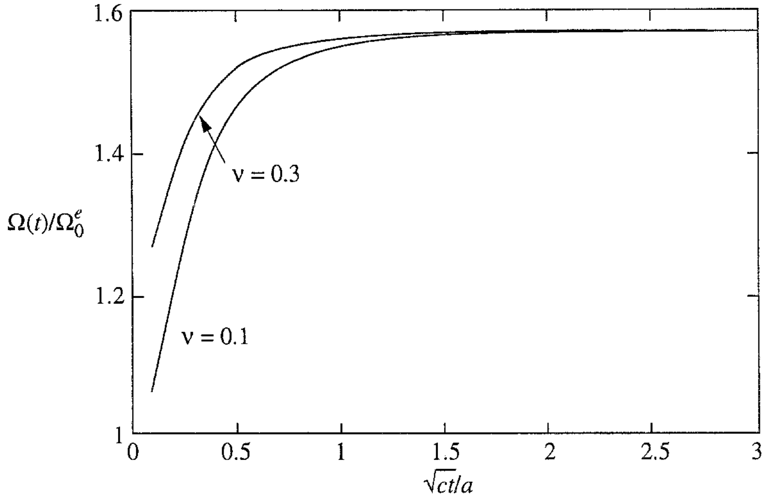

Figure 5. Interaction between a rigid indentor and an externally placed normal force: Variation of the non-dimensional tilt angle $\Omega(t)/\Omega_0^e$ with respecty to time for $\frac{f}{a}$ = 1 and various value surface of the poroelastic half-space is completely permeable.

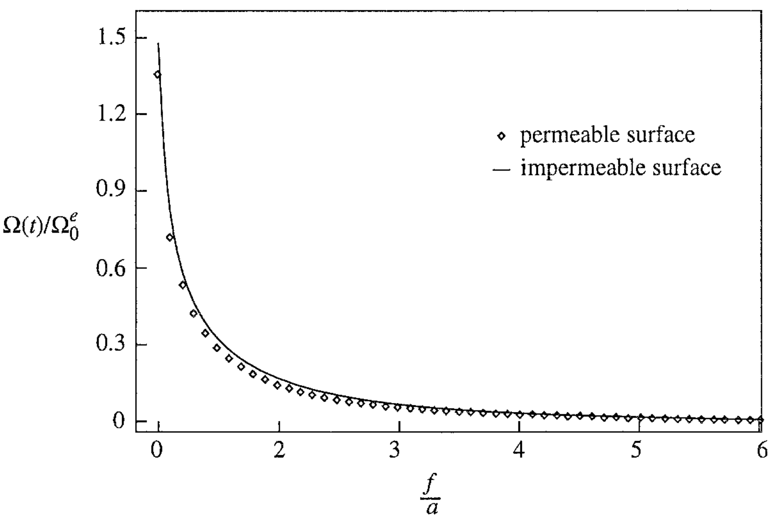

Figure 6. Interaction between a rigid indentor and an externally placed normal force: Variation of the non-dimensional tilt angle $\Omega(t)/\Omega_0^e$ with respect to the spatial ratio $\frac{f}{a}$ for $\nu = 0.3$

settlement does. For a given time, we also note from figure 6 that the tilt angle decreases to zero very rapidly as the spatial ratio $\frac{f}{a}$ increases.

Two identical indentor problems: Finally we consider the problem of two identical indentors penetrating a poroelastic medium to equal depths $\Delta(t)$. This problem can be interpreted, from a practical point of view, as a problem of the rotation free indentation of a poroelastic halfspace by two identical indentors, which are connected via a rigid structural element. The displacement boundary conditions of the problem are $f_0(x,t) = \bar{f}_0(x,t) = \Delta(t)$ and $f_n(x,t) = \bar{f}_n(x,t) = 0$ for $n \geq 1$. From the symmetry of this problem, we note that $X_n(\rho,t) = (-1)^n \bar{X}_n(\rho,t)$ in the time domain and consequently $\tilde{X}_n(\rho,s) = (-1)^n \tilde{\bar{X}}_n(\rho,s)$ in the Laplace domain. The governing integral equations for this problem become

