THE EVOLUTION OF THE LUNAR ORBIT REVISITED, II

F. MIGNARD

Centre d'Etudes et de Recherches Géodynamiques et Astronomique, Grasse, France

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Abstract. We present here a model for the tidal evolution of an isolated two-body system. Equations are derived, including the dissipation in the planet as in the satellite, in a frequency dependent lag model. The set of differential equations obtained is still valid for large eccentricity, as well as for all inclinations. The reference plane chosen enables us to study the evolution for both the orbital plane and the equatorial plane.

The results obtained show the Moon, after having approached the Earth with small variations for the inclination and the eccentricity, exhibits strong increase for the two parameters in the vicinity of the closest approach. In every case the eccentricity tends towards the value 1, whereas the variations of the inclinations are dependent on the magnitude of the dissipation in the satellite.

Some qualitative results are also investigated for the final behaviour of satellites such as Triton and the Galilean satellites.

0. Introduction

This paper is the second of a set of three which deal with the evolution of a planet–satellite system, with a particular emphasis on the Earth–Moon system. In the first paper, henceforth refered to as paper one, we studied the evolution of the Moon's orbit under the influence of the tides, by using simplifying hypotheses. First we neglected the eccentricity of the Moon, which made the validity of our results questionable, because of the enlargement of the Moon's eccentricity in the future (MacDonald, 1964) as well as in the distant past. Second, no allowance was made for the tides raised on the satellite by the planet and we show in this paper their important effect on both the evolution of the inclination and the eccentricity. Third, in paper one we kept only the second harmonic in the expansion of the tidal potential though integration was carried to less than 5 Earth radius. In this investigation the method used allow the different harmonics to be taken into consideration, but even with a closest approach less than 3 Earth radius no important modification appears.

The gravitational interaction between Sun and Moon, as well as slowing down of the Earth's rotation caused by the solar tides, are not covered in this paper because they lead to a quite different formulation, in particular during the averaging process. Therefore, these problems will be the subject of the third paper.

We present here a new calculation of the orbital evolution of an isolate two-body system with a frequency dependant lag. The reasons for such a choice are explained in paper one.

A recent paper (Lambeck, 1979) has clearly shown the difficulty which occurs in the case of large eccentricity, when the expansion of the potential is performed as in the paper by Kaula (1964). Consequently, in this paper we perform the calculation in such a way

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that we avoid this problem, and keep the equations in a finite form in eccentricity. A similar treatment was achieved by Kopal (1978) in the study of close binary stars. This method makes the equations valid for large eccentricity and facilitates the qualitative discussion of the solutions of the equations.

In the case of a system of two bodies, in which the angular momentum borne by the orbital motion and that contained in the rotation of the primary are of the same magnitude, a complex evolution takes place. As the satellite is receding, or approaching, its orbital plane is moving and the spin axis of the primary is changing its orientation because of the constancy of the total angular momentum. In this condition the equatorial plane does not remain an inertial plane and is by no means well suited as a reference plane. So, throughout this paper we will use the Laplacian plane of the system which in invariable as long as the total anular momentum does not vary.

The present paper is divided into four sections. In the first section the forces and the torques are computed, whereas the equations of evolution are derived in the second section. A discussion of the outstanding features of their solution follows during the third section, and numerical results are given for the Moon and Triton in the last section.

1. Forces

1.1. Potential

In view to get equations of evolution in the Gaussian form (Brouwer and Clemence, 1961) we are only concerned with the expression of the different forces which act upon the Moon and the Earth. So we investigate the forces which arise between two bodies, when the first one is distorted by its tidal interaction with the other one.

For the sake of simplicity we shall call the first body the Earth and the second the Moon, because there will be no difficulty in changing the respective situation of these two bodies in order to take account for the deformation of the Earth due to the Sun or the Moon's deformation due to the Earth.

As explained in Section 2 of paper one the instantaneous distortion of the Earth gives rise to an additional potential at point **r** written with the help of the Legendre polynomials as follows:

$$U(\mathbf{r}) = \sum_l k_l \frac{Gm^* R_e^{\,2l+1}}{r^{l+1} r^{*l+1}} P_l\left(\frac{\mathbf{r}\mathbf{r}^*}{rr^*}\right). \tag{1}$$

In this equation m^*, $\mathbf{r}^*$ are the mass and the vector position of the Moon, R_e the Earth's radius, k_l the Love number of order l and $P_l(x)$ is the Legendre polynomial. For an inelastic response of the Earth we introduce the time delay Δt between the stress of the Earth due to the Moon and the moment when the Earth gets its equilibrium figure.

In order to relate $U(\mathbf{r}, \mathbf{r}^*)$ given by Equation (1) to the expression $V(\mathbf{r}, \mathbf{r}^*)$ of the additional potential in the case of a dissipative medium, the following process is to be done. First assume the Earth free of rotation; during the delay Δt the Moon orbits around the Earth and the magnitude of the distortion at time t is due to the act of the Moon at time

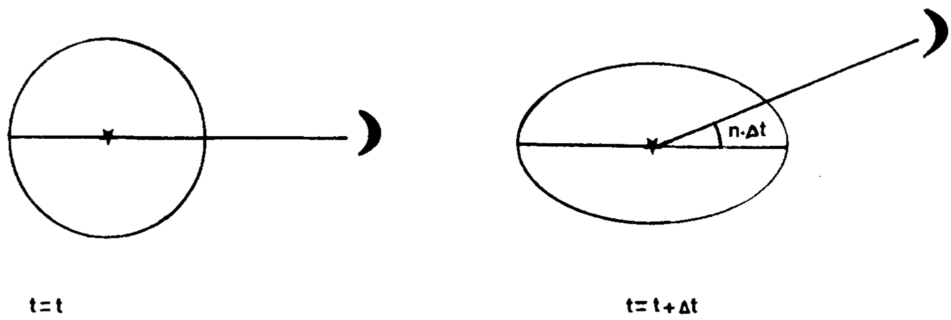

Fig. 1. The tidal bulge due to the time delay Δt and the orbital motion of the Moon.

