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Dirichlet problem for an annular disk

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Introduction

It is impossible even to mention all the publications related to the Dirichlet problem for a flat circular annulus. Their number is awesome. Tranter (1960) and Gubenko (1960) were among the first to consider the problem. One can find many references related to the mathematically equivalent contact problem in (Borodachev, 1976), other references related to the equivalent electrostatic problem can be found in Love (1976). Why is there any need for yet another paper on the subject? The main reason is that the majority of publications is devoted to the simplest flat centrally loaded annular punch problem. Though some results related to consideration of specific harmonics have been published (Williams, 1963; Cooke, 1963), no general solution to the problem has been attempted as yet. This kind of solution is now possible due to the new results in potential theory obtained by the author (Fabrikant, 1989). The problem is reduced to a set of two two-dimensional Fredholm integral equations with an elementary non-singular kernel which can be solved by iteration. This set can be easily uncoupled. The case of conducting circular annulus kept at constant potential and the problem of magnetic polarizability of such a disk are considered as examples. The governing integral equations are solved exactly in series involving the iterated kernels. Approximate formulae are derived for the case of a wide annulus.

Theory

It is convenient to reformulate the Dirichlet problem for a circular annulus as a mixed boundary value problem of potential theory for a half space $z \geq 0$. We need to find a harmonic function V vanishing at infinity and satisfying the following conditions at $z = 0$:

$$\begin{aligned} &V(\varrho, \phi, 0) = v(\varrho, \phi), \quad \text{for } b < \varrho < a, 0 \leq \phi < 2\pi; \\ &\frac{\partial V}{\partial z} = 0, \quad \text{for } \varrho < b \text{ or } \varrho > a, 0 \leq \phi < 2\pi. \end{aligned} \tag{1}$$

Here v is a known function. The approach proposed here is inspired by the elegant solution for the capacity of an annulus (Love, 1976) which is based on the method described in (Clement and Love, 1974) for solving axisymmetric problems. It looked very challenging to generalize the approach for non-axisymmetric case. Such a generalization has been found after several trials and errors, and it is presented here. The general approach is based on the recent results of the writer (Fabrikant, 1989). Let us introduce two harmonic functions

$$V_1(\varrho, \phi, z) = -\frac{1}{\pi^2}\int_0^{2\pi}\int_0^b \frac{\sqrt{\varrho_0^2 - l_1^2(\varrho_0)}}{R_0^2} f_1(\varrho_0, \phi_0)\, d\varrho_0\, d\phi_0$$
$$= -\frac{2}{\pi}\int_0^b \frac{\sqrt{\varrho_0^2 - l_1^2(\varrho_0)}}{l_2^2(\varrho_0) - l_1^2(\varrho_0)} \mathscr{L}\left(\frac{l_1(\varrho_0)}{l_2(\varrho_0)}\right) f_1(\varrho_0, \phi)\, d\varrho_0; \tag{2}$$

$$V_2(\varrho, \phi, z) = \frac{1}{\pi^2}\int_0^{2\pi}\int_a^\infty \frac{\sqrt{l_2^2(\varrho_0) - \varrho_0^2}}{R_0^2} f_2(\varrho_0, \phi_0)\, d\varrho_0\, d\phi_0$$
$$= \frac{2}{\pi}\int_a^\infty \frac{\sqrt{l_2^2(\varrho_0) - \varrho_0^2}}{l_2^2(\varrho_0) - l_1^2(\varrho_0)} \mathscr{L}\left(\frac{l_1(\varrho_0)}{l_2(\varrho_0)}\right) f_2(\varrho_0, \phi)\, d\varrho_0. \tag{3}$$

Here f_1 and f_2 are the as yet unknown functions, and the following notations were introduced:

$$l_1(x) = \tfrac{1}{2}\{\sqrt{(\varrho + x)^2 + z^2} - \sqrt{(\varrho - x)^2 + z^2}\},$$
$$l_2(x) = \tfrac{1}{2}\{\sqrt{(\varrho + x)^2 + z^2} + \sqrt{(\varrho - x)^2 + z^2}\}, \tag{4}$$
$$R_0 = \sqrt{\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0\cos(\phi - \phi_0) + z^2}.$$

In (Fabrikant, 1989) the $\mathscr{L}$-operator was introduced as follows:

$$\mathscr{L}(k)f(\varrho, \phi) = \frac{1}{2\pi}\int_0^{2\pi} \lambda(k, \phi - \phi_0) f(\varrho, \phi_0)\, d\phi_0$$
$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} k^{|n|} e^{in\phi}\int_0^{2\pi} e^{-in\phi_0} f(\varrho, \phi_0)\, d\phi_0$$
$$= \sum_{n=-\infty}^{\infty} k^{|n|} f_n(\varrho)\, e^{in\phi}. \tag{5}$$

Here f_n is the nth Fourier coefficient of the function f, and

$$\lambda(k, \psi) = \frac{1 - k^2}{1 - 2k\cos\psi + k^2}. \tag{6}$$

One can easily verify that the potential in (2) vanishes on the plane $z = 0$ for $\varrho > b$, while the potential in (3) vanishes on the boundary for $\varrho < a$. These properties allow us to reformulate the problem as the Dirichlet problem for a half space, with the potential prescribed all over the plane $z = 0$, namely,

$$V(\varrho, \phi, 0) = V_1(\varrho, \phi, 0), \quad \text{for } 0 \le \varrho < b, 0 \le \phi < 2\pi;$$
$$V(\varrho, \phi, 0) = v(\varrho, \phi), \quad \text{for } b \le \varrho \le a, 0 \le \phi < 2\pi; \tag{7}$$
$$V(\varrho, \phi, 0) = V_2(\varrho, \phi, 0), \quad \text{for } a < \varrho < \infty, 0 \le \phi < 2\pi.$$

Thus, the first boundary condition in (1) is satisfied. The unknown functions f_1 and f_2 are to be chosen in such a way that the second boundary condition in (1) is satisfied too.