$$\tilde{X}_n(\rho,s) + \int_0^a \tilde{X}_n(y,s)K_n(\rho,y,s)dy + \sum_{m=0}^{\infty}{}' \int_0^a \tilde{X}_m(y,s)K^p_{nm}(\rho,y,s)dy = \tilde{U}_n(\rho,s), \tag{59}$$

with

$$4G\sqrt{2\pi}\int_0^a \tilde{X}_0(\rho,s)d\rho = P_z/s; \quad 4G\sqrt{2\pi}\int_0^{\bar{a}} \tilde{\bar{X}}_0(\rho,s)d\rho = \bar{P}_z/s, \tag{60}$$

in the Laplace transform domain. Unlike the previous cases, the unknowns are coupled. It is unlikely that an exact solution can be found for the above infinite system of coupled integral equations. Considering that each integral equation in the infinite set of equations has an infinitely long interval, we have to truncate them even before resorting to any numerical methods. From the definitions of the kernels, we can see that the non-dimensional kernels K^p_{nm} (K^2_{nm} and K^3_{nm} in the time domain) can be expanded as power series in terms of $\epsilon_f = \frac{a}{f}$ and are of the order $O(\epsilon_f^{m+n+1})$. Therefore, all the equations for $n \geq 3$ and all these terms with $m \geq 3$ in the first three equations($n = 0,1,2$) can be ignored if we are seeking solutions accurate to the order of $O(\epsilon_f^3)$. The resulting truncated equations can then be solved iteratively by using the same numerical technique as described for the first case. Computational experiments show that solutions converge after three or four iterations. Some of the results are shown in Figure 7. It is worth noting that an extra moment of the order $O(\epsilon_f^2)$ is required to maintain the indentor without any tilt.

5. Conclusions

In this paper a general formulation for the three-dimensional problem in poroelasticity is presented, which can be considered as a generalization of Muki's formulation [15] in elasticity. By using the methodology proposed in [16] for the two-indentor problem for an elastic layer, we reduce the problem of two interacting indentors on a poroelastic half space to an infinite set of integral equations. Analytical solution is obtained by a semi-inverse method for the resultant force and moment required to maintain the first indentor with no normal displacement when the radius of the second indentor approaches zero. For the other cases discussed, a very efficient and accurate numerical scheme is developed to evaluate the

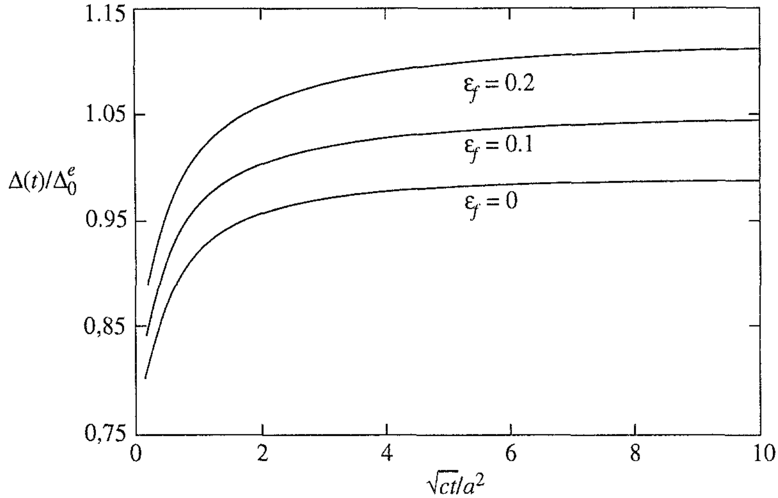

Figure 7. Symmetric settlement of two interacting indentors: Effect of the spatial ratio on the non-dimensional settlement $\Delta(t)/\Delta_0^e$ when the surface of the poroelastic half-space is completely permeable ($\nu = 0.3$).

time-dependent solutions of the resulting integral equations. Computational approximation shows that the Poisson's ratio and the spatial ratio ϵ_f play significant roles in the settlement of the indentors.

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