THE LUNAR ORBIT REVISITED, III

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Abstract. In this paper we present an investigation on the tidal evolution of a system of three bodies: the Earth, the Moon and the Sun. Equations are derived including dissipation in the planet caused by the tidal interaction between the planet and the satellite and between the planet and the sun. Dissipation within the Moon is included as well. The set of differential equations obtained is valid as long as the solar disturbances dominate the perturbations on the satellite's motion due to the oblateness of the planet, namely a/R_e greater than 15, and closer than that point equations derived in a preceding paper are used.

The result shows the Moon was closer to the Earth in the past than now with an inclination to the ecliptic greater than today, whereas the obliquity was smaller. Toward the past, the inclination to the Earth's equator begins decreasing to 12° for $a/R_e = 12$ and suddenly grows. During the first stage the results are weakly dependant on the magnitude of the dissipation within the satellite, whereas the distance of the closest approach and the prior history are strongly dependent on that dissipation. In particular, the crossing of the Roche limit can be avoided.

1. Introduction

This paper is the third and the last of a set of papers devoted to the study of the evolution of a planet-satellite system under the influence of the tides. In the two previous papers, hereafter referred to as Paper I and Paper II, we discussed the tidal evolution of an isolated two body system, that means we disregarded the effects of the solar disturbances. It is well known that today the most important departures of the orbit of the Moon from the Kepler laws are caused by the gravitational action of the Sun. For our purpose we only retain the precession of the node along the ecliptic and the nearly constant inclination of the orbital plane of the Moon to the ecliptic. The last property gives rise to a periodic variation in the inclination with respect to the Earth's equator.

In the past the Moon was closer to the Earth than now and the solar disturbances weaker, whereas the effect of the Earth's oblateness was larger. Thus there is a critical distance a_c obtained by equalling the two disturbances (Goldreich, 1966). With the current conditions a_c is found equal to $10R_e$, where R_e is one Earth's radius. If allowance is made for the constancy of the total angular momentum of the Earth–Moon system this value increases to $15R_e$ because of the fact that the rate of rotation of the Earth was greater in the past, as well as its flattening (Munk and McDonald, 1960).

From a purely theoretical point of view it seems our results of Paper II are no longer valid in the Moon's case because we have neglected the solar gravity. However we have shown in this paper that the most outstanding changes in the lunar orbit have occurred at distances smaller than $15R_e$, inside the critical sphere. The problem lies in the knowledge of the value of the different parameters when the semi-major axis of the Moon was

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equal to $15R_e$. But the backward integration from the present state to the state with a semi-major axis of $15R_e$ did not exhibit important changes in the inclination and in the eccentricity, which made our results on Paper II similar to those obtained by Goldreich (1966). Then the integration performed by neglecting the Sun's action was probably fairly good for the purposes of investigating closer to the Earth than $15R_e$. Nevertheless it was of interest to check this assumption by looking for new equations which would take the gravitational solar torque into account.

In fact other advantages come from this new formulation. In the case of a study of an isolated two-body system there is no possibility for binding the orbital plane of the Moon and the Earth's equator to the ecliptic. Then in this case the best-suited reference plane is the plane normal to the total angular momentum of the system, which is inertial, insofar as the act of the Sun is ignored. Unfortunately such a plane is quite independent of the ecliptic. But by introducing the Sun we introduce the ecliptic as a reference plane which allows us to investigate the evolution of the obliquity resulting from the tidal forces among the three bodies. Second, since the Moon's orbit undergoes a steady precession on the ecliptic, a good manner to represent the orientation of that orbit will be to determine the value of its inclination to the ecliptic, the past value of which is of great interest for theories of the origin of the Moon. Third, the current solar torque exerted on the Earth contributes to the lengthening of the day by 20%, the remainder being due to the Moon. In the future the Moon's torque will decrease faster than the solar one, which will become relatively more important. Therefore a study of the recent past as well as the future history of the Earth–Moon system requires a modelling of the forces including the three bodies.

A preceding investigation similar to that presented here has been carried out by Goldreich (1966). In this important paper the author presents a method for calculating the past history of the Earth–Moon system by considering the tidal interaction between the Earth and Moon as well as between Earth and Sun. But the method used to treat the precessional equations prevents him taking the eccentricity of the Moon's orbit into consideration, and its result is only valid for a circular orbit. For this reason, Goldreich does not include dissipation within the Moon, since it influences the evolution only in case of elongated orbits. This assumption is true as far as the semi-major axis is concerned, but we shall see later in this paper that effects occur in the inclination of the Moon's orbit to the ecliptic on account of the dissipation in the Moon. In the work of McDonald (1964) the author allows for the eccentricity but solar tides are not included. The phase lag is introduced in a geometrical way which is quite irrealistic when the eccentricity becomes significant. Moreover, if the computation relating to the evolution of the inclination to the Equator is valid, the one concerning the obliquity is only an approximation, principally because the author assumes that the normal to the ecliptic and both the angular momentum of the Moon and the Earth are coplanar.

It is useful to notice that the Sun modifies the set of equations given in Paper II in two ways. First, a *new* tidal dissipation occurs within the Earth and variations in the total angular momentum of the Earth–Moon system are involved. Such an effect gets significant from $50R_e$ and increases with the Moon's semi-major axis. The gravitational effect of the Sun is quite different. The precession of the orbit of the Moon obliges us to introduce an intermediate time-scale between the orbital periods of the Sun and Moon and the time defined by the rate at which the dissipation changes the orbital parameters of the Moon. Consequently all equations must be averaged over a precessional period. As explained at the beginning of the introduction, the gravitational effect will be included from $15R_e$ and the larger the semi-major axis, the more important is its influence.

The present paper is divided in three sections. We recall the notations and equations obtained in Paper II in Section 2. Section 3 deals with the derivation of three new differential equations: two for the orbital inclination of the Moon to the ecliptic and for the obliquity of the ecliptic, the third equation accounting for the variation of the length of day. A numerical integration is performed in the course of Section 4 and the results are discussed.