$t - \Delta t$ (Figure 1). Second, the Earth rotates with the angular velocity ω and the Moon is stopped. The tidal bulge produced by the Moon is carried out on the Earth–Moonline by the Earth rotation (Figure 2). Thus the additional potential is given by expression (1) by twisting the vector $\mathbf{r}^*$ of an angle $\omega\Delta t \times \mathbf{r}^*$ because of the new symmetry of the tidal bulge.

This discussion allows us to achieve the change of variable to compute $V(\mathbf{r}, \mathbf{r}^*)$ from $U(\mathbf{r}, \mathbf{r}^*)$:

$$V(\mathbf{r}, \mathbf{r}^*) = U(\mathbf{r}_1, \mathbf{r}^*),$$

with

$$\left.\begin{aligned} \mathbf{r}_1 &= \mathbf{r}, \\ \mathbf{r}_1{}^* &= \mathbf{r}^*(t - \Delta t) + \omega\Delta t \times \mathbf{r}^*. \end{aligned}\right\} \tag{2}$$

Current value of Δt for the Earth should be about 10 min if the whole effect on the evolution in the Moon's orbit was caused by dissipation within the Earth, which is not the case, since the most important part occurred inside the Oceans (Cazenave and Daillet, 1977). For the Moon, Mars and Neptune Δt remains largly unknown but can be estimated after determination of Q for these bodies (Goldreich and Soter, 1966). However, in subsequent computations we shall assume the value of Δt to be small with respect to the orbital period and we shall expand the additional potential to the first order in Δt. Furthermore, in this paper we are only concerned with the secular effects in the evolution of the Earth–Moon system, so only the terms in Δt are retained in the additional potential. That part of the potential due to the time delay is easily computed by writing Equation (1) as follows:

$$\begin{aligned} U(\mathbf{r}_1, \mathbf{r}_1^*) &= \sum_l U_l(\mathbf{r}_1, \mathbf{r}_1^*), \\ U_l(\mathbf{r}_1, \mathbf{r}_1^*) &= k_l G m^* R_e{}^{2l+1} \varphi_l(r_1, r_1^*) P_l(x); \end{aligned} \tag{3}$$

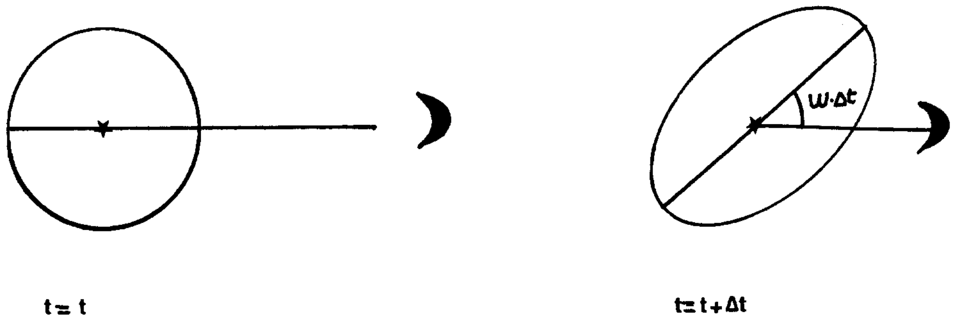

Fig. 2. The tidal bulge due to the time delay Δt and the rotation of the Earth.

with

$$r = |\mathbf{r}|, \quad x = \frac{\mathbf{r}_1 \mathbf{r}_1^*}{r_1 r_1^*}, \quad \varphi_l(r_1, r_1^*) = \frac{1}{r_1^{l+1} r_1^{*l+1}}.$$

Then

$$V(\mathbf{r}, \mathbf{r}^*) = U^0(\mathbf{r}, \mathbf{r}^*) + U^1(\mathbf{r}, \mathbf{r}^*),$$

where U^0 is the part of V which is not dependent on Δt and U^1 is of the first order in Δt.

$$U^1(\mathbf{r}, \mathbf{r}^*) = \nabla_{\mathbf{r}^*} [U(\mathbf{r}, \mathbf{r}^*)] \; [\omega \times \mathbf{r}^* - \mathbf{V}^*] \, \Delta t \tag{4}$$

In these equations $\mathbf{V}^*$ means the velocity of the Moon. The secular terms only proceed from $U_1(\mathbf{r}, \mathbf{r}^*)$ and we shall call U_1 the potential. By using Equation (3) we get an expression for $\nabla_{\mathbf{r}^*} U$

$$\nabla_{\mathbf{r}^*} U_l = k_l G m^* R_e^{2l+1} \left[P_l(x) \frac{\partial \varphi_l}{\partial r^*} \frac{\mathbf{r}^*}{r^*} + \right.$$
$$\left. + \varphi_l(r, r^*) \frac{\mathrm{d}P_l}{\mathrm{d}x} \left(\frac{\mathbf{r}}{r r^*} - \frac{(\mathbf{r}\mathbf{r}^*)\mathbf{r}^*}{r r^{*3}} \right) \right]. \tag{5}$$

1.2. Forces and Torques

The forces $\mathbf{F}$ acting upon the Moon is obtained by taking the gradient of U_1 with respect to $\mathbf{r}$. We study the case of a tidal rising object which is also the body of which the motion is investigated: namely, after differentiation $\mathbf{r}^* = \mathbf{r}$ and a dramatic simplification occurs in the formula. All algebraical calculations have been performed, yielding

$$\mathbf{F} = \sum_l \mathbf{F}_l,$$

$$\mathbf{F}_l = -k_l G m^* \Delta t R_e^{2l+1} \left[\mathbf{r} \frac{(\mathbf{r}\mathbf{v})}{\mathbf{r}^2} \left(\frac{\partial^2 \varphi_l}{\partial r \partial r^*} - \varphi_l \frac{P_l'(1)}{r^2} \right) + \right.$$

$$+\varphi_l P_l'(1)\left(\frac{\mathbf{r}\times\omega}{r^2}+\frac{\mathbf{v}}{r^2}\right)\Bigg].$$

