OSCILLATOR WITH A CENTRIFUGAL BARRIER

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We study the wave functions and energy levels of a one-dimensional oscillator with a centrifugal barrier of the form $s(s + 1)/x^2$. It is shown that a barrier of this kind automatically implies the point potential $A(s)\delta(x)/x$. It is also shown that two different sets of even states are physically admissible, but that only one of these sets transforms continuously into the set of states of harmonic oscillator when the centrifugal barrier is allowed to vanish and is preferable for this reason.

An oscillator with a centrifugal barrier has been considered in several papers (for example, [1-6]). All of these authors choose the same wave functions for the radial motion (odd in the one-dimensional case $-\infty < x < \infty$) [7, p. 158] but differ in the construction of the even states ($-\infty < x < \infty$).

We choose the potential in the form ($\hbar = 2m = 1$)

$$V(x) = x^2 + \frac{s(s+1)}{x^2}, \quad s > -\frac{1}{2}, \tag{1}$$

in order to exclude the case where the particle falls to the bottom of the well [7, pp. 143-146].

The difficulty in constructing the even eigenstates of the Schrödinger equation is due to the discontinuity of the wave functions and their derivatives at $x = 0$ because of the singularity of the potential (1).

Indeed, the independent solutions of the stationary Schrödinger equation with the potential (1) in the region $x > 0$ will be [7, p. 158; 4]

$$\psi_+(x) = x^{-s} e^{-\frac{x^2}{2}} F\left(\frac{-2s+1-E}{4}, -s+\frac{1}{2}, x^2\right), \tag{2}$$

$$\psi_-(x) = x^{s+1} e^{-\frac{x^2}{2}} F\left(\frac{2s+3-E}{4}, s+\frac{3}{2}, x^2\right), \tag{3}$$

where $F(u, v, z)$ is the degenerate hypergeometric function [7, pp. 741-745]; E is the energy of the state. These solutions satisfy boundary conditions independent of E in the limit $x \to 0$.

When $s > 1/2$ only solution (3) is quadratically integrable about zero; therefore, we must continue this solution into the region $x < 0$ to construct both the even and odd solutions. The energy levels then are doubly degenerate. From the continuity in s and the continuity of $\psi(x)$ in x, this set of even functions can also be extended to the region $-1/2 < s < 1/2$ [2, 3]. The discontinuity in $\psi'(x)$ as $x \to 0$, however, cannot be avoided for $-1/2 < s < 0$. This even function does not transform into the even harmonic oscillator function when $s = 0$. In [4] the functions (2) were chosen as the even functions for $-1/2 < s < 1/2$; these functions transform continuously in s into the even harmonic oscillator functions.

Because of these problems we attempt to construct the even functions $\psi_\ell(x)$ on the basis of a self-adjoint expansion of the differential Hamiltonian operator [8, pp. 478-489] of the oscillator (1):

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$$-\hat{H}\psi = \frac{d^2\psi}{dx^2} - \left(x^2 + \frac{s(s+1)}{x^2}\right)\psi. \tag{4}$$

Because the singularity of the potential coincides with the singularity of the Bessel operator [8, pp. 535-539], the expansion of the Hamiltonian (4) is classified as an expansion of the Bessel operator, and the even functions of the Schrödinger equation (1) and (2) can be (for $-1/2 < s < 1/2$) any functions of the form ($\psi_\alpha(x) \in L_2(0, \infty)$)

$$\psi_\alpha(x) = \cos\alpha\psi_+(|x|) + \sin\alpha\psi_-(|x|), \tag{5}$$

where the mixing angle $\alpha(s)$, $-\pi/2 < \alpha \leq \pi/2$, is chosen from physical considerations and the Hamiltonian (4) will be self-adjoint on the functions $\varphi(x) \in L_2(0, \infty)$ satisfying the same boundary conditions as $x \to +\infty$ as the functions (5).

As in the nonsingular case ($|V(0)| < \infty$), these conditions are conveniently written in terms of the logarithmic derivative of $\psi(+0)$ [9]. For the functions (5) and (3) the boundary conditions have form

$$\lim_{x\to+0}\frac{1}{(2s+1)x^{2s}}\left(\frac{\psi_\alpha'(x)}{\psi_\alpha(x)} + \frac{s}{x}\right) = \mathrm{tg}\,\alpha, \quad \alpha \neq \frac{\pi}{2}, \tag{6}$$

$$\lim_{x\to+0}\left(\frac{\psi_-'(x)}{\psi_-(x)} - \frac{s+1}{x}\right) = 0, \quad \alpha = \frac{\pi}{2}. \tag{7}$$