2. Review of Equations Previously Obtained

In Paper II we derived a set of differential equations to study the tidal evolution of a system composed of a planet and a satellite. These equations are obtained by assuming a time delay Δt between the moment when the planet feels the tidal force exerted by the satellite and the one when the planet is distorted in its equilibrium figure. The forces and torques which lead to the differential equations are worked out at the outset of Paper II, and here we only give their expressions for $l = 2$, which will be useful in the second section of this paper: namely,

$$\mathbf{F}_{\mathrm{EM}} = -3k^2 \frac{Gm^{*2}R_{\mathrm{E}}^5}{r^{10}} \Delta t[2\mathbf{r}(\mathbf{r}\cdot\mathbf{v}) + \mathbf{r}^2(\mathbf{r}\times\boldsymbol{\omega} + \mathbf{v})], \tag{1}$$

$$\mathbf{T}_{\mathrm{EM}} = +3k_2 \frac{Gm^{*2}R_{\mathrm{E}}^5}{r^8} \Delta t[(\mathbf{r}\cdot\boldsymbol{\omega})\mathbf{r} - \mathbf{r}^2\boldsymbol{\omega} + \mathbf{r}\times\mathbf{v}]. \tag{2}$$

The meaning of the various parameters involved in Equations (1) and (2) will be given later. $\mathbf{F}_{\mathrm{EM}}$ and $\mathbf{T}_{\mathrm{EM}}$ are the force and torque experienced by the Moon and due to the distorted Earth. The torque acting on the Earth is exactly the opposite. In the case when the Sun is the tide raising object we must change $m \to M_\odot$ and the orbital parameters into that of the Sun.

The equations for the evolution of the semi-major axis and the eccentricity are unchanged when allowance is made for the solar disturbances and they are taken from the set S1 of Paper II:

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} = {} & 6\times 4\pi^2 k_2 \frac{m}{M}\frac{m}{\mu}\frac{\Delta t}{p^2}\frac{1}{X^7} \\ & \times\left[-\frac{1}{(1-e^2)^{15/2}}\left(1+\frac{31}{2}e^2+\frac{255}{8}e^4+\frac{185}{16}e^6+\frac{25}{64}e^8\right)\right. \\ & \left.+\frac{\omega}{n}\frac{1}{(1-e^2)^6}\left(1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6\right)\right],\end{aligned} \tag{3}$$

$$\frac{de}{dt} = 3 \times 4\pi^2 k_2 \frac{m}{M}\frac{m}{\mu}\frac{\Delta t}{P^2}\frac{1}{X^8}$$
$$\times \left[-\frac{1}{(1-e^2)^{13/2}} \left(9e + \frac{135}{4}e^3 + \frac{135}{8}e^5 + \frac{45}{64}e^7\right) + \frac{\omega}{n}\frac{\cos I}{(1-e^2)^5}\left(\frac{11}{2}e + \frac{33}{4}e^3 + \frac{11}{16}e^5\right)\right]. \tag{4}$$

The third equation of the set S1 was concerned by the evolution of the orbital inclination of the Moon to the plane normal to the angular momentum which was the plane of the precession. Now this plane is replaced by the ecliptic and we shall derive the corresponding equation in the next section.

COMMENTS AND EXPLANATIONS

We rapidly recall the meanings of the various parameters embodied in Equations (1)–(4). In this set only the second harmonic of the tidal potential is retained because we intend to stop the integration at $X = 10$.

The semi-major axis is represented by $X = a/R_e$ where R_e means the Earth's radius. M and m are respectively the Earth's and the Moon's mass and the so-called reduced mass of the system $\mu = Mm/(M + m)$. The Love number is k_2 its value being taken equal to 0.3 for the Earth, and Δt is the time delay. If the major part of the secular acceleration of the Moon can be modelized by means of Equation (1) then the value of Δt is of the order of $10mn$. Indeed, the current evolution of the Earth–Moon system mainly takes place under the influence of the ocean tide. Then the previous value of Δt is overestimated for the solid tide. It is thought there was a smaller extent of the shallow sea in the past so the contribution of the dissipation within the oceans, to the evolution of the Moon's orbit, was less important than now. The history of Δt remains unknown and prevents us from giving a scenario with a chronology. Then instead of the time as independant variable we shall choose the semi-major axis and we shall compute a relative chronology.

P is the period of a light satellite orbiting the Earth at $1R_e$, the grazing satellite, and ω means the rate of rotation of the Earth, n the mean motion of the Moon, I the inclination of the Moon's orbit to the Earth's equator. In fact because of the intermediate time scale only the average value of $\cos(I)$ will be used in these equations.

The dissipation within the satellite is taken into account by expressing the ratio of the effects of the tide in the satellite and in the planet as explained in Paper II. Henceforth A will denote this ratio.

3. Supplementary Equations of Motion

As we have noted it is better to use the inclination of the orbit of the Moon with respect to the ecliptic than to the equator. In addition other variables are required to obtain a thorough description of the system: the orientation of the Earth's equator with respect to the ecliptic will be given by the obliquity of the ecliptic and the last unknown is the magnitude of the total angular momentum. Therefore, three new differential equations are to be derived.

3.1. Equation for the Inclination of the Orbit of the Moon on the Ecliptic

In Paper II we derived an expression for W, the component of the force, normal to the oscillating plane. For $l = 2$, its expression is given by

$$W = -G\frac{m^2}{\mu} k_2 \Delta t R_e^5 \, 3\frac{\omega}{a^7}\left(\frac{a}{r}\right)^7 \cos(\tilde{\omega} + v)\sin I. \tag{5}$$

In this Equation the orbit is referred to the equator of the planet, and the meaning of I, $\tilde{\omega} + v$ is shown on Figure 1.

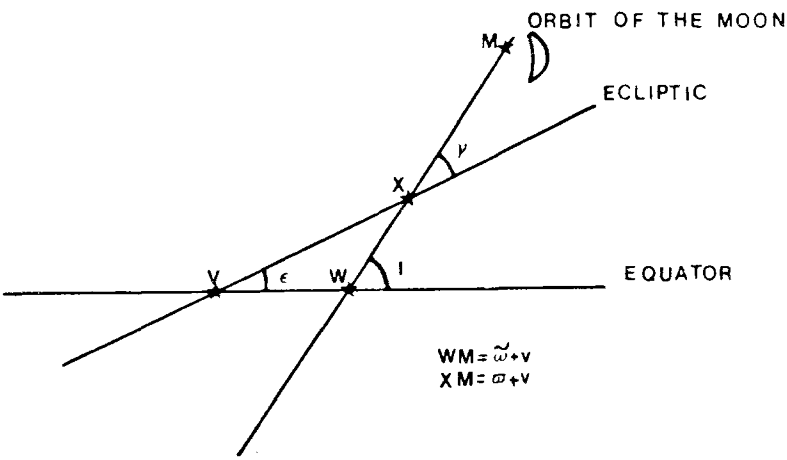

Fig. 1. Coordinate system used in this paper.