Formulae (2) and (3) on the plane $z = 0$ take the form

$$V_1(\varrho, \phi, 0) = -\frac{2}{\pi} \int_\varrho^b \mathscr{L}\left(\frac{\varrho}{\varrho_0}\right) \frac{f_1(\varrho_0, \phi)\, d\varrho_0}{\sqrt{\varrho_0^2 - \varrho^2}},$$
$$V_2(\varrho, \phi, 0) = \frac{2}{\pi} \int_a^\varrho \mathscr{L}\left(\frac{\varrho_0}{\varrho}\right) \frac{f_2(\varrho_0, \phi)\, d\varrho_0}{\sqrt{\varrho^2 - \varrho_0^2}}. \tag{8}$$

On the other hand, the following expression has been obtained in (Fabrikant, 1989) for the potential which is nonzero in the interval $[0, b]$ and zero outside this interval:

$$V_1(\varrho, \phi, 0) = 4 \int_\varrho^b \frac{dx}{\sqrt{x^2 - \varrho^2}} \int_0^x \frac{\varrho_0\, d\varrho_0}{\sqrt{x^2 - \varrho_0^2}} \mathscr{L}\left(\frac{\varrho\varrho_0}{x^2}\right) \sigma_1(\varrho_0, \phi). \tag{9}$$

Here σ_1 denotes the charge density distribution which is defined in the traditional way as $(-1/2\pi)\,\partial V_1/\partial z$. Comparison of (7) and (9) yields the following relationship between σ_1 and f_1:

$$f_1(\varrho, \phi) = -2\pi \int_0^\varrho \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho^2 - \varrho_0^2}} \mathscr{L}\left(\frac{\varrho_0}{\varrho}\right) \sigma_1(\varrho_0, \phi). \tag{10}$$

The inverse relationship is readily available, and is

$$\sigma_1(\varrho, \phi) = -\frac{1}{\pi^2 \varrho} \mathscr{L}\left(\frac{1}{\varrho}\right) \frac{d}{d\varrho} \int_0^{\min(\varrho, b)} \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho^2 - \varrho_0^2}} \mathscr{L}(\varrho_0) f_1(\varrho_0, \phi). \tag{11}$$

By comparing in the same manner the expression (Fabrikant, 1989)

$$V_2(\varrho, \phi, 0) = 4 \int_a^\varrho \frac{dx}{\sqrt{\varrho^2 - x^2}} \int_x^\infty \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho_0^2 - x^2}} \mathscr{L}\left(\frac{x^2}{\varrho\varrho_0}\right) \sigma_2(\varrho_0, \phi), \tag{12}$$

with (8), we obtain

$$f_2(\varrho, \phi) = 2\pi \int_\varrho^\infty \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho_0^2 - \varrho^2}} \mathscr{L}\left(\frac{\varrho}{\varrho_0}\right) \sigma_2(\varrho_0, \phi). \tag{13}$$

The inverse to (13) takes the form

$$\sigma_2(\varrho, \phi) = -\frac{\mathscr{L}(\varrho)}{\pi^2 \varrho} \frac{d}{d\varrho} \int_{\max(\varrho, a)}^\infty \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho_0^2 - \varrho^2}} \mathscr{L}\left(\frac{1}{\varrho_0}\right) f_2(\varrho_0, \phi). \tag{14}$$

We have presented the resultant field as a superposition of three fields, namely, the field due to potential equal to $V_1(\varrho, \phi, 0)$ on the interval $0 \le \varrho \le b$, and zero outside the interval; the field due to potential equal to $v(\varrho, \phi)$ in the interval $b \le \varrho \le a$, and zero outside this interval; and the field due to potential equal to $V_2(\varrho, \phi, 0)$ on the interval $a \le \varrho < \infty$, and zero outside the interval. The corresponding charge density distributions will be denoted as σ_1, σ_0, and σ_2 respectively. Now we can use the fact that the total charge $\sigma = \sigma_1 + \sigma_0 + \sigma_2 = 0$ in the intervals $\varrho \le b$ and $\varrho \ge a$. These conditions will give us two equations from which the as yet unknown functions f_1 and f_2 can be found. By using the result established in (Fabrikant, 1989), we can write

$$\sigma_0(\varrho, \phi) = -\frac{1}{\pi^2 \varrho} \mathscr{L}\left(\frac{1}{\varrho}\right) \frac{d}{d\varrho} \int_0^\varrho \frac{x\,dx}{\sqrt{\varrho^2 - x^2}} \mathscr{L}(x^2) \frac{d}{dx} \times \int_b^a \frac{\varrho_0\,d\varrho_0}{\sqrt{\varrho_0^2 - x^2}} \mathscr{L}\left(\frac{1}{\varrho_0}\right) v(\varrho_0, \phi), \quad \text{for } \varrho \le b. \tag{15}$$