By differentiating φ_l by using $P_l'(1) = l(l+1)/2$, and taking account of the fact that the Moon's mass is m, we obtain for the force and the torque

$$\mathbf{F}_l = -Gm^2 k_l \Delta t R_e^{2l+1} \frac{1}{r^{2(l+3)}}\left[\frac{(l+1)(l+2)}{2}(\mathbf{rv})\mathbf{r} + \right.$$

$$\left. + \frac{l(l+1)}{2} r^2 (\mathbf{v}+\mathbf{r}\times\omega)\right], \qquad (6)$$

$$\mathbf{T}_l = -Gm^2 k_l \Delta t R_e^{2l+1} \frac{1}{r^{2(l+2)}}\left[\frac{l(l+1)}{2}(\mathbf{r}\omega)\mathbf{r} - r^2\omega + \mathbf{r}\times\mathbf{v}\right]. \qquad (7)$$

For $l = 2$, Equations (6) and (7) change into Equations (5) and (6) of paper 1. In order to consider the act of the Sun on the Earth's rotation we shall use Equation (7) with the changes $M_\odot^2$ for m^2 and $(\mathbf{r}_\odot, \mathbf{v}_\odot)$ for $(\mathbf{r}, \mathbf{v})$. The effect of tide on the Moon may lead to important variations of the eccentricity of the Moon's orbit (MacDonald, 1964) which will play a dominant role in the evolution of the lunar orbit and will not be neglected. In this case, Equation (6) is to be used with the changes of M for m (Earth's mass) k_l' for k_l (Love number for the Moon), and R_M for R_e (the Moon's radius).

1.3. Components of the Forces

We intend to use perturbation equations in the Gaussian form in which the three mutually perpendicular components of the disturbing acceleration are used: R in the direction of the radius vector, S in the osculate plane directed as the increasing longitude, and W normal to the osculating plane. R, S, W form a right-handed frame.

Let us put $\mathbf{H}_M = \mathbf{r}\times\mathbf{V}$ and $H_M = na^2\sqrt{(1-e^2)}$ where n is the Moon's mean motion, and e and a are the eccentricity and the semi-major axis of the orbit respectively.

Then, the three components are

$$R = \frac{\mathbf{Fr}}{\mu r}, \qquad S = \frac{\mathbf{F}(\mathbf{H}_M\times\mathbf{r})}{\mu H_M r}, \qquad W = \frac{\mathbf{F}\mathbf{H}_M}{\mu H_M}. \qquad (8)$$

In these equations, $\mu = mM/(m+M)$ is the so-called reduced mass of the system. By using the Earth's equator as a reference plane for the orbit and the inclination I, the following expressions are obtained:

$$\mathbf{V}(\mathbf{H}\times\mathbf{r}) = n^2a^4(1-e^2),$$

$$(\mathbf{H}_M\times\mathbf{r})(\mathbf{r}\times\omega) = r^2na^2(1-e^2)^{1/2}\omega\cos I$$

$$\mathbf{H}_M(\mathbf{r}\times\omega) = rna^2(1-e^2)^{1/2}\omega\cos(\tilde{\omega}+v)\sin I. \qquad (9)$$

In these equations v means the true anomaly of the satellite whereas $\tilde{\omega}$ designates the longitude of the perihelion measured from the ascending node.

Gathering together Equations (7), (8) and (9), a straightforward computation leads to the three components expressed in terms of elliptical elements as

$$R_l = -G\frac{m^2}{\mu} k_l R_e^{2l+1} \Delta t (l+1)^2 \frac{n}{a^{2l+3}} \frac{e}{(1-e^2)^{1/2}} \left(\frac{a}{r}\right)^{2(l+2)} \sin v \tag{10}$$

$$S_l = -G\frac{m^2}{\mu} k_l R_e^{2l+1} \Delta t \frac{l(l+1)}{2} \left[\frac{n}{a^{2l+3}} (1-e)^{1/2} \left(\frac{a}{r}\right)^{2l+5} - \frac{\omega \cos I}{a^{2l+3}} \left(\frac{a}{r}\right)^{2l+3}\right], \tag{11}$$

$$W_l = -G\frac{m^2}{\mu} k_l R_e^{2l+1} \Delta t \frac{l(l+1)}{2} \frac{\omega}{a^{2l+3}} \left(\frac{a}{r}\right)^{2l+3} \cos(\tilde{\omega} + v) \sin I. \tag{12}$$

1.4. Comments

The torque, given by Equation (7), rules the evolution of the planet's rotation. In particular, the component along the spin axis makes the day longer or shorter, according to its sign. That component is easily calculated. Omitting a constant part and considering only $l = 2$, we obtain

$$\mathbf{T}\omega \propto \frac{1}{r^8}\left[(\mathbf{r}\omega)^2 - r^2\omega^2 + (\mathbf{r} \times \mathbf{v})\,\omega\right].$$

By averaging this equation on one orbital and apsidal period and equalling the result to zero, one obtains an equation providing the locus of points where $d\omega/dt = 0$.

This equation is of the form

$$\frac{\omega}{n} = \frac{1}{(1-e^2)^{3/2}} \frac{A_8}{A_6} \frac{2\cos I}{1+\cos^2 I},$$

where A_n is a polymonial of the eccentricity (computed later in the paper), and I represents the inclination of the orbit on the planet's equator.

The curve is plotted in the Figure 3. In this diagram, the upper curve represents the solution of the equation $da/dt = 0$, computed from Equation (S1) which will be achieved in the next section.

The point A, with $e = 0$ and $\cos I = 1$, is the syncronous point and the classical laws

$$\frac{d\omega}{dt} \lesseqgtr 0 \quad \text{and} \quad \frac{da}{dt} \gtreqless 0 \quad \text{for} \quad \frac{\omega}{n} \gtreqless 1$$

are seen. With different conditions for e and I, generalized laws are to be used. In particular, it is possible to have a direct approaching satellite and a slowing down planet, depending on the value of the eccentricity and inclination.

For an equatorial orbit, but highly eccentric, da/dt will be negative, even with $\omega/n > 1$. Such a case was that of Phobos in the past, for $I \simeq 0$, $e \simeq 0.6$ and $a/R_e \simeq 14$ (Mignard, 1980).