The Gauss equation for the variation of the inclination γ of an orbit referred to the ecliptic can be written as

$$\frac{d\gamma}{dt} = \frac{1}{na(1-e^2)^{1/2}}\frac{r}{a}\cos(\varpi + v)W, \tag{6}$$

where $\varpi + v$ is an argument measured from the node in the orbital plane (Figure 1).

Because of the precession of the node on the ecliptic. I and $\tilde{\omega}$ undergo a periodic variation with the same period and Equation (6) will be averaged over this precessional period. By introducing W in Equation (6),

$$\frac{d\gamma}{dt} = -G\frac{m^2}{\mu}k_2\Delta t R_e^5 \frac{3}{n(1-e^2)^{1/2}}\frac{\omega}{a^8}\left(\frac{a}{r}\right)^6 \cos(\overline{\omega} + v + WX)$$
$$\times \cos(\overline{\omega} + v)\sin I.$$

The average over one orbital period is carried out as in Paper II for deriving S1, by assuming $\bar{\omega}, I$ and WX are constant over such a period: i.e.,

$$\left\langle \left[\left(\frac{a}{r} \right)^6 \tfrac{1}{2} \sin I [\cos (2\bar{\omega} + v) + WX) + \cos WX] \right] \right\rangle = \\ = \frac{\sin I}{2} [H(6,0) \cos (WX) + H(6,2) \cos (2\bar{\omega} + WX)], \tag{8}$$

where $H(I,J)$ are functions of the eccentricity defined in Paper II.

In the spherical triangle VWX (Figure 1) the following relationships hold,

$$\begin{aligned} \cos I &= \cos\epsilon \cos\gamma + \sin\epsilon \sin\gamma \cos\Omega, \\ \sin I \cos WX &= \frac{\cos\epsilon - \cos I \cos\gamma}{\sin\gamma}, \\ \sin I \sin WX &= \sin\epsilon \sin\Omega. \end{aligned} \tag{9}$$

The average value of the left-hand side of these equations is

$$\begin{aligned} \langle \cos I \rangle &= \cos\epsilon \cos\gamma, \\ \langle \sin I \cos WX \rangle &= \cos\epsilon \sin\gamma, \\ \langle \sin I \sin WX \rangle &= 0. \end{aligned} \tag{10}$$

The last system leads for the secular variation to

$$\frac{d\gamma}{dt} = -\frac{3}{2} 4\pi^2 k_2 \frac{m}{M} \frac{m}{\mu} \frac{\Delta t}{p^2} \left[\frac{\mu}{m} \right]^{1/2} \frac{\omega}{n_G} \cos\epsilon \sin\gamma \frac{1}{X^{13/2}} \frac{H(6,0)}{(1-e^2)^{1/2}}, \tag{11}$$

where $n_G = 2\pi/P$ is the mean motion of the grazing satellite.

In order to include the effects of the dissipation within the satellite we assume that the inclination of the equatorial plane of the satellite to the ecliptic is small, as is shown by Cassini's laws. By computing W as in Equation (5), but with the Earth as tide-raising object, and by adding the two contributions to the equation for γ we obtain

$$\frac{d\gamma}{dt} = -\frac{3}{2} 4\pi^2 k_2 \frac{m}{M} \frac{m}{\mu} \frac{\Delta t}{P^2} \left(\frac{\mu}{m} \right)^{1/2} \left[\frac{\omega}{n_G} \cos\epsilon + \frac{A}{X^{3/2}} \right] \sin\gamma \frac{1}{X^{13/2}} \frac{H(6,0)}{(1-e^2)^{1/2}}. \tag{12}$$

The term with A will become significant when the semi-major axis is small.

Let us recall the expression of A given in Paper II, of the form

$$A = \frac{k_2' \Delta t'}{k_2 \Delta t} \left(\frac{M}{m} \right)^2 \left(\frac{R'}{Re} \right)^5.$$

Equation (12) has been obtained by taking the ecliptic as the precessional plane. Such a choice was imposed by the fact that the solar disturbances overwhelm those due to the

Earth's oblateness. In fact it can be shown that Equation (12) remains valid for every precessional plane provided that γ and ϵ mean the inclination of the lunar orbit and of the Earth's equator with respect to this precessional plane. This property will be used to extend the integration very close to the Earth.

3.2. Equation for the obliquity of the ecliptic

Rather than the obliquity of the ecliptic a better variable is the component of the rotational vector of the Earth along the normal to the ecliptic.

The evolution of the angular momentum of the Earth is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(C\boldsymbol{\omega}) = \mathbf{T}_{\mathrm{ME}} + \mathbf{T}_{\mathrm{SE}}, \tag{13}$$

where $\mathbf{T}_{\mathrm{ME}}$ and $\mathbf{T}_{\mathrm{SE}}$ are, respectively, the torques exerted on the Earth by the Moon and by the Sun.

If $\mathbf{E}$ is the unit vector along the normal of the ecliptic, a scalar equation can be obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}(C\boldsymbol{\omega}\cdot\mathbf{E}) = \mathbf{T}_{\mathrm{ME}}\cdot\mathbf{E} + \mathbf{T}_{\mathrm{SE}}\cdot\mathbf{E}. \tag{14}$$

But the equation of evolution for the orbital angular momentum of the Moon, $\mathbf{H}_{\mathrm{M}}$, can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu\cdot\mathbf{H}_{\mathrm{M}}) = -\mathbf{T}_{\mathrm{ME}}.$$

A scalar equation is obtained by forming the dot-product of this equation with $\mathbf{E}$ – i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu H_{\mathrm{M}}\cos\gamma) = -\mathbf{T}_{\mathrm{ME}}\cdot\mathbf{E}. \tag{15}$$

As $H_{\mathrm{M}} = (G(M+m)a(1-e^2))^{1/2}$, Equation (15) is intimately related to Equations (3), (4) and (11).

To get the right-hand side of Equation (15) we intend to make use of Equations (3), (4) and (11), rather than derive a supplementary one. But it must be pointed out that the torque $\mathbf{T}_{\mathrm{ME}}$ in Equation (14) is only due to the dissipation within the Earth, the Moon being regarded as a point-like mass. Therefore all equations required must be taken by putting $A = 0$. That can be explained as follows: let us consider the dissipation within the Moon only, caused by a spherical Earth. The Earth exerts a force at the center of mass of the Moon and a torque. The Moon exerts an opposite force at the center of mass of the Earth but the resulting torque is null for a perfect Earth. Then we can conclude that the tidal dissipation within the Moon affects the orbital evolution of the Earth–Moon system but does not act on the Earth rotational vector.