In the case of the even functions, the condition $\psi'(0)/\psi(0) = 0$ leads to a δ-function addition to the potential [10]. Therefore, in our case, choice of the condition (7) automatically includes the additional barrier

$$\delta V(x) = \frac{2(s+1)\delta(x)}{|x|}, \tag{8}$$

which leads to a break in $\psi_e'(x)$ at $s = 0$ and to a twofold degeneracy with respect to parity (E_{en}, E_{0n} are the energies of the even and odd states)

$$E_{en} = E_{0n} = 4n + 2s + 3, \quad n = 0, 1, 2\ldots \,. \tag{9}$$

Condition (6) for ψ_α gives a different additional singular potential

$$\delta V(x) = -\frac{2s\delta(x)}{|x|} + 2|x|^{2s}(2s+1)\delta(x)\,\mathrm{tg}\,\alpha. \tag{10}$$

The possible choice $\alpha(s) = 0$ gets rid of the weak barrier $2|x|^{2s}(2s + 1)\tan\alpha\cdot\delta(x)$. Therefore, $\alpha = 0$ is assumed to be physically justified [$\delta V(x) = 0$ when $s = 0$], together with the choice $\alpha = \pi/2$. The twofold degeneracy in (9) is removed when $\alpha = 0$:

$$E_{en} = 4n - 2s + 1 \neq E_{0n}, \quad n = 0, 1, 2\ldots \,. \tag{11}$$

Objections have been raised against the even states (2), (6), (11). In [3] the objection was the addition of the δ-function singularity (10) to the potential, but the same singularity (8) also occurs for the choice $\psi_e = \psi_-$ of (3) and (8). In [2] the state (6) ($\alpha = 0$), (11) was rejected because of the discontinuity of the quantity $\psi'(0)\psi(0)$, but the physically meaningful current $j \equiv 2\,\mathrm{Im}\,\psi^*\psi' \equiv 0$ is not discontinuous. Another reason for rejecting the state (11) in [2] was the divergence of the corrections to the energy levels (11) in the first order of perturbation theory under the transformation $s_0 \to s_0 + \delta s$. Such a divergence does occur for the perturbation $\delta W(x) = (2s_0 + 1)\delta s/x^2$, but a valid perturbation would be, according to (1) and (10)

$$\delta W = \delta s\left(\frac{2s_0+1}{x^2} - 2\frac{\delta(x)}{|x|}\right)$$

and this perturbation, with an accurate integration using the normalized wave functions of [4, 5], leads to finite corrections which coincide with the exact result following from (11): $\delta E_n = -2\delta s$. (The integral $I = 2\int_0^\infty \psi_{en}^2\delta W(x)\,dx$ is evaluated from a ($a > 0$) to $+\infty$ upon the substitution $\delta(x) \to \delta(a - \varepsilon)$, $\varepsilon > 0$. After evaluating the integral, we first take the limit

$\varepsilon \to +0$, and then the limit $a \to +0$. The divergences in the two terms of I cancel one another when the latter limit is taken.) The argument of [4] for the advantage of the functions (2) as the ψ_ℓ (continuous transition with respect to s from ψ_e of (2) to the even harmonic oscillator functions) is correct but insufficient: One can choose the mixing angle α in the form of the function $\alpha(s)$, $\alpha(0) = 0$; then the continuity with respect to s will be guaranteed and only an additional argument, such as discarding the weak barrier in (10), makes the choice $\alpha \equiv 0$ unique.

The functions (2) and (11) have a number of interesting and apparently paradoxical features which are the result of the mutual effect of the potentials (1) and (10) with $\alpha = 0$. The levels (11) decrease as s increases. The ground-state level [n = 0 in (11)] lies below the minimum in the potential (1) when $s \to 1/2 - 0$. The density of particles $\rho = \psi_e^2$ of (2) increases under the barrier ($s \to 1/2 - 0$). For the functions (2) the mean kinetic energy $\bar{T} = +\infty$ and potential energy $\bar{V} = -\infty$ both diverge, but the physically meaningful total energy $\bar{E}$ is given by $\bar{E} = E_{en}$ from (11). Here $\bar{E}$ is calculated with the help of regularization and the use of the limit $\to +0$, as was done for the integral I.

When $\alpha \neq 0$, $\pm\pi/2$ the functions (5) are physically meaningful as the even wave functions of an oscillator with a δ-function barrier [when s = 0 we have

$$V(x) = x^2 + 2\,\mathrm{tg}\,\alpha\delta(x) \tag{12}$$

instead of $V(x) = x^2$] to which the centrifugal barrier $s(s + 1)/x^2$ is switched on adiabatically. The equations for the eigenvalues (9) and (11) become transcendental equations for this choice of the $\psi_e(x)$ (compare [11])

$$\frac{\Gamma\left(\frac{3+2s-E_n}{4}\right)\Gamma\left(\frac{1}{2}-s\right)}{\Gamma\left(\frac{1-2s-E_n}{4}\right)\Gamma\left(\frac{3}{2}+s\right)} + \mathrm{tg}\,\alpha = 0, \tag{13}$$