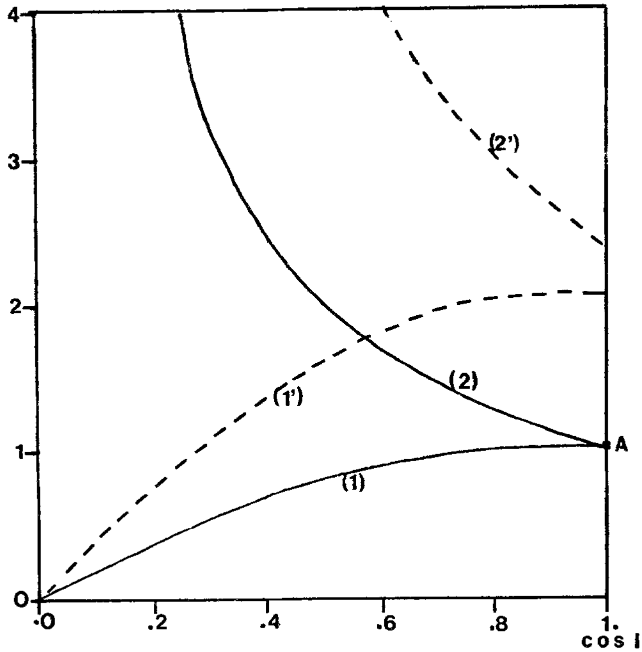

Fig. 3. ω/n is plotted vs. cos (I). 1, 1′ show the points where $d\omega/dt = 0$ with $e = 0$ and $e = 0.4$. 2, 2′ show the points where $da/dt = 0$ with $e = 0$ and $e = 0.4$. dw/dt is positive under the curve 1 and 1′. da/dt is positive above the curve 2 and 2′.

Equation (12) shows that the tidal bulge being dragged by the planet's rotation out of the orbital plane is a normal component of the force generated, and is always negative. This means the orbital plane and equator of the planet undergo an evolution which tends to unite them into one. Close satellites of giant planets provide evidence for such a phenomena.

2. Equations of Evolution

2.1. Problem

As explained in the introduction, two steps are required to study the evolution of a satellite orbit. In the solar system, some satellites do not strongly feel the solar disturbance and show quite a regular orbit. Other satellites have a motion very influenced by the gravitational act of the Sun. Phobos, Triton, and Galilean satellites are examples of the first group, whereas the Moon belongs to the second group.

More precisely, the knowledge of the distance, where both oblateness and solar disturbances are equal, provides a quantitative limit; in the Moon's case, this distance is equal to about ten Earth's radius.

Both situations are likely to have successively occurred in the Moon's history depending on the Earth–Moon distance which underwent significant variations in the past.

So, the first step of our study consists in the calculation of the evolution of an isolate two-body system which keeps a constant angular momentum. For the second step, we shall not be allowed to neglect the Sun's existence.

Unfortunately, the system of differential equations is quite different according to the first or the second case taking place.

2.2. Equations of Evolution

By means of Equations (10), (11) and (12), incorporated in the Gaussian equations of perturbation, we are able to write the differential system which rules the evolution of the two-body system. In this paper we are only concerned with the secular solutions of the equations and we have to average the second part of the differential system: all time-dependent terms are of the pattern of $(a/r)^j$, $(a/r)^j \cos v$, $(a/r)^j \cos 2v$, $j > 2$ where v is the true anomaly. We shall keep the constant terms in their Fourier expansions. It is well known in the theory of the Hansen coefficients that the constant part can be obtained in a finite form in eccentricity. Let us put the following notations:

$$\langle (a/r)^j \rangle = H(j, 0), \quad \langle (a/r)^j \cos v \rangle = H(j, 1),$$
$$\langle (a/r)^j \rangle \cos 2v = H(j, 2). \tag{13}$$

The properties of the Hansen coefficients (Tisserand, 1888) lead to the two recurrence relations

$$H(n, 0) = \frac{1}{1-e^2}\left[H(n-1, 0) + eH(n-1, 1)\right], \tag{14}$$

$$H(n, 1) = \frac{n-2}{n-1}\frac{1}{1-e^2}\left[eH(n-1, 0) + H(n-1, 1)\right]; \tag{15}$$

while $H(n, 2)$ is obtained with the aid of the following equation:

$$H(n, 2) = -\frac{2}{e(n-1)}\left[(1-e^2)H(n+1, 1) + H(n, 0)\right].$$

The solution of Equations (14) and (15) can be expressed as

$$H(n, 0) = \frac{A_n}{(1-e^2)^{n-3/2}}, \qquad H(n, 1) = \frac{B_n}{(1-e^2)^{n-3/2}}, \tag{16}$$

where A_n and B_n are polymonials in eccentricity which satisfy the next relations:

$$A_n = A_{n-1} + eB_{n-1}, \tag{17}$$

$$B_n = \frac{n-2}{n-1}\left[eA_{n-1} + B_{n-1}\right], \tag{18}$$

with $A_2 = 1$ and $B_2 = 0$.

This form is useful in the numerical integration of the differential system. The polynomials of significance in our problem are given in Table I. The analytical form of the

TABLE I

Some useful A_n and B_n

n	A_n	B_n
6	$1 + 3e^2 + \frac{3}{8}e^4$	$2e + \frac{3}{2}e^3$
7	$1 + 5e^2 + \frac{15}{8}e^4$	$\frac{5}{2}e + \frac{15}{4}e^3 + \frac{5}{16}e^5$
8	$1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6$	$3e + \frac{30}{4}e^3 + \frac{15}{8}e^5$
9	$1 + \frac{21}{2}e^2 + \frac{105}{8}e^4 + \frac{35}{16}e_6$	$\frac{7}{2}e + \frac{105}{8}e^3 + \frac{105}{16}e^5 + \frac{35}{128}e^7$
10	$1 + 14e^2 + \frac{210}{8}e^4 + \frac{35}{4}e^6 + \frac{35}{128}e^8$	$4e + 21e^3 + \frac{35}{2}e^5 + \frac{35}{16}e^7$

equations can be now derived by expressing the different terms of the Gauss equation. For example