By using (11), (14), and (15) the following equation may be written for $\varrho \le b$:

$$-\frac{1}{\pi^2 \varrho} \mathscr{L}\left(\frac{1}{\varrho}\right) \frac{d}{d\varrho} \int_0^\varrho \frac{\varrho_0\,d\varrho_0}{\sqrt{\varrho^2 - \varrho_0^2}} \mathscr{L}(\varrho_0) f_1(\varrho_0, \phi) - \frac{\mathscr{L}(\varrho)}{\pi^2 \varrho} \frac{d}{d\varrho} \times \int_a^\infty \frac{\varrho_0\,d\varrho_0}{\sqrt{\varrho_0^2 - \varrho^2}} \mathscr{L}\left(\frac{1}{\varrho_0}\right) f_2(\varrho_0, \phi)$$
$$-\frac{1}{\pi^2 \varrho} \mathscr{L}\left(\frac{1}{\varrho}\right) \frac{d}{d\varrho} \int_0^\varrho \frac{x\,dx}{\sqrt{\varrho^2 - x^2}} \mathscr{L}(x^2) \frac{d}{dx} \times \int_b^a \frac{\varrho_0\,d\varrho_0}{\sqrt{\varrho_0^2 - x^2}} \mathscr{L}\left(\frac{1}{\varrho_0}\right) v(\varrho_0, \phi) = 0. \tag{16}$$

Application of an operator

$$\int_0^r \frac{\varrho\,d\varrho}{\sqrt{r^2 - \varrho^2}} \mathscr{L}\left(\frac{\varrho}{r}\right)$$

to both sides of (16) yields

$$\frac{1}{2\pi} f_1(r, \phi) + \frac{1}{\pi^2} \int_0^r \frac{\varrho\,d\varrho}{\sqrt{r^2 - \varrho^2}} \int_a^\infty \frac{\varrho_0\,d\varrho_0}{(\varrho_0^2 - \varrho^2)^{3/2}} \mathscr{L}\left(\frac{\varrho^2}{\varrho_0 r}\right) f_2(\varrho_0, \phi)$$
$$+\frac{1}{2\pi} r \int_b^a \frac{\varrho_0\,d\varrho_0}{(\varrho_0^2 - r^2)^{3/2}} \mathscr{L}\left(\frac{r}{\varrho_0}\right) v(\varrho_0, \phi) = 0. \tag{17}$$

We can interchange the order of integration in the second term of (17) and perform the integration with respect to ϱ. Then equation (17) for $0 \le \varrho \le b, 0 \le \phi < 2\pi$ will take the form

$$f_1(\varrho, \phi) + \frac{1}{\pi^2} \int_0^{2\pi} \int_a^{\infty} Q_1(\varrho, \varrho_0, \phi - \phi_0) f_2(\varrho_0, \phi_0)\, d\varrho_0\, d\phi_0 = g_1(\varrho, \phi), \quad (18)$$

where

$$g_1(\varrho, \phi) = -\varrho \int_b^a \frac{\varrho_0\, d\varrho_0}{(\varrho_0^2 - \varrho^2)^{3/2}} \mathscr{L}\left(\frac{\varrho}{\varrho_0}\right) v(\varrho_0, \phi), \quad (19)$$

$$Q_1(\varrho, \varrho_0, \phi - \phi_0)$$
$$= 2\Re\left\{\frac{\sqrt{\varrho\varrho_0\, e^{i(\phi - \phi_0)}}}{(\varrho_0\, e^{i(\phi - \phi_0)} - \varrho)R} \tan^{-1}\left[\frac{e^{i(\phi - \phi_0)} - (\varrho/\varrho_0)}{(\varrho_0/\varrho) - e^{i(\phi - \phi_0)}}\right]^{1/2}\right\} - \frac{\varrho}{R^2}, \quad (20)$$

with

$$R = \sqrt{\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \cos(\phi - \phi_0)}. \quad (21)$$

Here the following integral was used

$$\int_0^r \frac{\varrho\, d\varrho}{\sqrt{r^2 - \varrho^2}(y^2 - \varrho^2)^{3/2}(1 - m\varrho^2)}$$
$$= \frac{m}{(my^2 - 1)^{3/2}\sqrt{1 - mr^2}} \tan^{-1}\left[\frac{r\sqrt{my^2 - 1}}{y\sqrt{1 - mr^2}}\right] - \frac{r}{y(y^2 - r^2)(my^2 - 1)}. \quad (22)$$

Everywhere in this paper the branch with positive real part is taken when computing a square root of a complex number. The $\tan^{-1}$ of a complex number is computed according to the formulae given in Dwight (1961).