Since $\mathrm{d}(\mu\mathbf{H}_{\mathrm{M}}\cdot\mathbf{E})/\mathrm{d}t$ can be computed with the help of (3), (4) and (11), we are only terested now in expressing the part due to the action of the Sun on the Earth.

A straightforward transformation of Equation (2) leads to

$$\begin{aligned} \mathbf{T}_{SE}\cdot\mathbf{E} &= 3k_2 G M_\odot^2 R_e^5 \Delta t \frac{1}{R^8}\left[-R^2\,\boldsymbol{\omega}\cdot\mathbf{E} + n_\odot R^2\right] \\ &= -3k_2 G M_\odot^2 R_e^5 \Delta t \frac{1}{R^6}\left[\omega\cos\epsilon - n_\odot\right], \end{aligned} \tag{16}$$

where $M_\odot$ and R are the mass of the Sun and the semi-major axis of the Earth's orbit around the Sun.

Let us evaluate now the different parts of the Equation (15) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu\mathbf{H}_\mathrm{M}\cdot\mathbf{E}) = \mu\cos\gamma\frac{\mathrm{d}H_\mathrm{M}}{\mathrm{d}t} + \mu H_\mathrm{M}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\gamma).$$

The equation of evolution of γ has previously been computed (Equation (11)) and an easy handling of Gauss's equation or a direct use of the angular momentum theorem (Burns, 1976) leads us to expect that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_\mathrm{M} = a\left[\frac{r}{a}\right]S, \tag{17}$$

where S is the orthoradial component of the force given in Paper II.

Finally, it remains to average the three different parts which contributes to the change of $\omega\cos\epsilon$.

In Equation (16), no periodic term appears if the Earth is assumed to orbit the Sun in a circle.

Omitting the constant multiplicating term we can transform Equation (17) into

$$\begin{aligned} a\left[\frac{r}{a}\right]S &\propto \frac{(1-e^2)^{1/2}}{a^6}\left[\frac{a}{r}\right]^8 - \frac{\omega}{n}\frac{\cos I}{a^6}\left[\frac{a}{\mathrm{r}}\right]^6, \\ \left\langle a\left[\frac{r}{a}\right]S\right\rangle &\propto \frac{(1-e^2)^{1/2}}{X^6}H(8,0) - \frac{\omega}{n}\frac{\cos I}{X^6}H(6,0). \end{aligned} \tag{18}$$

By inserting (10), (11), (16), (18) in the right hand side of (14) we reach the final result after some algebraic transformations. Let us put $U = (\omega/n_G)\cos\epsilon$; then

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t} = -3\frac{4\pi^2}{\alpha}k_2\frac{\Delta t}{P^2}\Bigg\{&\left[\frac{M_\odot}{M}\right]^2\left[\frac{Re}{R}\right]^6\left[U - \frac{n_\odot}{n_G}\right] - \left[\frac{m}{\mu}\right]^{1/2}\left[\frac{m}{M}\right]^2\frac{1}{X^{15/2}}\times \\ &\times\left[(1-e^2)^{1/2}H(8,0) - UX^{3/2}\cos\gamma H(6,0)\right]\cos\gamma + \\ &+\frac{1}{2}\left[\frac{m}{M}\right]^2 U\sin^2\gamma\frac{1}{X^6}H(6,0)\Bigg\}. \end{aligned}$$

In this equation, the first term on the right-hand side is due to the action of the solar terms are due to the Moon tides. Currently, the solar effect is five times smaller than the lunar one.

3.3. Equation for the Evolution of the Length of the Day

Equations (3), (4), and (19) constitute a set of four equations with four variables which can be integrated. But this set of variables does not allow us to separate the evolution of the rate of rotation of the Earth from the evolution of the obliquity of the ecliptic. That is an important drawback that we wish to solve by looking for an equation for the evolution of the length of the day.

At a first glance it seems that the best new variable is the total angular momentum of the Earth–Moon system because of its constancy if no allowance is made for the act of the Sun. Thus the effect of the Sun on that parameter would be significant only in the future. However this choice has been proved to be ineffective, the rate of rotation of the Earth being obtained in computing the difference between two quantities very large with respect to the result and in turn the obliquity of the ecliptic was determined badly. Consequently we retain the rate of rotation as a well suited variable and we derive the corresponding differential equation.

For that purpose we start from the Equation (13)

$$\frac{\mathrm{d}(C\boldsymbol{\omega})}{\mathrm{d}t} = \mathbf{T}_{\mathrm{ME}} + \mathbf{T}_{\mathrm{SE}}\,. \tag{13}$$

By forming the dot-product of this equation with $\boldsymbol{\omega}$

$$\frac{1}{2}\frac{\mathrm{d}(C\omega^2)}{\mathrm{d}t} = \mathbf{T}_{\mathrm{ME}}\cdot\boldsymbol{\omega} + \mathbf{T}_{\mathrm{SE}}\cdot\boldsymbol{\omega}\,. \tag{20}$$

The analytical expression of the right-hand side of (20) is obtained from (2) as:

$$\mathbf{T}_{\mathrm{ME}}\cdot\boldsymbol{\omega} = 3Gm^2k_2\Delta t\,\frac{R_e^5}{r^8}\left[(\mathbf{r}\cdot\boldsymbol{\omega})^2 - r^2\omega^2 + (\mathbf{r}\times\mathbf{v})\boldsymbol{\omega}\right]. \tag{21}$$

If we refer the lunar orbit to the equator the following relations hold:

$$\begin{aligned}(\mathbf{r}\,\boldsymbol{\omega})^2 &= r^2\omega^2\sin^2 I\sin^2(\tilde{\omega}+v),\\ (\mathbf{r}\times\mathbf{v})\cdot\boldsymbol{\omega} &= H_{\mathrm{M}}\omega\cos I.\end{aligned} \tag{22}$$

We now insert these relationships in (21) and average over one orbital period, i.e.,

$$\begin{aligned}\langle\mathbf{T}_{\mathrm{ME}}\cdot\boldsymbol{\omega}\rangle &= 3Gm^2k_2\Delta t\,\frac{R_e^5}{a^6}\\ &\quad\times\left[-\omega^2H(6,0)\,\frac{(1+\cos^2 I)}{2} + \frac{H_M}{a^2}\omega H(8,0)\cos I\right].\end{aligned} \tag{23}$$

This equation has been used in Paper II to investigate the effect of the lunar or solar torque on the Earth's rotation for various values of the inclination and the eccentricity.