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{n(1-e^2)^{1/2}}\left[Re\sin v + (1-e^2)\frac{a}{r}S\right].$$

By using Equations (10) and (13), we get

$$\langle R_l e \sin v\rangle = -G\frac{m^2}{\mu}k_l R_e^{2l+1}\,\Delta t\,\frac{(l+1)^2}{2l+3}(1-e^2)^{1/2}e\,\frac{n}{a^{2l+3}}H(2l+5,1),$$

$$\langle (a/r)S_l\rangle = -G\frac{m^2}{\mu}k_l R_e^{2l+1}\,\Delta t\,\frac{l(l+1)}{2}\,\frac{n}{a^{2l+3}}\times$$

$$\times\left[(1-e^2)^{1/2}H(2l+6,0) - \frac{\omega}{n}\cos I H(2l+4,0)\right].$$

With $l = 2$, it becomes

$$\langle R_2 e \sin v\rangle = -\tfrac{9}{2}G\frac{m^2}{\mu}k_2 R_e^5\,\Delta t\,\frac{ne^2}{a^7(1-e^2)^7}\times$$

$$\times\left[1 + \tfrac{15}{4}e^2 + \tfrac{15}{8}e^4 + \tfrac{5}{64}e^6\right],$$

$$\langle (a/r)S_2\rangle = -3G\frac{m^2}{\mu}k_2 R_e^5\,\Delta t\,\frac{n}{a^7}\left[\frac{1}{(1-e^2)^8}(1 + 14e^2 +\right.$$

$$\left. + \tfrac{105}{4}e^4 + \tfrac{35}{4}e^6 + \tfrac{35}{128}e^8) + \frac{\omega\cos I}{n(1-e^2)^{13/2}}(1 + \tfrac{15}{2}e^2 + \tfrac{45}{8}e^4 + \tfrac{5}{16}e^6)\right].$$

If we restrict now to the second harmonic in the potential, namely $l = 2$, the evolution equations are written, by putting $X = a/R_l$, as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 6\times 4\pi^2 k_2\frac{m}{M}\frac{m}{\mu}\frac{\Delta t}{P^2}\frac{1}{X^7}\left[-\frac{1}{(1-e^2)^{15/2}}(1 + \tfrac{31}{2}e^2 + \tfrac{255}{8}e^4 +\right.$$

$$\left. + \tfrac{185}{16}e^6 + \tfrac{25}{64}e^8) + \frac{\omega}{n}\frac{\cos I}{(1-e^2)^6}(1 + \tfrac{15}{2}e^2 + \tfrac{45}{8}e^4 + \tfrac{5}{16}e^6)\right],$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = 3 \times 4\pi^2 k_2 \frac{m}{M}\frac{m}{\mu}\frac{\Delta t}{P^2}\frac{1}{X^8}\left[-\frac{1}{(1-e^2)^{13/2}}(9e + \tfrac{135}{4}e^3 + \right.$$

$$\left. + \tfrac{135}{8}e^5 + \tfrac{45}{64}e^7) + \frac{\omega}{n}\frac{\cos I}{(1-e^2)^5}(\tfrac{11}{2}e + \tfrac{33}{4}e^3 + \tfrac{11}{16}e^5)\right],$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{3}{2} \times 4\pi^2 k_2 \frac{m}{M}\frac{m}{\mu}\frac{\Delta t}{P^2}\frac{1}{\alpha}\frac{\mu}{M}\frac{1}{X^{13/2}}\frac{T\sin i}{(1-e^2)^5}(1 + 3e + \tfrac{3}{8}e^4). \quad \text{(S1)}$$

The constancy of the total angular momentum allow us to express ($\omega \cos I/n$) in terms of X, e, i and

$$\mathbf{H} = \mathbf{H}_M + \mathbf{H}_E ,$$

where $\mathbf{H}$, $\mathbf{H}_M$ and $\mathbf{H}_E$ are the total angular momentum and the Moon's and the Earth's angular momentum, respectively. Then, by using the coordinate system shown in Figure 4, we obtain the following two relationships

$$H_E \cos J + H_M \cos i = H, \qquad H\cos i - H_E \cos I = H_M,$$

$$H_E \sin J - H_M \sin i = 0, \qquad H \sin i - H_E \sin I = 0.$$

If particular units are used, such as the unit of angular momentum is that of a satellite orbiting the Earth at 1 R_e (grazing satellite), then the period of this is

$$P = 2\pi \left(\frac{R_e^3}{GM}\right)^{1/2} .$$

In this case we put

$$T = H \frac{1}{(GMm^2R_e)^{1/2}} .$$

Finally the expected relation is obtained:

$$\frac{\omega}{n}\cos I = \frac{\mu}{M}\frac{1}{\alpha}\left[TX^{3/2}\cos i - X^2(1-e^2)^{1/2}\right].$$

In the set (S1), $\alpha = C/MR_e^2$, where C is the moment of inertia of the Earth with respect to its spin axis.

6. Discussion of the Solution

Some remarks may be made concerning these equations and the outstanding features of their solutions.

For null eccentricity and inclination, the equation for X reduces to the classical one for an equatorial secular orbit with a frequency dependent lag and we do not need to make assumption for the sign according to the value of $\omega - n$.

In the case of zero eccentricity dX/dt and di/dt are the only two relevant equations and are reduced to equations (17) in paper 1.

In this system, we have only included the distortion of the planet caused by the satellite. As the satellite raises a tide on the planet, so does the planet raise a tide on the satellite and this new dissipation contributes to the orbital evolution.

