These restrictions and also the absence of a general solution do not allow us to use the solution found in [6] for the realization of a complete classification of (N.1) Stackel spaces.

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THERMODYNAMIC CHARACTERISTICS OF FERMI GASES IN A MAGNETIC FIELD

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UDC 523.877

Within the framework of statistical thermodynamics of equilibrium systems, general expressions are obtained for the chemical potential, pressure, and magnetic susceptibility for degenerate ideal nonrelativistic electron, proton, and neutron gases in magnetic fields, which exert no pronounced influence on the anomalous magnetic moments of the fermions.

Investigating the influence of the magnetic field on the energy and magnetic characteristics of Fermi gases is of definite interest both for solid-state and semiconductor physics and for theoretical astrophysics. According to current concepts in superdense astrophysical objects, in conditions similar to those for an electron gas in a metal, a relativistic electron gas and nonrelativistic proton and neutron gases may exist [1]. In the present work, using the method outlined in [2], general relations applicable for the description of degenerate ideal nonrelativistic electron, proton, and neutron gases in a magnetic field within the framework of the statistical thermodynamics of equilibrium systems are obtained, taking account of the static anomalous magnetic moments of microparticles; in addition, a comparative analysis is made of the influence of the magnetic field on the pressure, chemical potential, and magnetic susceptibility of these gases.

Following [2], by standard calculations, an expression may readily be obtained for the large thermodynamic potential of extremely degenerate ideal nonrelativistic Fermi gas in a constant and homogeneous magnetic field with induction B

$$\Omega_\kappa(B) = \Omega_\kappa(0)\cdot R_{5/2}(x_\kappa)\cdot R_{3/2}^{-5/3}(x_\kappa), \tag{1}$$

where

$$\Omega_\kappa(0) = -0{,}4N_\kappa\,\zeta_\kappa(0),\quad \kappa = e,\ p,\ n, \tag{2}$$

$$\zeta_\kappa(0) = (3\pi^2)^{2/3}\cdot\hbar^2 n_\kappa^{2/3}\cdot(2m_\kappa)^{-1},\quad n_\kappa = N_\kappa/V_\kappa, \tag{3}$$

A. S. Pushkin Pedagogical Institute, Brest. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 5, pp. 21-25, May, 1987. Original article submitted February 8, 1985.

0038-5697/87/3005-0375$12.50

N_κ is the number of fermions in volume V_κ; m_κ is the fermion mass; $\hbar$ is the Planck constant; the subscripts e, p, n correspond to the electron, proton, and neutron; $R_{5/2}(x_\kappa)$ and $R_{3/2}(x_\kappa)$ are functions of the parameter

$$x_\kappa = \zeta_\kappa(B)\cdot(\mu B)^{-1}, \tag{4}$$

and ζ_κ is the chemical potential; for electrons, μ is the Bohr magneton; for nucleons, μ is the nuclear magneton. For charged fermions

$$R_{3/2}(x_\kappa) = 1{,}5\cdot\sum_{n=0}^{l}[(x_\kappa - 2n - 1 - \sigma_\kappa)^{1/2} + (x_\kappa - 2n - 1 + \sigma_\kappa)^{1/2}], \tag{5}$$

$$R_{5/2}(x_\kappa) = 2{,}5\cdot\sum_{n=0}^{l}[(x_\kappa - 2n - 1 - \sigma_\kappa)^{3/2} + (x_\kappa - 2n - 1 + \sigma_\kappa)^{3/2}], \tag{6}$$

summation in Eqs. (5) and (6) is performed until the radicand becomes negative; n is the number of the Landau quantum level. For neutrons

$$R_{3/2}(x_n) = 0{,}5\,[(x_n - \sigma_n)^{3/2} + (x_n + \sigma_n)^{3/2}], \tag{7}$$

$$R_{5/2}(x_n) = 0{,}5\,[(x_n - \sigma_n)^{5/2} + (x_n + \sigma_n)^{5/2}]. \tag{8}$$