In (23) the parts depending on the inclination of the Moon's orbit to the Earth's equator are to be averaged over one precessional period; i.e.,

$$\begin{aligned} \langle \cos I \rangle &= \cos\epsilon \cos\gamma, \\ \langle \cos^2 I \rangle &= \cos^2\epsilon \cos^2\gamma + \tfrac{1}{2}\sin^2\epsilon \sin^2\gamma. \end{aligned} \tag{24}$$

By carrying out a similar calculation for the solar torque we obtain

$$\langle \mathbf{T}_{SE}\cdot\boldsymbol{\omega}\rangle = 3GM_\odot^2 k_2 \Delta t \frac{R_e^5}{R^6}\left[-\omega^2\frac{(1+\cos^2\epsilon)}{2} + n_\odot\omega\cos\epsilon\right]. \tag{25}$$

Some straightforward algebraic manipulations in Equations (20), (23), (24) and (25) lead to

$$\begin{aligned} \frac{d}{dt}\left(\frac{\omega}{n_G}\right)^2 &= \frac{4\pi^2}{P^2} k_2 \Delta t \frac{m\,m}{M\,\mu} \times \\ &\times \left\{\frac{3}{\alpha}\frac{m}{M}\frac{1}{X^6}\left[-H(6,0)\left[\left(\frac{\omega}{n_G}\right)^2\frac{2+\sin^2\gamma}{2} + U^2\frac{3\cos^2\gamma - 1}{2}\right] + \right.\right. \\ &\left.\left. + \frac{2H_M}{X^2} U\cos\gamma\, H(8,0)\right] + \frac{3}{2}\frac{M}{m}\left(\frac{M_\odot}{M}\right)^2\left(\frac{R_e}{R}\right)^6\left[-\left(\frac{\omega}{n_G}\right)^2 - U^2 + 2\frac{n}{n_G}U\right]\right\}, \end{aligned} \tag{26}$$

which is the desired result. Recall now the meaning of the different notations

$$U = \frac{\omega}{n_G}\cos\epsilon, \qquad \alpha = \frac{C}{MR_e^2}.$$
$$H_M = [X(1-e^2)]^{1/2},$$

In Equation (26) the part containing the Moon's mass as a factor proceeds from the Moon and the other from the Sun. The ratio of these two terms is expressed as

$$\frac{\text{Moon}}{\text{Sun}} = \left(\frac{m}{M_\odot}\right)^2\left(\frac{R}{a}\right)^6\frac{\omega - n}{\omega} = 4.8\left(\frac{X_0}{X}\right)^6.$$

This ratio means that today the Moon is slowing down the rate of rotation of the Earth five times more than the Sun does. But in the future, when the Moon is more distant from the Earth than now, the action of the Sun will increase. In fact, the action of the Moon drastically decreases because of its removal and the approach of the synchronous state when $(\omega/n)\cos I$ approaches 1. In conjunction with Equation (3) the last ratio allows us to obtain a useful relationship between the secular acceleration of the Moon and the lengthening of the day, of the form

$$10^9\,\dot{\omega}/\omega = (0.92 + 1/4.8)\dot{n} = 1.13\,\dot{n}.$$

3.4. COMMENTS

A qualitative discussion of the future evolution of the Moon's orbit is possible by assuming a small eccentricity as well as a small obliquity and inclination. With these simplifying hypotheses Equations (3) and (26) reduce to

$$\frac{\mathrm{d}X}{\mathrm{d}t} \propto -\frac{1}{X^7}\left[1-\frac{\omega}{n}\right], \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\omega}{n_G}\right] \propto \left[-4.8\frac{\omega-n}{n_G}\left[\frac{X_0}{X}\right]^6\frac{\omega}{n_G}\right]. \tag{28}$$

In the last equation the mean motion of the Sun has been neglected with respect to that of the Earth. Today ω/n is of the order of 27 and decreases as the Moon recedes from the Earth toward the synchronous point.

Before reaching this point the first part of the right-hand side of (28) will become smaller than the second: namely, the effects caused by the Sun will exceed those of the Moon to reduce the rate of rotation of the Earth. The closeness of ω/n to 1 prevents the orbit of the Moon to evolve during this stage. After a very long time the solar tide will have removed a part of the angular momentum of the Earth and ω/n will become smaller than 1, then the Moon will reverse its evolution and will start approaching the Earth definitively. Such a scenario has already been pointed out by Darwin (1880) and by Goldreich.

During the last stage ω remains smaller than n and the first part of (28) again becomes greater than the second one. Then the month will be shorter than the day but both decrease as the Moon approaches the Earth.

This qualitative result remains roughly unchanged if the equations are numerically integrated, except for the obliquity, which does not maintain a small value. The exact integration also indicates that the time scale is really prohibitive. In addition we can say that the extreme smallness of the torques in the vicinity of the reversing point makes the model questionable, residual torques being probably able to lock the Earth–Moon system in the synchronous state.

4. Integration of the Equations of Motion

4.1. OUTLINE

The differential set is composed of Equations (3), (4), (12), (14) and (26). Equations (3) and (4) are to be modified as in Paper II to account for the dissipation within the Moon. The routine used is based upon an Adams–Moulton–Cowell method developed by N. Borderies and L. Castel at C.N.E.S. (1975). The values of functions $H(I, J)$ are computed at each step by means of the recurrence relation derived in Paper II. The following starting values have been used in the various trials:

$$x = 60, \quad e = 0.055, \quad \gamma = 5.145, \quad U = 0.0538, \quad \omega/n_G = 0.0587\,.$$

Several assumptions have been made for the value of A and the curves are plotted with the two extreme values, $A = 0$ and $A = 20$.

As we have already pointed out the time is not well suited as an independant variable, due to our poor knowledge of the rate of dissipation, and also for numerical reasons. The equations previously derived are strongly dependent on the value of the semi-major axis of the Moon. For example the rate of variation of the eccentricity is proportional to $1/X^8$ and in the course of the integration X varies from 70 to values smaller than 10. Consequently if we keep the time as an independant variable, the size of the time-step should be variable and the integration would be highly time-consuming.