The second equation is obtained from the condition that $\sigma = 0$ for $\varrho > a$. We write from (Fabrikant, 1989)

$$\sigma_0(\varrho, \phi) = -\frac{1}{\pi^2 \varrho} \mathscr{L}(\varrho) \frac{d}{d\varrho} \int_\varrho^\infty \frac{x\, dx}{\sqrt{x^2 - \varrho^2}} \mathscr{L}\left(\frac{1}{x^2}\right) \frac{d}{dx}$$
$$\times \int_b^a \frac{\varrho_0\, d\varrho_0}{\sqrt{x^2 - \varrho_0^2}} \mathscr{L}(\varrho_0) v(\varrho_0, \phi), \quad \text{for } \varrho > a. \quad (23)$$

By using (11), (14), and (23) the following equation can be obtained for $\varrho \ge a$:

$$-\frac{1}{\pi^2 \varrho} \mathscr{L}(\varrho) \frac{d}{d\varrho} \int_\varrho^\infty \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho_0^2 - \varrho^2}} \mathscr{L}\left(\frac{1}{\varrho_0}\right) f_2(\varrho_0, \phi) + \frac{1}{\pi^2}$$
$$\times \int_0^b \frac{\varrho_0\, d\varrho_0}{(\varrho^2 - \varrho_0^2)^{3/2}} \mathscr{L}\left(\frac{\varrho_0}{\varrho}\right) f_1(\varrho_0, \phi)$$

$$-\frac{1}{\pi^2\varrho}\mathscr{L}(\varrho)\frac{d}{d\varrho}\int_\varrho^\infty \frac{x\,dx}{\sqrt{x^2-\varrho^2}}\mathscr{L}\left(\frac{1}{x^2}\right)\frac{d}{dx}$$

$$\times\int_b^a \frac{\varrho_0\,d\varrho_0}{\sqrt{x^2-\varrho_0^2}}\mathscr{L}(\varrho_0)v(\varrho_0,\phi)=0. \tag{24}$$

Let us apply the operator

$$\int_r^\infty \frac{\varrho\,d\varrho}{\sqrt{\varrho^2-r^2}}\mathscr{L}\left(\frac{r}{\varrho}\right)$$

to both sides of (24). The result is

$$\frac{1}{2\pi}f_2(r,\phi)+\frac{1}{\pi^2}\int_r^\infty \frac{\varrho\,d\varrho}{\sqrt{\varrho^2-r^2}}\int_0^b \frac{\varrho_0\,d\varrho_0}{(\varrho^2-\varrho_0^2)^{3/2}}\mathscr{L}\left(\frac{\varrho_0 r}{\varrho^2}\right)f_1(\varrho_0,\phi)$$

$$+\frac{1}{2\pi}\mathscr{L}\left(\frac{1}{r}\right)\frac{d}{dr}\int_b^a \frac{\varrho_0\,d\varrho_0}{\sqrt{r^2-\varrho_0^2}}\mathscr{L}(\varrho_0)v(\varrho_0,\phi)=0. \tag{25}$$

Again we can interchange the order of integration in (25) and perform the integration with respect to ϱ, with the result

$$f_2(\varrho,\phi)+\frac{1}{\pi^2}\int_0^{2\pi}\int_0^b Q_2(\varrho,\varrho_0,\phi-\phi_0)f_1(\varrho_0,\phi_0)\,d\varrho_0\,d\phi_0=g_2(\varrho,\phi). \tag{26}$$

Here $\varrho \geq a$, $0 \leq \phi < 2\pi$,

$$g_2(\varrho,\phi)=\varrho\int_b^a \frac{\varrho_0\,d\varrho_0}{(\varrho^2-\varrho_0^2)^{3/2}}\mathscr{L}\left(\frac{\varrho_0}{\varrho}\right)v(\varrho_0,\phi), \tag{27}$$

$$Q_2(\varrho,\varrho_0,\phi-\phi_0)$$
$$=2\Re\left\{\frac{\sqrt{\varrho\varrho_0\,e^{i(\phi-\phi_0)}}}{(\varrho\,e^{i(\phi-\phi_0)}-\varrho_0)R}\tan^{-1}\left[\frac{e^{i(\phi-\phi_0)}-(\varrho_0/\varrho)}{(\varrho/\varrho_0)-e^{i(\phi-\phi_0)}}\right]^{1/2}\right\}-\frac{\varrho_0}{R^2}, \tag{28}$$

and R is defined by (21). The following integral was used:

$$\int_r^\infty \frac{\varrho^3\,d\varrho}{\sqrt{\varrho^2-r^2}(\varrho^2-y^2)^{3/2}(\varrho^2-m)}$$
$$=\frac{m}{(m-y^2)^{3/2}\sqrt{r^2-m}}\tan^{-1}\left[\frac{m-y^2}{r^2-m}\right]^{1/2}-\frac{y^2}{(r^2-y^2)(m-y^2)}. \tag{29}$$

We note that $Q_1(\varrho,\varrho_0,\phi-\phi_0)=Q_2(\varrho_0,\varrho,\phi-\phi_0)$. This circumstance will allow us to decouple the equations by introduction of new variables. Indeed, substituting in (18) $t\sqrt{ab}$ instead of ϱ and $\sqrt{ab}/x$ instead of ϱ_0 yields

$$F_1(t,\phi)+\int_0^{2\pi}\int_0^k K(tx,\phi-\phi_0)F_2(x,\phi_0)\,dx\,d\phi_0=G_1(t,\phi). \tag{30}$$