Here $\sigma_\kappa = \mu_\kappa/\mu$, where μ_κ is the intrinsic magnetic moment of the fermion. In deriving these formulas, it is taken into account that the energy spectrum of the free charged fermion in the magnetic field is

$$\varepsilon = p_{z\kappa}^2\cdot(2m_\kappa)^{-1} + \mu B\cdot(2n + 1 \pm \sigma_\kappa), \tag{9}$$

while for the neutron

$$\varepsilon = p_n^2\cdot(2m_n)^{-1} \pm \sigma_n\mu B, \tag{10}$$

P_z is the projection of the fermion momentum p onto the direction of induction of the field. It is assumed in Eqs. (9) and (10) that σ_κ does not depend on B; i.e., only the static anomalous magnetic moment of the fermion is taken into account. This limits the applicability of the relations obtained to the conditions where $B \ll 4.414\cdot10^{13}$ G for the electron gas [3] and $B \lesssim 10^{16}$ G for nucleon gases. When $B \gtrsim 10^{16}$ G, the cyclotron energy of the pion exceeds the energy of β^- decay of the neutron, and the magnetic moment of the neutrons depends on B. Thus, let $\sigma_e \approx 1.00116$; $\sigma_p \approx 2.793$; $\sigma_n \approx 1.913$ [3].

Using the above relations, it is found that

$$P_\kappa(B) = -\Omega_\kappa(B)\cdot V_\kappa^{-1} = P_\kappa(0)\cdot R_{5/2}(x_\kappa)\cdot R_{3/2}^{-5/3}(x_\kappa), \tag{11}$$

$$\varkappa_\kappa(B) = -\frac{1}{V_\kappa B}\cdot\left[\frac{\partial\Omega_\kappa(B)}{\partial B}\right]_{\zeta_\kappa} = \varkappa_{\kappa 0}\,\frac{R_{5/2}(x_\kappa) - x_\kappa R_{3/2}(x_\kappa)}{R_{3/2}^{1/3}(x_\kappa)}, \tag{12}$$

where P_κ is the pressure; $\varkappa_\kappa$ is the magnetic susceptibility; $P_\kappa(0) = 0.4n_\kappa\cdot\zeta_\kappa(0)$, $\varkappa_{\kappa 0} = n_\kappa\mu^2\cdot\zeta_\kappa^{-1}(0)$. It also follows from the condition $N_\kappa = -\left(\frac{\partial\Omega_\kappa}{\partial\zeta_\kappa}\right)_{V_\kappa B}$ that

$$\zeta_\kappa(B) = \zeta_\kappa(0)\cdot x_\kappa\cdot R_{3/2}^{-2/3}(x_\kappa). \tag{13}$$

Note that the expressions for the chemical potential and magnetic susceptibility of the proton gas were obtained in [4] and [5], respectively; when $\sigma_e = 1$, the relations in the present work transform to the formulas of [2].

In the limiting case of weak magnetic fields ($x_\kappa \gtrsim 10$), for charged fermions

$$R_{3/2}(x_\kappa) \approx x_\kappa^{3/2}\cdot[1 + 0{,}125x_\kappa^{-2}\cdot(3\sigma_\kappa^2 - 1)], \tag{14}$$

$$R_{5/2}(x_\kappa) \approx x_\kappa^{5/2}\cdot[1 + 0{,}625x_\kappa^{-2}\cdot(3\sigma_\kappa^2 - 1)] \tag{15}$$

and

$$\zeta_\kappa(B) \approx \zeta_\kappa(0)\cdot\left[1 - \frac{3\sigma_\kappa^2 - 1}{12}\cdot\frac{\mu^2B^2}{\zeta_\kappa^2(0)}\right], \tag{16}$$