Equation (12) proves that the orbital inclination of the Moon to the ecliptic is always increasing backwards, in the time and, moreover, the equation $dX/d\gamma$, $de/d\gamma$ are slowly varying. Therefore γ is a convenient parameter to replace the time and we actually integrate four equations. The time scale is determined step-by-step by computing $d\gamma/dt$.

We shall discuss the results in three stages: The past and distant past, the current state and the future.

4.2. Past and Distant Past

The results of several numerical integrations are displayed in Figures 2–7 with the semi-major axis along the abscissa. The greatest influence of the dissipation within the satellite

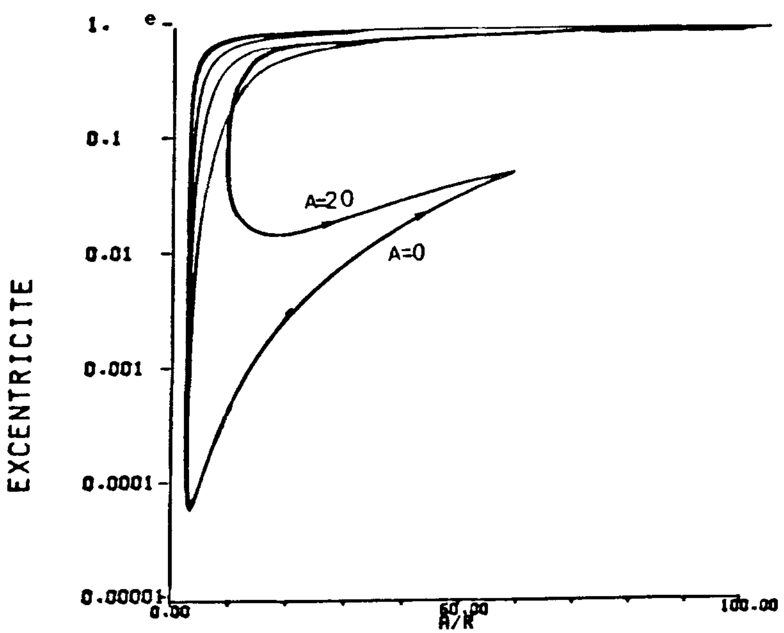

Fig. 2. History of the Moon's eccentricity versus the semi-major axis. Arrows are pointing present. At $X = 10$ we used $I = 11°$, $8°$, $6°$ and $4°$ to integrate toward the past.

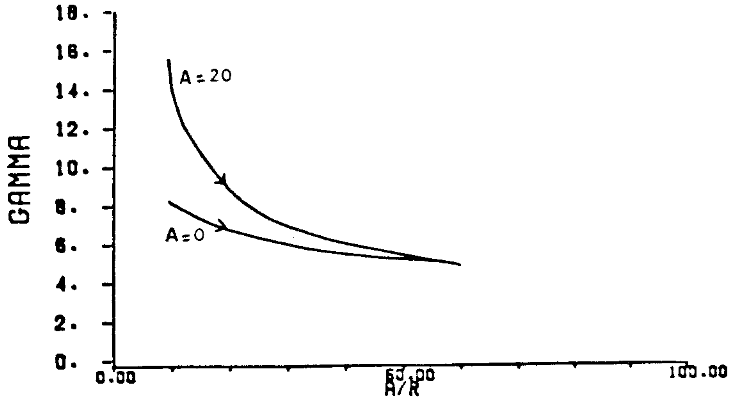

Fig. 3. History of the inclination of the Moon to the ecliptic versus the semi-major axis. Arrows are pointing to the present.

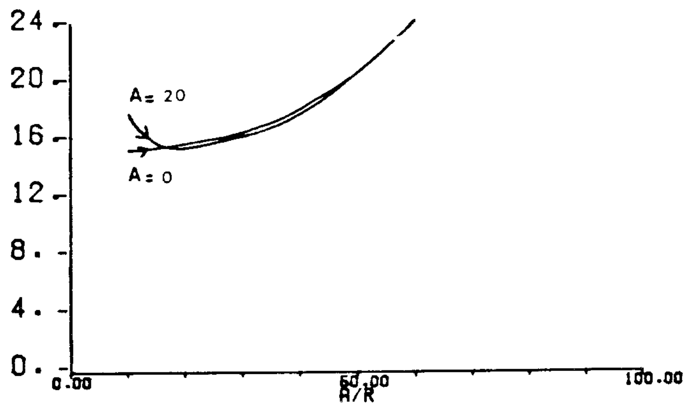

Fig. 4. History of the inclination of the Moon to the equator of the Earth versus the semi-major axis. $10R_e \times 60R_e$.

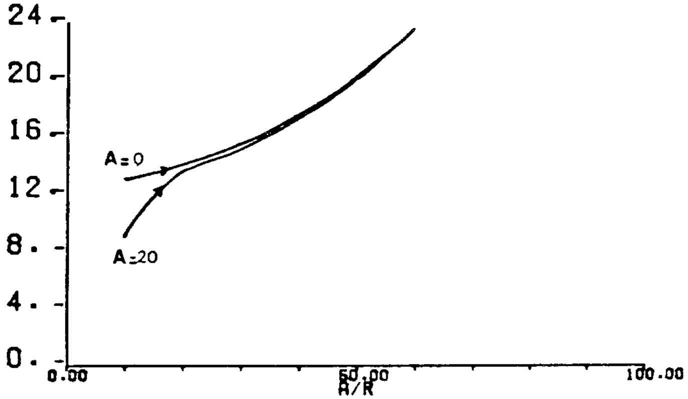

Fig. 5. History of the obliquity of the ecliptic versus the semi-major axis. Arrows are pointing to the present.

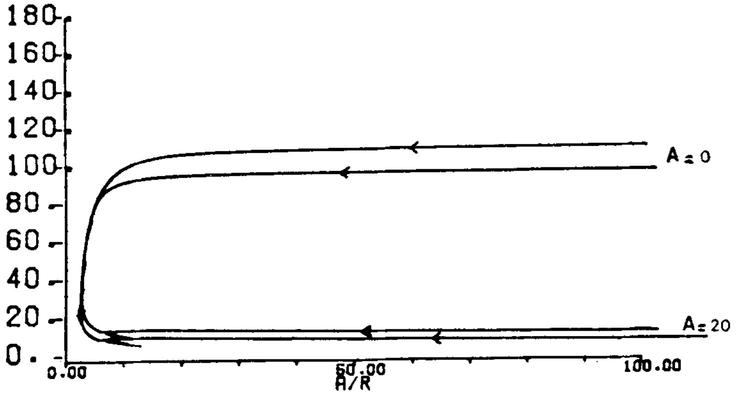

Fig. 6. History of the inclination of the Moon to the equator of the Earth versus the semi-major axis in the distant past. Arrows are pointing to the present.

