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Solutions of the Einstein Equation with Heat Flow

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We study a shear-free spherically symmetric cosmological model with heat flow and present a general method to generate solutions. Using this solution generating scheme we obtain new classes of solutions to the Einstein equations.

1. INTRODUCTION

Consider a shear-free spherically symmetric cosmological model. The energy-momentum tensor takes the form

$$T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu + p g_{\mu\nu} + q_\mu U_\nu + q_\nu U_\mu \tag{1}$$

where U_μ is the four-velocity of the fluid and q_μ is the heat flux. Due to spherical symmetry and the shear-free condition, the line element can be written as

$$ds^2 = -A^2 dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2}$$

where A and B are functions of r and t.

In the comoving system, following Glass [1] and Bergmann [2], the pressure isotropy condition gives rise to the following equation:

$$F\frac{\partial^2 A}{\partial x^2} + 2\frac{\partial F}{\partial x}\frac{\partial A}{\partial x} - \frac{\partial^2 F}{\partial x^2} A = 0 \tag{3}$$

where $F(r, t) = B^{-1}$ and $x = r^2$.

Once A and F are obtained by solving Eq. (3), the other components of the Einstein equation just give expressions for ρ, p, and q_μ. A number of

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authors have obtained various solutions to Eq. (3), among which is a particular class

$$A = \frac{c(t)\,x + d(t)}{a(t)\,x + b(t)} \tag{4}$$

$$F = a(t)\,x + b(t) \tag{5}$$

where a, b, c, and d are arbitrary functions of t. Sanyal [3] and Modak [4] obtained this class along with other solutions, while Bergmann [2] and Maiti [5] obtained special cases of the class. In this paper, we develop a general method to generate solutions and obtain a new class of solutions which includes Eqs. (4) and (5) as special cases.

2. THE METHOD OF GENERATING SOLUTIONS

Since Eq. (3) contains no time derivatives, it can be considered an ordinary differential equation with respect to x. Notice also that it is a second-order linear equation of A if F is a known function, and vice versa. This provides a convenient way of solving the equation. Substituting a simple function $A_1(x, t)$ for $A(x, t)$ (a constant, for example), we get a linear differential equation for F. We can then obtain the general solution $F_1(x, t)$ of it easily. The pair

$$A = A_1 \qquad \text{and} \qquad F = F_1 \tag{6}$$

thus solves Eq. (3). Now putting $F_1(x, t)$ back into Eq. (3) we get an equation for A, to which A_1 is already a solution. A second solution A_2, linearly independent of A_1, can then be constructed and the pair

$$A = A_3 = \alpha A_1 + \beta A_2, \qquad F = F_1 \tag{7}$$

is another solution of Eq. (3). Equation (6) is just a special case ($\beta = 0$) of Eq. (7).

The merit of this method rests upon the fact that knowing a first solution A_1 of a second-order differential equation one can always write down a second solution A_2, which is linearly independent of A_1, in terms of A_1 and the coefficients of the equation.

Now putting A_3 back into Eq. (3), we get an equation for F, to which F_1 is a solution. Thus an F_2 can be generated. By alternating between A and F, this process can go on forever, generating infinite series of solutions expressed in terms of integrals.

3. THE SOLUTIONS

We start with a simple case,

$$A_1 = 1 \tag{8}$$

Equation (3) then becomes

$$F'' = 0 \tag{9}$$

where a prime means derivative with respect to x. Equation (9) can be solved directly:

$$F_1 = a(t)\,x + b(t) \tag{10}$$

with a and b arbitrary functions of t. Equations (8) and (10) yield the solution obtained by Bergmann. Substituting Eq. (10) into Eq. (3) we get

$$A'' + \frac{2a}{ax+b}A' = 0 \tag{11}$$

The general solution of Eq. (11) is easily found:

$$A = A_2 = \frac{c(t)\,x + d(t)}{a(t)\,x + b(t)} \tag{12}$$

with c and d being arbitrary functions. Equations (10) and (12) are none other than Eqs. (4) and (5). Thus we obtain this class in a very simple manner.

The next step is to continue the process and generate new solutions. Putting Eq. (12) into Eq. (3), we get

$$F'' - \left[2\frac{\partial}{\partial x}\left(\frac{cx+d}{ax+b}\right)\bigg/\left(\frac{cx+d}{ax+b}\right)\right]F' - \left[\frac{\partial^2}{\partial x^2}\left(\frac{cx+d}{ax+b}\right)\bigg/\left(\frac{cx+d}{ax+b}\right)\right]F = 0 \tag{13}$$

Since F_1 given by Eq. (10) is a solution of Eq. (13), a second solution is obtained in the following way:

$$F_2 = F_1(x)\int^x \frac{dy}{[F_1(y)]^2}\exp\left[\int^y dz\,\frac{\partial}{\partial z}\ln\left(\frac{cz+d}{az+b}\right)^2\right] \tag{14}$$

where the dependence on t is not explicitly shown. Performing the integration we get

$$F_2 = \frac{-1}{3a}\left[\left(\frac{cx+d}{ax+b}\right)^2 + \frac{c}{a}\frac{cx+d}{ax+b} + \frac{c^2}{a^2}\right] \tag{15}$$

The general solution to Eq. (13) is a linear combination of F_1 and F_2, which are linearly independent of each other. Thus with Eqs. (10), (12), and (15) we have obtained a new solution, namely,

$$
\begin{aligned}
A(x,t) &= A_2 = \frac{c(t)\,x + d(t)}{a(t)\,x + b(t)} \\
F(x,t) &= F_3 = g(t)\,F_1 + h(t)\,F_2 \\
&= g(t)(ax+b) - \frac{h(t)}{3a}\left[\left(\frac{cx+d}{ax+b}\right)^2 + \frac{c}{a}\frac{cx+d}{ax+b} + \frac{c^2}{a^2}\right]
\end{aligned}
\tag{16}
$$

where g and h are another pair of arbitrary functions.

This solution generating scheme can be continued. The next generation of solutions is found to be

$$
\begin{aligned}
F(x,t) &= g(t)(ax+b) - \frac{h(t)}{3a}\left[\left(\frac{cx+d}{ax+b}\right)^2 + \frac{c}{a}\frac{cx+d}{ax+b} + \frac{c^2}{a^2}\right] \\
A &= l(t)\frac{cx+d}{ax+b} + m(t)\frac{cx+d}{ax+b}\int^x \left(\frac{ay+b}{cy+d}\right)^2 \frac{dy}{[F(y,t)]^2}
\end{aligned}
\tag{17}
$$

where $l(t)$ and $m(t)$ are again arbitrary. It can be seen that the problem now becomes that of performing integrations. As we go further in the process, the integrations become harder. Since the integrand in Eq. (17) is a rational function, the integrations can be carried out in principle, but the calculation can become very tedious. We do not pursue it further here.

4. CONCLUSION

We have developed a general method to obtain solutions of Eq. (3) and found the solutions given by Eqs. (16) and (17). To the best of our knowledge, they represent new classes of solutions. Equation (16) contains, as special cases, not only Eqs. (4) and (5) but also some other solutions obtained by the authors in the references.

Having obtained the solutions to Eq. (3), it remains to determine that they are physically acceptable. The functions a, b, c, d, g, h, l, and m have not been determined so far. They will be determined by substituting them into the other components of the Einstein equations, which will express them in terms of ρ, p, and q_μ on which relevant physical conditions can be imposed.

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