$$P_{\kappa}(B) \approx P_{\kappa}(0) \cdot \left[1 + \frac{5(3\sigma_{\kappa}^2 - 1)}{12} \cdot \frac{\mu^2 B^2}{\zeta_{\kappa}^2(0)}\right]; \tag{17}$$

$$\varkappa_{\kappa}(B) \approx \varkappa_{\kappa 0} \cdot \frac{3\sigma_{\kappa}^2 - 1}{2}\left[1 - \frac{3\sigma_{\kappa}^2 - 1}{24} \cdot \frac{\mu^2 B^2}{\zeta_{\kappa}^2(0)}\right]; \tag{18}$$

for neutrons

$$R_{3/2}(x_n) \approx x_n^{3/2} \cdot (1 + 0{,}375 x_n^{-2} \sigma_n^2), \tag{19}$$

$$R_{5/2}(x_n) \approx x_n^{5/2} \cdot (1 + 1{,}875 x_n^{-2} \sigma_n^2), \tag{20}$$

$$\zeta_n(B) \approx \zeta_n(0) \cdot [1 - 0{,}25\sigma_n^2 \mu^2 B^2 \cdot \zeta_n^{-2}(0)], \tag{21}$$

$$P_n(B) \approx P_n(0) \cdot [1 + 1{,}25\sigma_n^2 \mu^2 B^2 \cdot \zeta_n^{-2}(0)], \tag{22}$$

$$\varkappa_n(B) \approx \varkappa_{n0} \cdot 1{,}5\sigma_n^2 \cdot [1 - 0{,}125\sigma_n^2 \mu^2 B^2 \cdot \zeta_n^{-2}(0)]. \tag{23}$$

Thus, as $B \to 0$, it is found that $\zeta_K(B) \to \zeta_K(0)$, $P_K(B) \to P_K(0)$, and also $\varkappa_e(B)/\varkappa_e(0) \to 1.00348$, $\varkappa_p(B)/\varkappa_p(0) \to 11.2013$, $\varkappa_n(B)/\varkappa_n(0) \to 5.4899$.

In the quantum limit of superstrong magnetic fields, for charge fermions $n = 0$ and the spin is directed along the field, i.e., $x_K \leq 3 - \sigma_K$. Then

$$R_{3/2}(x_{\kappa}) = 1{,}5(x_{\kappa} - 1 + \sigma_{\kappa})^{1/2}, \tag{24}$$

$$R_{5/2}(x_{\kappa}) = 2{,}5(x_{\kappa} - 1 + \sigma_{\kappa})^{3/2}, \tag{25}$$

and the thermodynamic characteristics of Fermi gases with specified concentration n_K depend on the induction of the field B explicitly

$$\zeta_{\kappa}(B) = \left(\frac{2}{3}\right)^{2/3} \frac{\zeta_{\kappa}(0) \cdot x_{\kappa}}{(x_{\kappa} - 1 + \sigma_{\kappa})^{1/3}} = \frac{\pi^4 \hbar^6 n_{\kappa}^2}{2m_{\kappa}^3 \mu^2 B^2} + (1 - \sigma_{\kappa})\mu B, \tag{26}$$

$$P_{\kappa}(B) = \frac{5}{3} \cdot \left(\frac{2}{3}\right)^{2/3} \cdot P_{\kappa}(0) \cdot (x_{\kappa} - 1 + \sigma_{\kappa})^{2/3} = \frac{\pi^4 \hbar^6 n_{\kappa}^3}{3m_{\kappa}^3 \mu^2 B^2}, \tag{27}$$

$$\varkappa_{\kappa}(B) = \varkappa_{\kappa 0}\left(\frac{2}{3}\right)^{1/3} (x_{\kappa} - 1 + \sigma_{\kappa})^{1/2} \cdot \left[x_{\kappa} + \frac{5}{2}(\sigma_{\kappa} - 1)\right] =$$

$$= \frac{P_{\kappa}(B)}{B^2} + \frac{(\sigma_{\kappa} - 1)\mu n_{\kappa}}{B}. \tag{28}$$