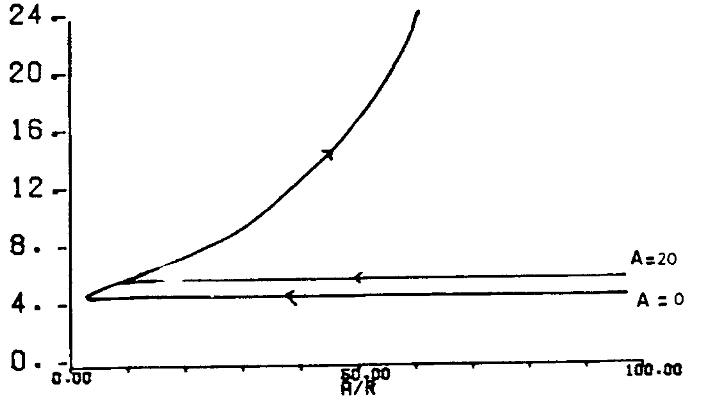

Fig. 7. History of the length of the day. Arrows are pointing to the present.

appears in the evolution of eccentricity as it was expected after the results of Paper II. But in spite of this influence eccentricity remains smaller than its current value in a region where X is greater than 10. In case of large dissipation within the Moon, the eccentricity is strongly increasing in the vicinity of the closest approach. Indeed Equations (3) and (4) are like those used in Paper II and the influence of the Sun is only felt by means of $\cos(I)$. The solutions given here and in Paper II do not show important differences in the behaviour of $\cos(I)$. Then the evolution of X and e computed in Paper II remains plenty valid. In particular the eccentricity tends to 1 in the very distant past.

The past history of the Moon's inclination to the ecliptic, for $A = 0$ is very similar to the mean curve given by Goldreich, whereas the case $A = 20$ exhibits a larger increase for small distance. However in Equation (12) we have assumed the equator of the Moon joined with the ecliptic. If such assumption is left and the equator of the Moon kept on its orbit, the two curves in Figure 3 meet. At distances as close as $10R_e$ the Earth's angular momentum (rotational) is larger than that of the Moon (orbital) and the precessional motion of the Moon takes place on the equatorial plane of the Earth. So the inclination of the equator of the Moon to its orbit tends to 0, as well as the normal componant of the force resulting from the dissipation within the Moon.

Consequently, the departure of the curve $A = 20$ from the curve $A = 0$ is overestimated in Figure 3 in the vicinity of $X = 10$. A similar conclusion can be drawn for the obliquity of the ecliptic and the inclination to the equator in Figures 4 and 5.

At this stage it is of interest to view the evolutionary tracks inside the critical distance, namely for $X \leqslant 15$. Our purpose is to carry on the integration with the help of the differential set developed in Paper II. There is no difficulty with X and e, but some problems arise with the various inclinations.

Let us assume a satellite orbiting the Earth which is much lighter than the Moon in order to satisfy the inequality

$$H_E \gg H_M ,$$

where H_E and H_M are the rotational and orbital angular momentum of the planet and of the satellite. For a semi-major axis greater than $20R_e$ the ecliptic is the precessional plane and if $X \simeq 10R_e$ that is the Earth's equator. At intermediate distance the precession takes place along the Laplacian plane (Laplace, 1805), which lies between the equator and the ecliptic. As the satellite approaches the Earth the Laplacian plane leaves the ecliptic and evolves toward the equator by taking the orbital plane off. This last plane keeps a constant inclination to the Laplacian plane in the course of the change of precessional plane, except small variations caused by tidal forces normal to the orbital plane.

This scenario does not apply to the Moon directly. The law of the conservation of the angular momentum can be written as follows for an equatorial and circular orbit

$$H = H_E + H_M = \alpha \frac{M\,\omega}{m\,n_G} + \left[\frac{a}{R}\right]^{1/2} .$$

The current value of H is 9.15 and was slightly larger in the past, because of the angular momentum removed by the Sun, namely $H = 9.4$. The equality of the angular momentum of the Earth and Moon is obtained for $(a/R)^{1/2} = 4.7$ that is to say $X = 22$. Inside this distance the angular momentum of the Earth is larger than that of the Moon.

At a distance greater than $20R_e$ the Moon precesses on the ecliptic and keeps a constant mean inclination to this plane. But for semi-major axis of the order of $10R_e$ the precessional plane is not the equator but a plane normal to the total angular momentum vector. In fact the Moon tends to keep also an invariable inclination to the equator but only the plane normal to the total angular momentum is nearly inertial because of the precession of the Earth's spin axis around the total angular momentum.

Then during the transition which goes from $X = 25$ to $X = 10$ the precessional plane lies between the ecliptic and the plane normal to the total angular momentum and gradually changes from the first plane into the second. So from $25R_e$ the Moon does not keep a constant inclination neither to the ecliptic, nor to the equator as it can be seen in the paper by Goldreich. A useful discussion of this point is made by Rubincam (1975).

As we have noted Equation (12) remains valid to investigate the variation of the inclination to the precessional plane even during the transition period. Except for the variations given by Equation (12) the inclination of the Moon's orbit to the precessional plane remains constant during the transition which allows us to know the value of the inclination of the Earth's equator and of the Moon's orbit to the plane normal to the total angular momentum as the semi-major axis reaches $10R_e$. Such a quantity is required to perform an integration of the set S1 which rules the evolution of an isolated two-body system.

At $X = 10$ the value $\gamma = 8$ and $I = 15$ lead to $i = 10$ and $J = 5$ where i and J are, respectively, the inclination of the Moon's orbit and of the Earth's equator to the invariable plane. From now on the ecliptic is theoretically lost. But since the angular momentum of the Earth is greater than that of the Moon, the inclination J is small and will become smaller as the Moon will approach the Earth. Therefore the angle between the normal to the ecliptic and the total angular momentum, namely the normal to the invariable plane, is not very different from the obliquity of the ecliptic, more precisely as long as the Earth bears the major part of the angular momentum of the Earth–Moon system, its spin axis is nearly invariable and the obliquity of the ecliptic keeps an average value close to 12° and oscillates between 8° and 16° because of the common precession of the Earth's spin axis and the Moon's angular momentum around the total angular momentum.