Here $0 \le t \le k, 0 \le \phi < 2\pi$,

$$F_1(t, \phi) = f_1(t\sqrt{ab}, \phi), \quad F_2(t, \phi) = f_2(\sqrt{ab}/t, \phi)/t, \quad k = \sqrt{b/a}; \tag{31}$$

$$G_1(t, \phi) = g_1(t\sqrt{ab}, \phi),$$
$$K(xt, \phi - \phi_0) = \frac{2}{\pi^2} \Re\left\{\frac{\gamma}{R_{xt}} \tan^{-1}\left(\frac{xt}{\gamma R_{xt}}\right)\right\} - \frac{xt}{\pi^2 R_{xt}^2}, \tag{32}$$

$$R_{xt} = \sqrt{1 + x^2t^2 - 2xt\cos(\phi - \phi_0)}, \quad \gamma = \frac{\sqrt{xt\, e^{i(\phi - \phi_0)}}}{e^{i(\phi - \phi_0)} - xt}. \tag{33}$$

Equation (26) can be transformed by substitution $\varrho = \sqrt{ab}/t$ and $\varrho_0 = x\sqrt{ab}$ to exactly the same form as (30), namely,

$$F_2(t, \phi) + \int_0^{2\pi} \int_0^k K(tx, \phi - \phi_0) F_1(x, \phi_0)\, dx\, d\phi_0 = G_2(t, \phi). \tag{34}$$

Here $0 \le t \le k, 0 \le \phi < 2\pi$,

$$G_2(t, \phi) = g_2(\sqrt{ab}/t)/t, \tag{35}$$

and all the remaining notations are given by (31)–(33). Equations (30) and (34) can be easily uncoupled by summation and subtraction

$$F_+(t, \phi) + \int_0^{2\pi} \int_0^k K(tx, \phi - \phi_0) F_+(x, \phi_0)\, dx\, d\phi_0 = G_+(t, \phi), \tag{36}$$

$$F_-(t, \phi) - \int_0^{2\pi} \int_0^k K(tx, \phi - \phi_0) F_-(x, \phi_0)\, dx\, d\phi_0 = G_-(t, \phi), \tag{37}$$

where $0 \le t \le k, 0 \le \phi < 2\pi$,

$$F_\pm = F_1 \pm F_2, \quad G_\pm = G_1 \pm G_2. \tag{38}$$

Thus the problem has been reduced to two independent integral equations (36) and (37) with elementary non-singular kernels which can be solved by iteration. Convergence of the iteration procedure is not guaranteed for k very close to unity which corresponds to the case of a very narrow annulus. Direct computation of the norm of the kernel in space L_2 gave the result of 0.41 for $k = 0.9$, and it was less than 0.8 for $k = 0.95$. It is then recommended to use an asymptotical solution for $k > 0.95$. We note that the arguments of the kernel x and t do not enter it independently but only as a product xt. The following integral representation is useful for computation of various integrals of the kernel:

$$K(y, \psi) = \frac{y}{2\pi^2} \int_0^1 \frac{\lambda(yz, \psi)\, dz}{\sqrt{1 - z}(1 - y^2 z)^{3/2}}. \tag{39}$$

We recall that λ is defined by (6). Expression (39) shows that the kernel for each particular harmonic will also be an elementary function. For example, the kernels for the zero and first harmonic will be respectively

$$K_0(xt) = \frac{2}{\pi} \frac{xt}{1 - x^2t^2}, \tag{40}$$

$$K_1(xt) = \frac{2}{\pi} \left[\frac{1}{1 - x^2t^2} - \frac{1}{2xt} \ln \left(\frac{1 + xt}{1 - xt} \right) \right]. \tag{41}$$

Expression (40) is in agreement with the result of Clement and Love (1974). It is important to notice that various integral characteristics of interest can be expressed directly through the function f_2 (or F_2). For example, the total charge Q can be written as a limit

$$Q = \lim_{\varrho \to \infty} \{\varrho V_2(\varrho, \phi, z)\}. \tag{42}$$

Substitution of (3) in (42) leads to

$$Q = \frac{1}{\pi^2} \int_0^{2\pi} \int_a^{\infty} f_2(\varrho, \phi)\, d\varrho\, d\phi = \frac{\sqrt{ab}}{\pi^2} \int_0^{2\pi} \int_0^k F_2(x, \phi) \frac{dx}{x}\, d\phi. \tag{43}$$

The quantities proportional to magnetic polarizability can be found from

$$\tau_x = \lim_{\substack{\varrho \to \infty \\ \phi = 0}} \left[\varrho^2 \frac{\partial V_2}{\partial \phi} \right], \tag{44}$$

$$\tau_y = \lim_{\substack{\varrho \to \infty \\ \phi = \pi/2}} \left[\varrho^2 \frac{\partial V_2}{\partial \phi} \right]. \tag{45}$$