and the normalized wave functions take the form

$$\psi_l(x) = \left|\Gamma\left(\frac{1-2s-E_n}{4}\right)\right|\left[\sum_{\kappa=0}^{\infty}\frac{\Gamma\left(\kappa+\frac{1}{2}-s\right)}{\kappa!\left(\kappa+\frac{1-2s-E_n}{4}\right)^2}\right]^{-\frac{1}{2}} \times$$

$$\times e^{-\frac{x^2}{2}} x^{-s} U\left(\frac{1-2s-E_n}{4},\ \frac{1}{2}-s,\ x^2\right), \tag{14}$$

where $U(a, b, y)$ is the degenerate hypergeometric function of the second kind [12]. When s = 0 the Eqs. (14), (13) and (3), (9) give the solution for the problem of an oscillator with a δ-function wall (12) (compared to the analogous problem of a potential box in [13]).

Hence, a centrifugal barrier $s(s + 1)/x^2$ for the one-dimensional motion in the field of an oscillator (or in any even confining field $V(\pm\infty) = +\infty$) must, in the quantum-mechanical treatment, involve the strongly singular point potential $A(\delta(x)/|x|)$ whose magnitude depends on the barrier parameter s and on the choice of the boundary condition for the even functions. Such a barrier (well) does not occur in the classical treatment of the motion in the field $s(s + 1)/x^2$, and hence it is a purely quantum effect.

Physically there are two possible quantum mechanics of the oscillator (1). The first is given by (2) and (6) with $\alpha = 0$ and (11), and it transforms continuously into the quantum mechanics of the harmonic oscillator in the limit $s \to 0$. The second is given by (3), (7), (8), (9), and when the centrifugal barrier is switched off it leads to a distorted (because of the barrier $2(\delta(x)/|x|)$ harmonic oscillator: the Klauder phenomenon [3, 14]. Each of the choices for $\psi_\ell(x)$ has its advantages and disadvantages, as discussed here. The choice (2), (6), $\alpha = 0$, (11) seems to us to be preferable because it continuously (with respect to s) transforms into the even states of the harmonic oscillator because of the absence of degeneracy and the Klauder effect, and because of its sensitivity to the δ-function wall (12) [the solution (3), (9) is not sensitive to the substitution $x^2 \to x^2 + 2\tan\alpha\delta(x)$ in the potential (1)].

Similar results also occur for the one-dimensional ($-\infty < x < \infty$) motion in an even retaining field with the singularity $V(x) \approx A/|x|$, $1 \leq \nu < 2$ for $|x| \to 0$. In this case the additional barrier arising upon the choice of even states which transform continously into oscillator states [the analog of the potential (10), $\alpha = 0$] is less singular than the additional barrier $\delta V = 2(\delta(x)/|x|)$, which appears upon the choice of "truncated" odd states as the even states [the analog of the potential (8)]. This fact is an additional argument in favor of the choice of the functions (2) for the ψ_e.

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LITERATURE CITED

1. V. D. Krivchenkov and V. D. Kukin, Dokl. Akad. Nauk SSSR, 204, 60 (1972).
2. F. Calagero, J. Math. Phys., 10, 2191 (1969).
3. J. Klauder, Acta Phys. Austriaca Suppl., 11, 341 (1973).
4. I. A. Malkin and V. I. Man'ko, Dynamical Symmetry and Coherent States of Quanum Systems [in Russian], Nauka, Moscow (1979), pp. 99-109.
5. K. B. Wolf, Preprint No. 285, IMAS, UNAM, Mexico (1981).
6. V. B. Gostev, V. S. Mineev, and A. R. Frenkin, VMU, Ser. 3, 23, No. 2, 75 (1982).
7. L. D. Landau and E. M. Lifshitz, Quantum Mechanics, Pergamon Press, Oxford (1974).
8. N. I. Akhiezer and I. M. Glazman, Theory of Linear Operators in Hilbert Space [in Russian, Nauka, Moscow (1966).
9. A. I. Baz', Ya. B. Zel'dovich, and A. M. Perelomov, Scattering, Reactions, and Decay in Nonrelativistic Quantum Mechanics [in Russian], Appendix A, Nauka, Moscow (1971), pp. 513-529.
10. P. M. Morse and H. Feshbach, Methods of Theoretical Physics (2 vols.), Vol. 2, McGraw-Hill, New York (1953), pp. 595-596.
11. V. A. Alessandrini and I. Giambiagi, Nuovo Cim., 29, 1353 (1963).
12. Handbook of Mathematical Functions, M. Abramowitz and I. Stegun (eds.), National Bureau of Standards, Washington, D. C. (1964), Chap. 13.
13. S. Flügge, Practical Quantum Mechanics (2 vols.), Springer-Verlag, Berlin (1971).
14. M. Reed and B. Simon, Methods of Modern Mathematical Physics [Russian translation], Vol. 4, Mir, Moscow (1982), p. 77.