On the other hand the orbit of the Moon undergoes large variations in its orientation. Indeed the tidal torque acts strongly upon the small angular momentum of the Moon and causes it to change greatly.

We have carried out numerical integration with two sets of starting values surrounding those previously given in order to account for the uncertainty in their determination. The results are very similar to those obtained in Paper II. As the Moon approaches the Earth the eccentricity of its orbit increases to value larger than 0.9 and its angular momentum keeps a constant magnitude. The results relating to the minimum distance and the inclination to the Equator are greatly influenced by the ratio of the dissipation within the Moon and the Earth. By taking $A = 0$ the closest approach is found to be about $3R_e$ and the asymptotic value of the inclination I to the equator to be 100° that means the Moon was on a retrograde orbit. In case of important dissipation inside the Moon, $A = 20$, the main change occurs in the closest approach, which becomes $8R_e$, quite outside the Roche limit, whereas the inclination I remains smaller than 20° and the orbit direct. Various additional runs have been made with different initial conditions and indicate that while the closest approach is only depending on A, the final value of I decreases with its initial value. For example with $A = 0$,

$$
\begin{aligned}
I\,\text{initial} &= 17^\circ & I\,\text{final} &= 125^\circ \\
I\,\text{initial} &= 13^\circ & I\,\text{final} &= 110^\circ \\
I\,\text{initial} &= 10^\circ & I\,\text{final} &= 90^\circ.
\end{aligned}
$$

Such modifications occurs also with $A = 20$.

In every case the length of day evolves in accordance with Figure 8 of Paper II and greater is A, larger the length of day in the distant past: 4 hours with $A = 0$ and 6 hours with $A = 20$. Of course, in the very distant past when the eccentricity was close to 1 and the semi-major axis larger than $10R_e$ our results are only indicative and a careful investigation of the gravitational motion of a highly eccentric satellite is needed before giving definitive conclusions.

4.3. Current State

The immediate past and future are well represented by the values of the various derivatives used in the differential equations. All quantities given in Table I are computed with $\Delta t = 10\,\text{mn}$. In order to account for the observed secular acceleration of the Moon's mean longitude. As explained in the introduction the true effect of the bodily tides is weaker than that computed with $\Delta t = 10\,\text{mn}$.

TABLE I

Current variations of the lunar orbit.

	$\frac{da}{dt}$ (cm y^{-1})	$\frac{de}{dt}$ (deg y^{-1})	$\frac{d(\text{day})}{dt}$ (s y^{-1})	$\frac{d\gamma}{dt}$ (deg y^{-1})	$\frac{d\epsilon}{dt}$ (deg y^{-1})	$\frac{dI}{dt}$ (deg y^{-1})
$A = 0$	3.9	1.4×10^{-11}	2.5×10^{-5}	-1.3×10^{-10}	3.2×10^{-9}	3.2×10^{-9}
$A = 20$	3.8	6.4×10^{-12}	2.5×10^{-5}	-2.4×10^{-10}	3.2×10^{-9}	3.2×10^{-9}

4.4. Future

In addition to the scenario sketched at the end of Section 3, we give numerical precisions. The maximum distance of the Moon from the Earth is found to be $75R_e$, and 10^{10} years are required to reach it. The length of day at this stage is of the order of 15 days and goes on growing up to 55 days to end decreasing very slowly after. In Paper I we have shown that the equilibrium point was distant $84R_e$ from the Earth which seems to disagree with the result given here. In fact the synchronous point can be located at the distance where the total angular momentum is incorporated in the orbital motion of the Moon. As the sun in continuously removing angular momentum from the system the theoretical maximum distance is continuously decreasing.

The major improvement to the equatorial scenario previously described concerns the inclinations. Both the obliquity and the inclination to the equator reach values as large as 55° while the inclination to the ecliptic remains moderate at 4.8°. Immediately after the Moon will have gone through the stationary point the obliquity suddenly decreases toward 0°. Then the system evolves to a situation in which the orbital plane, the equator and the ecliptic are coplanar and orbit circular, the Moon approaching the Earth.

5. Conclusions

Once more the relative dissipation within the Moon with respect to the Earth has been proved to be a critical parameter in the past history of the Earth–Moon system. In the conclusion of Paper I we intended to look for a mechanism allowing the inclination of the Moon to the proper plane of the system to increase, in order to avoid an approach to the Earth closer than $2R_e$. Then we proposed to investigate the act of the Sun and the influence of the eccentricity of the Moon's orbit. In fact, neither the Sun nor an elliptical orbit lead to such a result. They particularly influence the history of the inclination. But, on the other hand, moderate dissipation within the Moon is able to increase the distance of the closest approach and to provide an answer to the question raised. In the opinion of the present author, this result constitutes the major achievement of the theory developed here.

Besides, the dissipation within the Moon gives rise to the possibility for placing the birth or the capture of the Moon in a direct orbit with a moderate inclination to the equator and an eccentricity close to one. The integration of the dynamical equations in the distant past gives a solution with a constant orbital angular momentum even in case of large inclination. This last property prevents the Earth from possessing an unrealistic initial angular-momentum density and allows an impediment for the capture theory to be cleared off.

The time-scale problem is not yet solved and no progress has been made by our calculations. However, it is nearly certain now that the observed secular acceleration is caused by tidal dissipation within the oceans and seas, the configuration of which is greatly changing over a period as short as five million years. Hence, the solution of the time-scale problem required the almost unattainable knowledge of the locus of tidal dissipation in the past.

Possible improvement that might be done in our calculations are of two kinds. First, an analytical solution of precessional equations where satellite and planet have an angular momentum of the same magnitude would be welcome and would permit the uncertainties of the computation during the transition period to be removed. Until now I did not succeed in obtaining such a solution. Second, our whole theory is based on the assumption of a time delay Δt between the stress and strain of Earth and Moon: that is to say, we have assumed a phase-lag proportional to the frequency. Such an assumption is very difficult to prove or disprove; the artificial satellites being quite insensitive to the lag and the current evolution of the Moon to the frequency law of the lag. In a paper now in progress we shall attempt to show that the inclination of Phobos could be used to discriminate between the different models.

Of course, a determination of the time-delay of the Moon will be of great interest for a choice of the most realistic one among all trajectories of the phase space.

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