Substitution of (3) in (44) and (45) yields respectively

$$\begin{aligned} \tau_x &= \frac{2}{\pi^2} \int_0^{2\pi} \int_a^{\infty} f_2(\varrho, \phi) \sin \phi \varrho\, d\varrho\, d\phi \\ &= \frac{2}{\pi^2} ab \int_0^{2\pi} \int_0^k F_2(x, \phi) \sin \phi \frac{dx}{x^2}\, d\phi, \end{aligned} \tag{46}$$

$$\begin{aligned} \tau_y &= -\frac{2}{\pi^2} \int_0^{2\pi} \int_a^{\infty} f_2(\varrho, \phi) \cos \phi \varrho\, d\varrho\, d\phi \\ &= -\frac{2}{\pi^2} ab \int_0^{2\pi} \int_0^k F_2(x, \phi) \cos \phi \frac{dx}{x^2}\, d\phi. \end{aligned} \tag{47}$$

Formulae (30)–(35), (43), and (46)–(47) are the main new results of this paper.

Examples

Conducting annular disk charged to a unit potential

The governing integral equations in this case will take the form

$$F_1(t) + \int_0^k K_0(xt)F_2(x)\,dx = G_1(t), \tag{48}$$

$$F_2(t) + \int_0^k K_0(xt)F_1(x)\,dx = G_2(t), \tag{49}$$

where K_0 is defined by (40), F_1, F_2, G_1, and G_2 are understood as zero harmonics of the relevant notations (31), (32), and (35). In this particular case

$$G_1 = -\frac{t}{\sqrt{k^2 - t^2}} + \frac{kt}{\sqrt{1 - k^2t^2}}, \tag{50}$$

$$G_2 = \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t\sqrt{1 - k^2t^2}}. \tag{51}$$

Equations (48) and (49) were solved by Love (1976). We present here a slightly different version though based on the same idea, as well as a simple approximate treatment of the problem.

Assuming convergence of the iteration procedure, we can write the formal solution in the form

$$F_1 = \sum_{n=0}^{\infty} K_0^{2n}(G_1 - K_0G_2), \tag{52}$$

$$F_2 = \sum_{n=0}^{\infty} K_0^{2n}(G_2 - K_0G_1). \tag{53}$$

Here K_0^m is understood as the mth iteration of the kernel. The first iteration in (53) ($n = 0$) yields

$$F_2^{(1)}(t) = \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t\sqrt{1 - k^2t^2}} - \frac{2}{\pi}t \times \int_0^k \left[\frac{k}{\sqrt{1 - k^2x^2}} - \frac{1}{\sqrt{k^2 - x^2}}\right]\frac{x^2\,dx}{1 - x^2t^2}$$
$$= \frac{k}{t\sqrt{k^2 - t^2}} - \frac{1}{t} + \theta(t), \tag{54}$$

where the notation was introduced

$$\theta(t) = \frac{2}{\pi t}\left[\sin^{-1}(k^2) - \frac{k}{\sqrt{k^2 - t^2}}\sin^{-1}\left(\frac{k\sqrt{k^2 - t^2}}{\sqrt{1 - k^2t^2}}\right)\right]. \tag{55}$$

Now we need to consider the action of K_0^2 on the first iteration (54).

$$K_0^2 F_2^{(1)}(t) = \frac{4}{\pi^2} \int_0^k \frac{ts\,ds}{1-t^2s^2} \int_0^k \left[\frac{k}{x\sqrt{k^2-x^2}} - \frac{1}{x}\right] \frac{sx\,dx}{1-s^2x^2} + K_0^2\theta(t)$$

$$= \frac{2}{\pi} kt \int_0^k \frac{s^2\,ds}{(1-t^2s^2)\sqrt{1-k^2s^2}} - \frac{2}{\pi^2}$$

$$\times \int_0^k \ln\left(\frac{1+ks}{1-ks}\right) \frac{ts\,ds}{1-t^2s^2} + K_0^2\theta(t)$$

$$= -\theta(t) - \int_0^k K_0^2(xt)\frac{dx}{x} + K_0^2\theta(t). \tag{56}$$

Substitution of (56) in (53) shows that all the expressions containing $\theta(t)$ will cancel out at each subsequent iteration. This allows us to write the exact solution in the form

$$F_2(t) = \frac{1}{t}\left[\frac{k}{\sqrt{k^2-t^2}} - 1\right] - \sum_{n=1}^{\infty} \int_0^k K_0^{2n}(xt)\frac{dx}{x}. \tag{57}$$

Since all iterated kernels are positive, the term in square brackets in (57) gives the upper bound for the solution. Substitution of (57) in (48) yields, after simplification,

$$F_1(t) = -\frac{t}{\sqrt{k^2-t^2}} + \sum_{n=0}^{\infty} \int_0^k K_0^{2n+1}(xt)\frac{dx}{x}. \tag{58}$$

Capacitance of the annulus can be found by substitution of (57) in (43), with the result

$$C = \frac{2}{\pi} a \left\{1 - k \sum_{n=1}^{\infty} \int_0^k \int_0^k K_0^{2n}(xt)\frac{dx}{x}\frac{dt}{t}\right\}. \tag{59}$$

Taking into consideration that

$$C_0 = \frac{2}{\pi} a \tag{60}$$

is the capacity of a circular disk of radius a, we can write the expression for the dimensionless capacity C^* which is defined as the ratio

$$C^* = \frac{C}{C_0} = 1 - k \sum_{n=1}^{\infty} \int_0^k \int_0^k K_0^{2n}(xt)\frac{dx}{x}\frac{dt}{t}. \tag{61}$$