For neutrons in a superstrong magnetic field, $x_n \leq \sigma_n$, and

$$R_{3/2}(x_n) = 0{,}5(x_n + \sigma_n)^{3/2}, \tag{29}$$

$$R_{5/2}(x_n) = 0{,}5(x_n + \sigma_n)^{5/2}, \tag{30}$$

$$\zeta_n(B) = \zeta_n(0) \cdot 2^{2/3} \cdot x_n (x_n + \sigma_n)^{-1} = 2^{2/3}\zeta_n(0) - \mu_n B, \tag{31}$$

$$P_n(B) = 2^{2/3} P_n(0), \tag{32}$$

$$\varkappa_n(B) = \varkappa_{n0} \cdot 2^{-2/3}\sigma_n (x_n + \sigma_n) = \mu_n n_n B^{-1}. \tag{33}$$

Thus, the chemical potential of the given Fermi gases when $\zeta_K(0) = \text{const}$ ($n_K = \text{const}$) decreases with increase in induction of the superstrong magnetic field, and reaches zero when

$$B = 2^{2/3}\zeta_{\kappa}(0) \cdot 3^{-2/3} \cdot (\sigma_{\kappa} - 1)^{-1/3} \cdot \mu^{-1} \tag{34}$$

for charged fermions and when

$$B = 2^{2/3}\zeta_n(0) \cdot \mu_n^{-1} \tag{35}$$

for neutrons. In the corresponding real conditions, higher B is hardly achievable, and therefore is not of practical interest at present. Note simply that the pressure of the electron and proton gases and the magnetic susceptibility of all three gases tends to zero as $B \to \infty$. The pressure of the neutron gas is unchanged over the whole region of superstrong magnetic fields (does not depend on B).

The results of numerical calculations are shown in Figs. 1-3. It is readily evident that, in contrast to an electron gas, the chemical potential of a nucleon gas in a magnetic

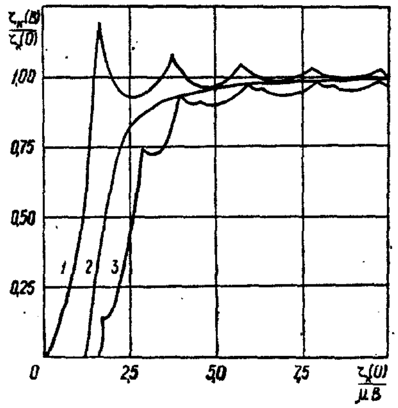

Fig. 1

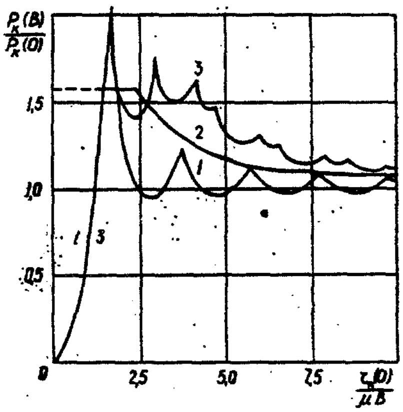

Fig. 2

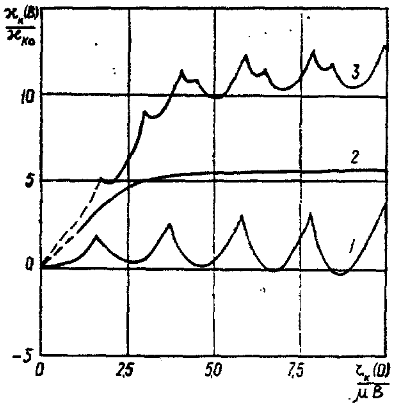

Fig. 3

Fig. 1. Dependence of the chemical potential of Fermi gases on the induction of the magnetic field: 1) electron gas; 2) neutron gas; 3) proton gas.