If we consider a synchronous satellite, as seems to be the most frequent case, in the solar system then, because of the shortness of the time scale of evolution into this state (Peale, 1976), the factor $\omega/n \cos I$ turns into $\cos I$ and generally I is a small angle. Then, the equation for X becomes:

$$\frac{dX}{dt} = 6 \times 4\pi^2 k_2' \frac{M}{m}\frac{M}{\mu}\frac{\Delta t'}{P'^2}\frac{R'^8}{R_e^8}\frac{1}{X^7}\left\{-\frac{1}{(1-e^2)^{15/2}}(1 + \tfrac{31}{2}e^2 + + \tfrac{255}{8}e^4 + \tfrac{185}{16}e^6 + \tfrac{25}{64}e^8) + \frac{1}{(1-e^2)^6}(1 + \tfrac{15}{2}e^2 + \tfrac{45}{8}e^4 + \tfrac{5}{16}e^6)\right\} .$$

The prime parameters refer to the satellite properties and $(R'/R_e)^8$ comes from the fact that X is still measured in unit of planet radius. In the Moon's case, the ratio of the effects of the tide in the Moon and in the Earth is

$$A = \frac{k_2'}{k_2}\frac{\Delta t'}{\Delta t}\frac{P^2}{P'^2}\left(\frac{M}{m}\right)^3\left(\frac{R'}{R_e}\right)^8 = \frac{k_2'}{k_2}\frac{\Delta t'}{\Delta t}\left(\frac{M}{m}\right)^2\left(\frac{R'}{R_e}\right)^5 .$$

The last ratio is unknown today, especially because of the lack of information of $\Delta t'$ in the Moon. We shall use different value for A from 0.0 to 20.0 corresponding to the different determinations of Q.

The equations for the evolution of i and e show interesting characteristics. For a small inner satellite, $m/M \ll 1$, the Moon in the solar system excepted, the angular momentum of the system is principally borne by the spin axis of the planet which will be very little changed in its orientation by tidal evolution.

Therefore, the equatorial plane can be taken as an inertial plane with the spin axis along the total angular momentum. In this case $i = I$ and later on the orbital plane and the equator will tend to merge in one plane because $dI/dt < 0$. Different results were given by Kaula and Burns (Kaula, 1964; Burns, 1976) by considering constant phase lag.

Another interesting feature is related to the evolution of the eccentricity.

Let us write the equation de/dt for a small eccentricity as

$$\frac{de}{dt} \propto \frac{1}{X^8}\left[-9(1 + A) + \left(\frac{\omega}{n}\cos I + A\right)\tfrac{11}{2}\right]e . \tag{19}$$

If we neglect the tide in the satellite body, $A = 0$, then tide generally causes increasing eccentricities, as pointed out by Jeffreys (Jeffreys, 1961), which disagree with observational evidence. A qualitative explanation of this discordance was given by Goldreich (1963) by introducing the tidal distortion of the satellite.

If the value of A is sufficiently large, the tidal lock of the satellite causes Equation (19) to become

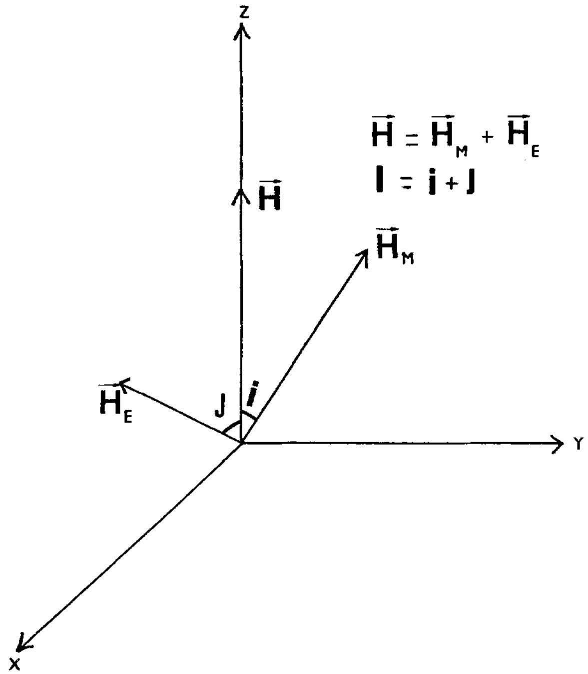

Fig. 4. Coordinate system.

$$\frac{\mathrm{d}e}{\mathrm{d}t} \propto -\frac{7}{2}\frac{A}{X^8}e\,,$$

and energy loss in the satellite tends to decrease the eccentricity.

In the solar system, the Galilean satellite of Jupiter, Mars's satellites, the inner Saturn's satellites, and Triton follow this kind of evolution and probably the newly discovered Pluto's satellite Charon.

It must be noted that the effect of the satellite distortion mainly appears in the evolution of the eccentricity. In fact, the synchronous lock of the rotation causes this effect to be negligible in the equation $\mathrm{d}X/\mathrm{d}t$ as far as e is a small quantity.

4. Numerical Integration

The qualitative features described in the previous section do not answer to the most important questions: How much did the eccentricity vary? Did the inclination drastically change? A numerical integration of the equations (S1) provides us with some information related to these problems. For this we used the routine AMC 1 prepared by Borderies and Castel (1975).

The dependance on X^{-8} in the differential equations leads to a very large variation in the value of the derivative, making the integration rather difficult if the time is taken as the independent parameter.