The symmetry of x and t in (61) allows us to reduce the order of iterated kernel as follows:

$$C^* = 1 - k \sum_{n=1}^{\infty} \int_0^k \left[\int_0^k K_0^n(xt)\frac{dx}{x}\right]^2 dt. \tag{62}$$

Yet another approximate solution for the dimensionless capacity C^* can be found by a different method. Indeed, from (43) we have

$$C^* = k \int_0^k F_2(t) \frac{dt}{t}. \tag{63}$$

Multiplying both sides of (49) by k/t and integrating with respect to t from 0 to k, we obtain

$$k \int_0^k F_2(t) \frac{dt}{t} + \frac{k}{\pi} \int_0^k \ln\left(\frac{1+kx}{1-kx}\right) F_1(x)\, dx = \sqrt{1-k^4}. \tag{64}$$

We can express F_1 from (48) and substitute it into (64). The result is

$$k \int_0^k F_2(t) \frac{dt}{t} - \frac{2k}{\pi^2} \int_0^k T(x) F_2(x) \frac{dx}{x}$$
$$= \frac{2}{\pi}\left[\cos^{-1}(k^2) + \tfrac{1}{2}\sqrt{1-k^4} \ln\left(\frac{1+k^2}{1-k^2}\right)\right], \tag{65}$$

where

$$T(x) = x^2 \int_0^k \ln\left(\frac{1+kt}{1-kt}\right) \frac{t\, dt}{1-x^2 t^2}. \tag{66}$$

Now we can use the mean value theorem to obtain

$$C^* = \frac{2}{\pi}\left[\cos^{-1}(k^2) + \tfrac{1}{2}\sqrt{1-k^4} \ln\left(\frac{1+k^2}{1-k^2}\right)\right]\left(1 - \frac{2}{\pi^2} T(X)\right)^{-1}. \tag{67}$$

According to the mean value theorem, we know about X only that it is located somewhere in the interval $[0, k]$. One needs to find an optimum value for X in order to make (67) useful. This exercise is beyond the scope of this paper. When $X = 0$, formula (67) coincides with the result of Smythe (1951) who obtained it from physical considerations. Since $T(X)$ is non-negative, the term in square brackets in (67) gives the lower bound. Note also that it is exact in two extreme cases, namely, $k = 0$ and $k = 1$.

Magnetic polarizability of a circular annulus

In this case we may assume, without loss of generality, that

$$v(\varrho, \phi) = v_1 \varrho \cos\phi, \tag{68}$$

where v_1 is a constant. The governing integral equations will take the form

$$F_1(t) + \int_0^k K_1(xt) F_2(x)\, dx = G_1(t), \tag{69}$$

$$F_2(t) + \int_0^k K_1(xt) F_1(x)\, dx = G_2(t), \tag{70}$$

where K_1 is defined by (41), F_1, F_2, and G_1, G_2 are understood as first harmonics of the relevant notations (31), (32), and (35). In this particular case

$$G_1(t) = v_1\sqrt{ab}t^2\left[\frac{k}{\sqrt{1-k^2t^2}} - \frac{1}{\sqrt{k^2-t^2}}\right], \tag{71}$$

$$G_2(t) = v_1\frac{\sqrt{ab}}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}} - \frac{2-k^2t^2}{\sqrt{1-k^2t^2}}\right]. \tag{72}$$

Equations (69) and (70) have not been considered before. We employ the same method as above. Assuming convergence of the iteration procedure, we can write the formal solution in the form

$$F_1 = \sum_{n=0}^{\infty} K_1^{2n}(G_1 - K_1 G_2), \tag{73}$$

$$F_2 = \sum_{n=0}^{\infty} K_1^{2n}(G_2 - K_1 G_1). \tag{74}$$

Here K_1^m is understood as the mth iteration of the kernel. The first iteration in (74) yields

$$F_2^{(1)}(t) = v_1\frac{\sqrt{ab}}{t^2}\left[\frac{2k^2-t^2}{k\sqrt{k^2-t^2}} - 2\right] + \theta_1(t). \tag{75}$$

Here the notation was introduced

$$\theta_1(t) = \frac{2v_1\sqrt{ab}}{\pi t^2}\left[2\sin^{-1}(k^2) - \frac{t}{2k}\sqrt{1-k^4}\ln\left(\frac{1+kt}{1-kt}\right)\right.$$
$$\left. - \frac{2k^2-t^2}{k\sqrt{k^2-t^2}}\sin^{-1}\left(\frac{k\sqrt{k^2-t^2}}{\sqrt{1-k^2t^2}}\right)\right]. \tag{76}$$

Now we need to compute

$$K_1^2F_2^{(1)}t = \int_0^k K_1(ts)\,ds\int_0^k K_1(sx)\frac{v_1\sqrt{ab}}{x^2}\left[\frac{2k^2-x^2}{k\sqrt{k^2-x^2}} - 2\right]dx + K_1^2\theta_1(t). \tag{77}$$