Fig. 2. Dependence of the Fermi-gas pressure on the magnetic induction.

Fig. 3. Dependence of the magnetic susceptibility of Fermi gases on the magnetic induction.

field is less than in the absence of a field, and the pressure (except for the region of superstrong magnetic fields) is higher. Contrary to the assertion of [5], the magnetic-susceptibility graph for a proton gas is markedly different from that for an electron gas. The presence of an anomalous magnetic moment exceeding the nuclear magneton at the proton leads to predominance of the paramagnetic susceptibility over the diamagnetic susceptibility in the whole range of variation of B. For an electron gas, as shown in [2] and as evident from Fig. 3, the inverse effect may also be found.

The characteristics of the neutron gas are not oscillatory, since the neutron motion in the magnetic field is not quantized. The oscillations of the proton-gas characteristics differ somewhat in form from those for the electron gas. Note, however, that each value of n also corresponds to two oscillational peaks in the electron gas. For example, for $n = 1$, $\zeta_e(B)/\zeta_e(0) = 1.21$ and $\zeta_e(B)/\zeta_e(0) = 1.19$ when $\zeta_e(0)/\mu B = 1.65$ and 1.69, respectively. In the scale chosen here, these peaks coalesce. Therefore, the graphs of the electron-gas characteristics in magnetic field in Figs. 1-3 are the same as in the case $\sigma_e = 1$ and coincide with the graphs of [2] for the chemical potential and magnetic susceptibility.

Note, in conclusion, that the decrease in chemical potential of the Fermi gas with increase in induction of the superstrong magnetic field may lead to lifting of its degeneracy [6-8]. This must be taken into account in using the formulas of the present work obtained for the case when $T = 0$ for specific physical problems.

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HAMILTONIZATION OF A SYSTEM OF PARTICLES WITH ANOMALOUS MOMENTS IN THE NULL-PLANE FORMALISM

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UDC 530.145

A Hamiltonization of a system of interacting Bose and Fermi fields is performed. It is assumed that the fermions possess anomalous magnetic and intrinsic dipole moments. The Hamiltonization is conducted using the null-plane formalism in the gauge $A^0 = A_3 = 0$, which is an analog of the Coulomb gauge in a Cartesian coordinate system.

There currently exist a fairly large number of papers on quantum field theory on a null-plane (see, for example, [1-3] and references therein). The key point of the theory is that quantum processes are considered in a special coordinate system introduced by Dirac [4], whose variables u^μ are related to the Cartesian coordinates x^μ by

$$\sqrt{2}u^0 = (x^0 - x^3);\ u^{1,2} = x^{1,2};\ \sqrt{2}u^3 = (x^0 + x^3). \tag{1}$$

The variable u^0 in this theory plays the role of time, and the evolution of states is considered from one plane u^0 = const to another. On these planes, called null-planes, are defined initial conditions and formulated commutation relations for fields. Such relations have been obtained in [5-7], where quantization was performed in a special gauge analogous to the Coulomb gauge used in quantization in Cartesian coordinate systems. Quantization in the Lorentz gauge has been considered in [8].

The purpose of this paper is to construct the formalism of quantum electrodynamics on a null-plane taking into account the anomalous magnetic and (hypothetical) electric moments of fermions. This problem was discussed in the case of a Cartesian coordinate system in [9, 10]. In these papers the Hamilton–Dirac formalism was constructed for the system of interacting photon and fermi fields, Dirac brackets of the field variables were found, and the quantization rule was postulated by requiring that the Dirac brackets be replaced by a commutator or an anticommutator upon quantization.

In this paper we shall perform an analogous procedure. We shall Hamiltonize the system in the gauge $A^0 = A_3 = 0$.

We start with the Lagrangian with the density

$$L = \frac{i}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi) - \bar{\psi}(e\gamma^\nu A_\nu - \mu\Gamma^{\nu\lambda}\partial_\nu A_\lambda + m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}; \tag{2}$$

where

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