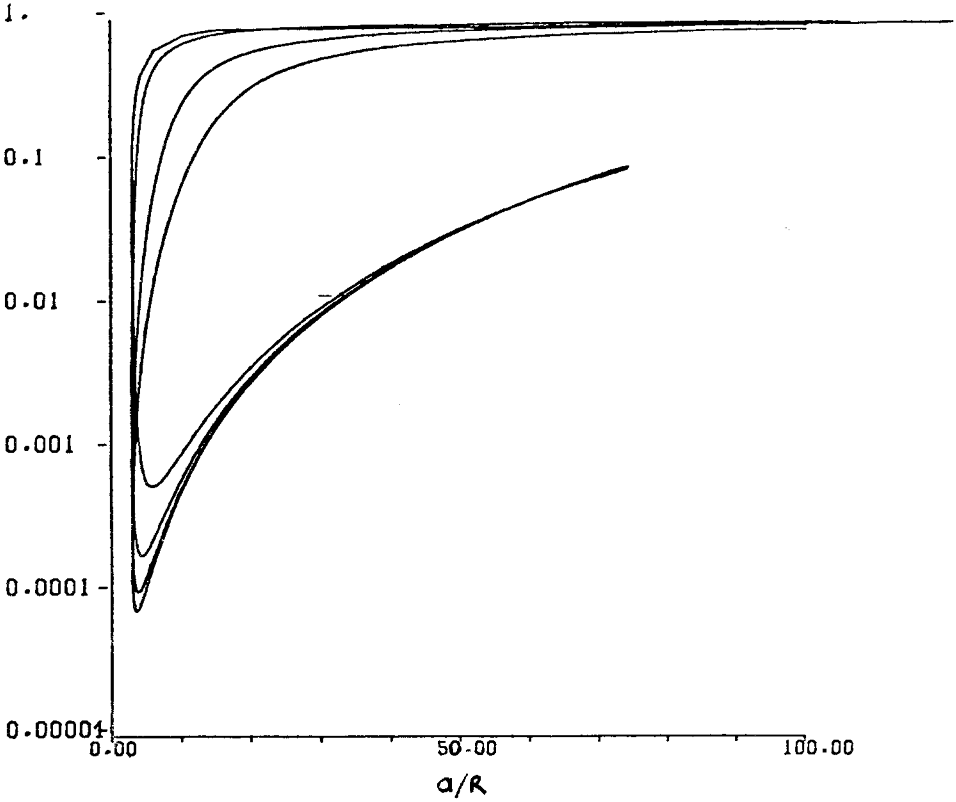

Fig. 5. Variation of the eccentricity of the Moon vs. the semi-major axis. The unity is one Earth-radius. $A = 0, 0.3, 1, 3$.

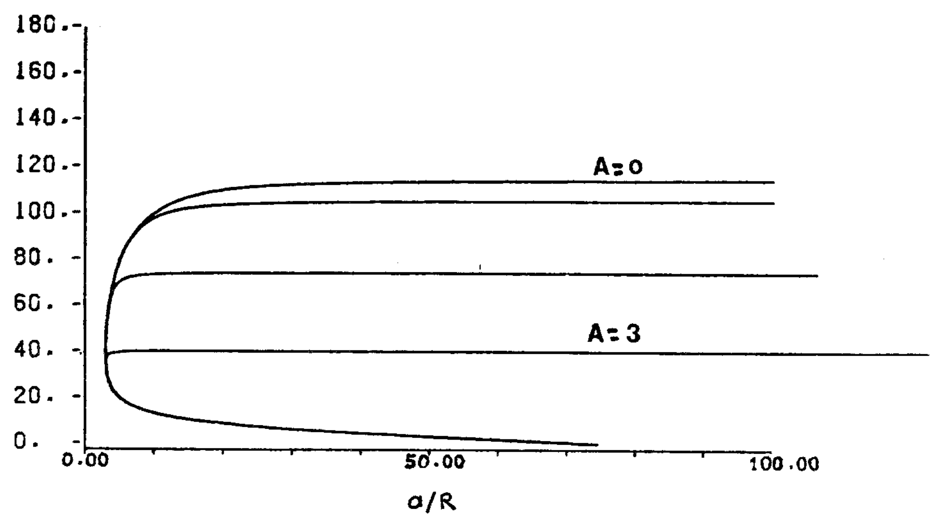

Fig. 6. Variation of the inclination of the Moon on the absolute plane vs. the semi-major axis. The unity is one Earth-radius. $A = 0, 0.3, 1, 3$.

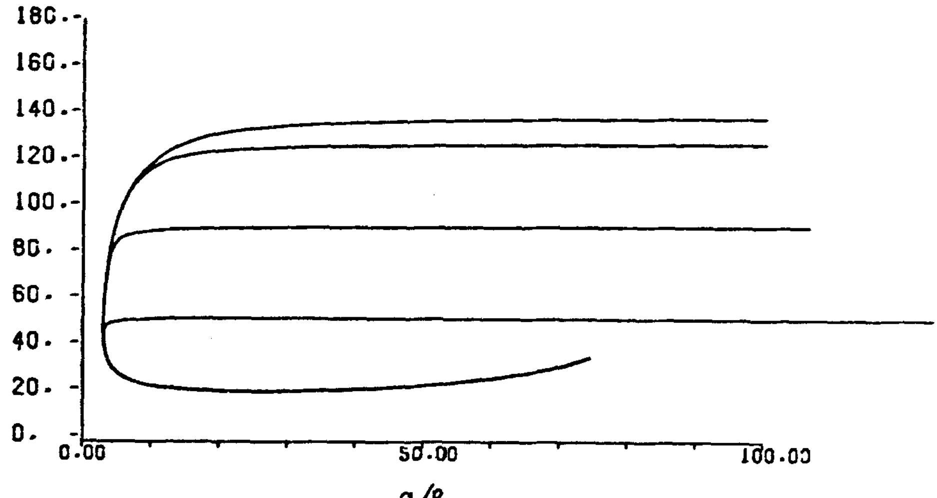

Fig. 7. Variation of the inclination of the Moon on the equator of the Earth vs. the semi-major axis. The unity is one Earth-radius. $A = 0.\ 0.3, 1, 3$.

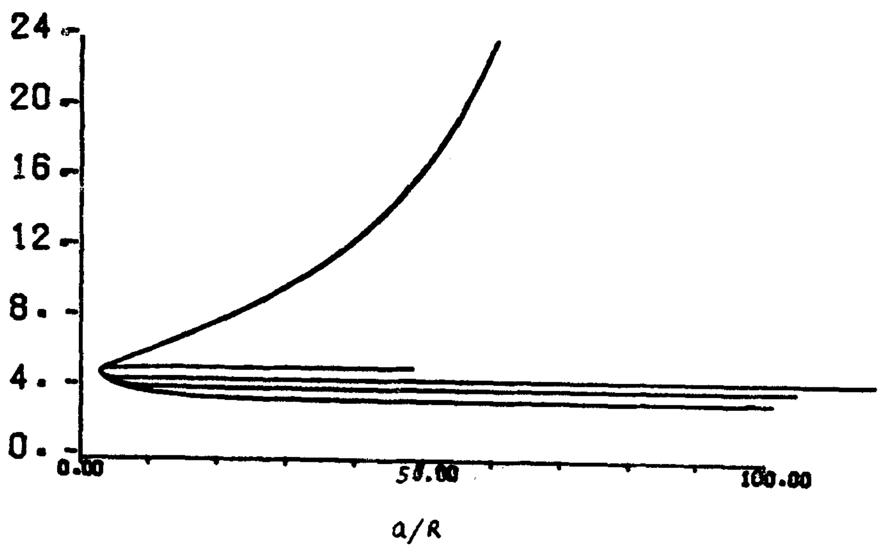

Fig. 8. Duration of the day vs. the semi-major axis. The unity is one Earth-radius. $A = 0, 0.5, 1, 5$.