Integration with respect to x in (77) yields

$$K_1^2F_2^{(1)}(t) = \frac{2}{\pi}v_1\sqrt{ab}\int_0^k\left[\frac{1}{k} + \frac{\pi}{2}\frac{ks^2}{\sqrt{1-k^2s^2}} - \frac{1+k^2s^2}{2k^2s}\right.$$
$$\left.\times\ln\left(\frac{1+ks}{1-ks}\right)\right]K_1(ts)\,ds + K_1^2\theta_1(t). \tag{78}$$

One can easily verify that

$$\int_0^k \frac{ks^2}{\sqrt{1-k^2s^2}}K_1(st)\,ds = -\frac{\theta_1(t)}{v_1\sqrt{ab}}. \tag{79}$$

Substitution of (79) in (78) gives

$$K_1^2 F_2^{(1)}(t) = -\theta_1(t) + \frac{2}{\pi} v_1 \sqrt{ab} \times \int_0^k \left[\frac{1}{k} - \frac{1+k^2s^2}{2k^2s} \ln\left(\frac{1+ks}{1-ks}\right) \right] K_1(st)\, ds + K_1^2\theta_1(t). \tag{80}$$

By using the identity

$$\frac{1}{k} - \frac{1+k^2s^2}{2k^2s} \ln\left(\frac{1+ks}{1-ks}\right) = -\pi \int_0^k K_1(xs) \frac{dx}{x^2}, \tag{81}$$

we can further simplify (80), namely,

$$K_1^2 F_2^{(1)}(t) = -\theta_1(t) - 2v_1\sqrt{ab} \int_0^k K_1^2(st) \frac{ds}{s^2} + K_1^2\theta_1(t). \tag{82}$$

Finally, substitution of (82) in (74) gives the solution

$$F_2(t) = v_1\sqrt{ab} \left\{ \frac{1}{t^2} \left[\frac{2k^2 - t^2}{k\sqrt{k^2 - t^2}} - 2 \right] - 2 \sum_{n=1}^{\infty} \int_0^k K_1^{2n}(st) \frac{ds}{s^2} \right\}. \tag{83}$$

Since iterated kernels are all positive, the first term in (83) gives the upper bound for F_2. Substitution of (83) in (69) allows us to find

$$F_1(t) = v_1\sqrt{ab} \left\{ -\frac{t^2}{\sqrt{k^2 - t^2}} + 2 \sum_{n=0}^{\infty} \int_0^k K_1^{2n+1}(st) \frac{ds}{s^2} \right\}. \tag{84}$$

Formulae (83) and (84) give the complete solution to the problem. We introduce the dimensionless magnetic polarizability τ^* as the ratio of magnetic polarizability of the annulus τ_y to that of a circular disk of radius a $\tau_0 = 4v_1 a^3/(3\pi)$. Substitution of (83) in (47) yields, after integration,

$$\tau^* = 1 - 3k^3 \sum_{n=1}^{\infty} \int_0^k \int_0^k K_1^{2n}(st) \frac{ds\, dt}{s^2 t^2}. \tag{85}$$

Again, the symmetry of (85) allows us to reduce the order of the kernel iteration by writing

$$\tau^* = 1 - 3k^3 \sum_{n=1}^{\infty} \int_0^k \left[\int_0^k K_1^n(st) \frac{ds}{s^2} \right]^2 dt. \tag{86}$$

Since all iterated kernels are positive, truncation in (86) gives the upper bound for τ^*. A simple approximate formula for small k can be derived by integration of (75), with the result

$$\tau^* = 1 - \frac{2}{\pi} \left\{ \sin^{-1}(k^2) + \frac{\sqrt{1-k^4}}{4} \left[k^2 - \frac{5+k^4}{2} \ln\left(\frac{1+k^2}{1-k^2}\right) \right] \right\}. \tag{87}$$

Formula (87) is exact in two extreme cases, namely, for $k = 0$ and for $k = 1$. The series expansion of (87) gives

$$\tau^* = 1 - \frac{2}{\pi}\left(\frac{k^{10}}{5} + \frac{9k^{14}}{70} + \frac{233k^{18}}{2520}\right) + O(k^{22}). \tag{88}$$

It is of interest to notice that all powers of k below 10 cancelled out as compared to the relevant expression for capacity where the series expansion starts with the sixth power of k. This means that τ^* will be not far away from unity even for not so small k. For example, τ^* is greater than 0.98 for $k = 0.8$. Convergence of the iteration procedure was investigated by computing the norm of K_1. It was found to be much less than 1 up to the ratio $b/a = 0.9$. Even for $b/a = 0.999$ the norm is equal to 0.6 which still assures a good convergence.

Conclusion

The new approach presented here was proven to be very effective in dealing with non-axisymmetric problems involving an annular disk. The main advantage of the new method is in reduction of the problem to a non-singular integral equation with an elementary kernel which can be solved by iteration. The method can be generalized for spherical and toroidal coordinates, so that similar problems for a spherical ring could be considered in general formulation.

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Abstract

A general formulation is given for the first time to the title problem. The method is based on the new results in potential theory obtained by the author earlier. The problem is reduced to a two-dimensional integral equation with an elementary non-singular kernel. Several specific examples are considered. Exact solution has been obtained in terms of the iterated kernel.

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