Then we carried out the integration of the equations dX/di and de/di, the time scale being computed step by step with the help of di/dt. Runs have been made for various hypotheses concerning the dissipation in the satellite, with $A = 0, 0.3, 1, 3, 10, 20$. Displays of the results are given in Figures 5, 6, 7 and 8.

Of course, the most striking fact is the possibility for Moon to reach (in the past) a very large eccentricity whatever the Moon's dissipation may be. A careful examination of the set (S1) provides an asymptotic solution for large e. For this purpose, we use the new variable $\chi = 1 - e$ and we estimate the value of the various polymonials with $e = 1$. Fortunately, the value of the two polynomials in the equation $\mathrm{d}X/\mathrm{d}t$ are respectively the same as in the equation for $\mathrm{d}e/\mathrm{d}t$ (indeed, that property depends more on the structure of the Gaussian equations than on the structure of the force where only the decrease of the tangential component with the distance is needed). The equations become

$$\frac{\mathrm{d}X}{\mathrm{d}\chi} = -\frac{X}{\chi}$$

whose solution is $X(1 - e) = \text{constant}$; implying that, in case of large eccentricity, the perigee distance is constant. Such a result seems amazing, But the tidal effect on the orbital evolutions is highly dependent on the distance between the two bodies. So far as the eccentric orbit is considered, the whole effect only occurs during the motion through the perigee and acts upon the satellite as an impulse. After one period, the satellite returns at the same perigee. This analytical solution for large e allows us to stop the numerical integration for $e = 0.8$.

Let us now insert the above results in the equation governing the evolution of the inclination and perform the integration of $\mathrm{d}X/\mathrm{d}i$. There is no difficulty in finding the following solution, valid for large e.

$$\frac{\mathrm{d}X}{\mathrm{d}i} = \frac{1}{\lambda}\frac{X^2}{\sin i},$$

$$\tan(i/2) = \tan(i_0/2)\exp\left[\left(\frac{1}{X_0} - \frac{1}{X}\right)\lambda\right],$$

$$\tan(i_\infty/2) = \tan(i/2)\exp\left(\frac{\lambda}{X_0}\right)$$

with (i_0, X_0) initial condition for $e = 0.8$ and λ is constant. So far $X \to \infty$, the inclination on the absolute plane tends to a limit value.

Several runs were made with different assumptions concerning the Love numbers k_2, k_3, and k_4. As we have noted in the introduction, the trajectories in the phase space are fairly unchanged if allowance is made for k_3 and k_4. Only the time scale in the vicinity of the closest approach is slightly shorter but the value of the closest approach remains about $3R_e$ and the eccentricity remains very small.

As in paper 1, a variation in the initial condition, in particular for the inclination on the absolute plane, leads to a closest approach outside the Roche limit. But this question seems less important since the discovery of a new satellite around Jupiter, which orbits only 57 000 km above the surface. Its existence asks for computation of the time scale to break up a satellite within the Roche limit.

The lack of knowledge of the current dissipation in the solid Earth prevented us from giving a precise time scale; we have represented in Table II the variation of the semi-major axis with respect to time. We have chosen an arbitrary unit of time such that it

TABLE 2

Variation of the semi-major axis with the time. Only a relative scale is considered

$\frac{a}{R_E}$	t	$\frac{a}{R_E}$	t
60	0	10	1.000
55	0.51	20	1.000
50	0.77	30	1.001
45	0.92	40	1.003
40	0.96	50	1.011
20	0.999	60	1.029
10	1.000	70	1.065
3	1.000	100	1.4

took 1.0 unit to come from the closest approach to the present state. We see that 95% of the time scale is needed to go through $40R_e$ to $60R_e$ and in the distant past the time scale was very short in comparison with the time required today to move the Moon by $1\,R_e$. The numerical values in Table II are computed with $A = 0$. For another value no important change takes place (we always keep the same time-unit) except in the distant past where the scale strongly decreases as the dissipation in the Moon increases.

The integration in the case of Triton does not lead to surprising results. In fact the eccentricity of Triton is very small and badly determined. However, for our purpose a null eccentricity is very different from a small one and $e = 0.0001$ have been taken with $a/R = 14.3$ for the present state.

As could have been anticipated, in the future Triton moves towards Neptune in a quasi-circular orbit, the inclination on the equator changing from 160° to 155° for $a/R = 5$. In the past we stopped the integration at $a/R = 30$ because of the impossibility of estimating the time scale. The eccentricity tends to 0.001 for $a/R = +30$. In both cases, past and future, the rotation of Neptune is slightly changed within the range of one hour.

5. Conclusion

With the model of evolution computed in this paper, an origin of the Moon by capture would be possible. The difficulty pointed out by Kaula (Kaula and Harris, 1975) is avoided.

For orbits with large eccentricity, the velocity close to the perigee is nearly the escape velocity and, as can be seen in Fig. 5, such a state can occur with a large semi-major axis and the perigee distance is significant. Then, a very small brake is needed to capture the satellite and no strong dynamical constraints are required at the moment of the capture.

The time scale problem remains unsolved and it will remain so for as long as the past dissipation in the oceans are unknown. The most recent determination of the oceanic tides parameters (Felsentreger *et al.*, 1979) indicates that the major part of the Moon's acceleration is caused by tides in the oceans. Thus the time lag Δt should be much smaller than 10 mn and the time scale larger than one billion years to come from $10R_e$ to the present state.

In conclusion, we shall say the Sun's act will change the scenario for distances comprised between $20R_e$ to $60R_e$ whereas the behaviour of the eccentricity and the inclination will remain qualitatively unmodified closer to the Earth.

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